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A projection principle concerning biholomorphic automorphisms

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1. Introduction

Let E denote a Banach space and D be a bounded domain in E. A mapping F of D onto itself is called a biholomorphic automorphism of D if the Fréchet derivative of F exists at each point $x \in D$ and is a bounded invertible linear E-operator. Our basic motivation in this article is the problem of describing $\text{Aut}\, B(E)$ the group of all biholomorphic automorphisms of the unit ball $B(E)$ of E. By recent results of W. KAUP [7] and J.-P. VIGUÉ [18], this problem stands in a close relationship with that of the classification of symmetric complex Banach manifolds which is solved since a long time in the finite dimensional case [2] but fairly not settled for infinite dimensions.

In 1979, E. VESENTINI [16] has shown that the unit ball of a nontrivial L^1-space admits only linear biholomorphic automorphisms. His proof goes back to investigations on Aut-invariant distances and a classical two dimensional result of M. KRITIKOS [9]. Using a characterization of polynomial vector fields tangent to $\partial B(E)$ (the boundary of $B(E)$) we found [11] an essentially two dimensional argument that enabled us to establish the sufficent and necessary condition for an L^p-space to have only linear unit ball automorphisms (for different approaches cf. also [1], [16]).

The purpose of Section 2 the general abstract part of this work is to clear up the deeper geometric background and connections of the seemingly different methods in treating L^p-spaces that occur in [16] and [11], respectively. Our main theorem provides a sufficent condition in terms of the Carathéodory (or Kobayashi) metric to reconstruct the biholomorphic automorphism group of Banach manifolds from those of its certain submanifolds via holomorphic projections. This result seems to be very well suited in calculating explicitly $\text{Aut}\, B(E)$ in various Banach spaces E admitting a sufficiently large family of contractive linear projections. In Section 3 we illustrate the use of this projection principle by two typical examples where the con-

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clusion seems hardly available with other already published methods: After numerous partial solutions, recently T. Franzoni [4] gave the complete description of $\mathrm{Aut}\, B(\mathscr{L}(H_1, H_2))$ where $\mathscr{L}(H_1, H_2) \equiv \{$bounded linear operators $H_1 \to H_2\}$ and H_1, H_2 are arbitrary Hilbert spaces. As we shall see, the projection principle makes it possible to obtain the exact description of $\mathrm{Aut}\, B(H_1 \otimes \ldots \otimes H_n)$ in an elementary way where $H_1 \otimes \ldots \otimes H_n \equiv \{$continuous n-linear functionals $H_1 \times \ldots \times H_n \to \to \mathbf{C}\}$. Note that $\mathscr{L}(H_1, H_2) \simeq H_1 \otimes H_2$ and for $n \geqq 3$, $H_1 \otimes \ldots \otimes H_n$ cannot be equipped with a suitable J^*-structure on which Franzoni's method is based. The key of the reduction by the projection principle is the fact that in finite dimensions the strong precompactness of $B(H_1 \otimes \ldots \otimes H_n)$ considerably simplifies the treatment of the space (Section 4). The second application concerns atomic Banach lattices. The unit balls of finite dimensional such spaces are exactly the convex Reinhardt domains. In 1974, T. Sunada [13] characterized $\mathrm{Aut}_0 D$ for all the bounded Reinhardt domains D. However, his proofs depend on the Cartan theory of finite dimensional semisimple Lie algebras thus cannot be carried out in infinite dimensions. If the finite dimensional ideals form a dense submanifold, the projection principle reduces even the most general case to some straightforward 2 dimensional considerations. We remark that in this way also Sunada's proof can be simplified and the method applies in parts to other Banach lattices (cf. [12]).

2. Projection principle

Our main abstract result concerns with holomorphic vector fields on complex Banach manifolds (for basic definitions see [17], [7, § 2]). If M denotes a complex Banach manifold, a vector field $v: M \to TM$ is complete in M iff for every $x \in M$, there exists a mapping $e_x: \mathbf{R} \to M$ such that $e_x(0) = x$ and $\frac{d}{dt} e_x(t) = v(e_x(t))$ $\forall t \in \mathbf{R}$. In this case we define $\exp(tv)(x) \equiv e_x(t)$. A function $\delta: TM \to \mathbf{R}_+$ is called a differential Finsler metric on M if for any fixed $x \in M$, the functional $T_x M \ni w \mapsto \delta(x, w)$ is convex and positive-homogeneous and for each coordinate-map (U, Φ), the function $f_v^{(U,\Phi)}: \Phi U \ni e \mapsto \delta(\Phi^{-1} e, v(\Phi^{-1} e))$ is locally bounded and lower semicontinuous whenever v is a holomorphic vector field on M. We shall write d_M for the Carathéodory distance [3], [17] on M, i.e. $d_M(x, y) \equiv \sup\{$areath $F(y)$: F is a holomorphic $M \to \triangle$ function, $F(x) = 0\}$ where $\triangle \equiv \{\zeta \in \mathbf{C}: |\zeta| < 1\}$. For a holomorphic mapping $F: M \to M$, we denote by F' its Fréchet derivative (recall that for any fixed $x \in M$, $F'(x)$ is a bounded linear $T_x M \to T_x M$ operator). For a Banach space E, we shall denote by E^*, $\| \ \|$, $^-$ and $B(E)$ its dual, norm, closure operation and open unit ball, respectively.

2.1. Theorem. *Let M be a complex Banach manifold, M' a (complex) submanifold of M and v a complete holomorphic vector field on M. Suppose P is a holomorphic mapping of M onto M' such that $P|_{M'}=\mathrm{id}_{M'}$ (the identity mapping on M').*

Suppose there exists a differential Finsler metric δ on M' such that

(i) *the vector field $P'v|_{M'}$ is δ-bounded (i.e. $\sup_{x\in M}\delta(x, P'(x)\,v(x))<\infty$)*

and by writing d for the intrinsic distance generated by δ on M',

(ii) *the topology of the metric d is finer than that of M',*

(iii) *for any sequence $x_1, x_2, \ldots\in M'$ which is a Cauchy sequence with respect to d but which is not convergent in M' we have $d_{M'}(x_1, x_n)\to\infty$ $(n\to\infty)$.*

Then the vector field $P'v$ is complete in M'.

Proof. For the sake of simplicity, the proof will be divided into three steps.

1) From the definition of Carathéodory distance we see immediately that $d_{M'}(x, y)\geqq d_M(x, y)$ $\forall x, y\in M'$ since $M'\subset M$. It is also well-known [2] that the mapping P is a $d_M\to d_{M'}$ contraction. Hence the relation $P|_{M'}=\mathrm{id}_{M'}$ entails $d_{M'}(x, y)\leqq d_M(x, y)$. Thus we obtained $d_{M'}=d_M|_{M'}$.

In the sequel, we set $a_x(t)\equiv\exp(tv)(x)$ $(x\in M, t\in\mathbf{R})$ and b_x will denote the maximal solution of the initial value problem $\left\{\frac{d}{dt}y=P'(y)v(y); y(0)=x\right\}$.

We show that for arbitrarily fixed $z\in M'$,

$$d_{M'}(Pa_z(h), b_z(h)) = o(h) \quad (h\to 0). \tag{1}$$

Indeed: Consider any coordinate-map (U, Φ) from the atlas of M' for which $z\in U$. We may assume without loss of generality that Φ is a biholomorphism between U and the open unit ball of some Banach space E. Then for all $h\in\left\{t\in\operatorname{dom} b_z\colon b_z(t)\in\Phi^{-1}\left(\frac{1}{2}B(E)\right)\right\}$ we have

$$d_{M'}(Pa_z(h), b_z(h)) \leqq d(Pa_z(h), b_z(h)) = d_{B(E)}(\Phi Pa_z(h), \Phi b_z(h)) \leqq$$
$$\leqq \mu\|\Phi Pa_z(h)-\Phi b_z(h)\|$$

where $\mu\equiv\sup\left\{d_{B(E)}(f, g)/\|f-g\|: f, g\in\frac{1}{2}B(E)\right\}$. It is easily seen that $\mu\leqq$ $\leqq 2\sup\left\{d_{B(E)}(f, 0)/\|f\|: f\in\frac{1}{2}B(E)\right\}=2\sup\left\{\|f\|^{-1}\,\text{areath}\,\|f\|: \|f\|\leqq\frac{1}{2}\right\}<\infty$.

The estimate $\|\Phi Pa_z(h)-\Phi b_z(h)\|=o(h)$ $(j\to 0)$ can be verified as follows: By definition, a is the solution of the initial value problem $\left\{\frac{d}{dt}y=v(y), y(0)=z\right\}$.

Therefore $\|\Phi a_z(h)-(\Phi z+h\Phi' v(z))\|=o(h)$. Thus $\frac{d}{dh}\Big|_0 [\Phi P a_z(h)-\Phi b_z(h)]=$
$=\frac{d}{dh}\Big|_0 \Phi P a_z(h)-\Phi' P' v(z)=\Phi' P' v(z)-\Phi' P' v(z)=0.$

An application of (1) directly yields that for any $x, y\in M'$,

$$\overline{\lim_{h\to 0}} \frac{1}{|h|}[d_{M'}(b_x(h), b_y(h))-d_{M'}(x, y)] = \overline{\lim_{h\to 0}} \frac{1}{|h|}[d_{M'}(Pa_x(h), Pa_y(h))-d_{M'}(x, y)] \leqq$$

$$\leqq \overline{\lim_{h\to 0}} \frac{1}{|h|}[d_M(a_x(h), a_y(h))-d_M(x, y)] = 0$$

(since P is a contraction $d_M\to d_{M'}$ and $d_{M'}=d_M|_{M'}$).

2) Henceforth we proceed by contradiction. Assume that the vector field $P'v$ is not complete in M'.

Now we may fix a point $x\in M'$ such that $\operatorname{dom} b_x\neq \mathbf{R}$. Let t_0 be a boundary point of the interval (or ray) dom b_x. Since $0\in\operatorname{dom} b_x$, we have $t_0\neq 0$. So (by passing to the vector field $\frac{1}{t_0}v$) we may assume $t_0=1$. Then consider the function

$$\varrho(t) \equiv d_{M'}\left(b_x(t), b_x\left(t+\frac{1}{2}\right)\right) \qquad \left(t\in\left[0, \frac{1}{2}\right)\right).$$

Since $b_x(t+h)=b_{b_x t}(h)$ and $b_x\left(t+\frac{1}{2}+h\right)=b_{b_x}\left(t+\frac{1}{2}\right)(h)$ whenever $t, t+h, t+\frac{1}{2}$, $t+\frac{1}{2}+h\in[0, 1)$, from step 3) it follows that

$$\overline{\lim_{h\to 0}} \frac{\varrho(t+h)-\varrho(t)}{|h|} \leqq 0 \quad \forall t\in\left[0, \frac{1}{2}\right).$$

We show that the function ϱ is locally Lipschitzian. Since the conclusion of the previous step can be interpreted as $\varrho'(t)=0$ for all such values t where $\varrho'(t)$ exists, hence we obtain that ϱ is constant i.e.

$$d_{M'}\left(b_x(t), b_x\left(t+\frac{1}{2}\right)\right) = d_{M'}\left(x, b_x\left(\frac{1}{2}\right)\right) \quad \forall t\in\left[0, \frac{1}{2}\right). \tag{2}$$

Proof. By triangle inequality, it suffices to see that for any $z\in M'$, the mapping $t\mapsto b_z(t)$ is locally Lipschitzian with respect to the metric $d_{M'}$. Denote by $\delta_{M'}$ the Carathéodory differential Finsler metric of the manifold M' (for definition see [2], [17]). Then the function $\gamma\colon \tau\mapsto\delta_{M'}(b_z(\tau), P'b(b_z(\tau)))$ is locally bounded (cf.

[17]). Hence if $\mathscr{I}$ is a compact subinterval of dom b_z then $\sup_{t\in\mathscr{I}} \gamma(t)<\infty$ and therefore

$$d_{M'}(b_z(t'), b_z(t'')) \leqq \left|\int_{t'}^{t''} \delta_{M'}(b_z(t), b'_z(t))\, dt\right| = \left|\int_{t'}^{t''} \gamma(t)\, dt\right| \leqq$$
$$\leqq \sup_{t\in\mathscr{I}} \gamma(t)\cdot|t''-t'| \quad \text{whenever} \quad t', t''\in\mathscr{I}.$$

3) Write $K\equiv\sup_{x\in M'} \delta(x, P'v(x))$ and consider the sequence $t_n\equiv\frac{1}{2}-\frac{1}{2n}$ $(n=1, 2, \ldots)$. For $m\leqq n$ we have

$$d\left(b_x\left(t_m+\frac{1}{2}\right), b_x\left(t_n+\frac{1}{2}\right)\right) \leqq \int_{t_m}^{t_n} \delta(b_x(t), b'_x(t))\, dt =$$
$$= \int_{t_m}^{t_n} \delta(b_x(t), P'v(b_x(t)))\, dt \leqq \int_{t_m}^{t_n} K dt = \frac{K}{2}\left(\frac{1}{m}-\frac{1}{n}\right).$$

Thus $\left\{b_x\left(t_n+\frac{1}{2}\right)\right\}_{n\in\mathbf{N}}$ is a Cauchy sequence with respect to the metric d. Suppose $d\left(b_x\left(t_n+\frac{1}{2}\right), z\right)\to 0$ $(n\to\infty)$ for some point $z\in M'$. Then we would have $P'v(b_x(t_n))\to P'v(z)$ $(n\to\infty)$, as a consequence of (ii). However, in this case the function $\tilde{b}(t)\equiv\begin{cases} b_x(t) & \text{if } t\in\operatorname{dom} b_x \\ b_z(t-1) & \text{if } 0\leqq(t-1)\in\operatorname{dom} b_z\end{cases}$ is a solution of the initial value problem $\left\{\frac{d}{dt}y=P'v(y), y(0)=x\right\}$ with $\operatorname{dom}\tilde{b}\supsetneqq\operatorname{dom} b_x$ which is excluded by the maximality of b_x. Thus $\left\{b_x\left(t_n+\frac{1}{2}\right)\right\}$ does not converge in the metric d.

By condition (iii), $d_{M'}\left(b_x\left(\frac{1}{2}\right), b_x\left(1-\frac{1}{2n}\right)\right)=d_{M'}\left(b_x\left(t_1+\frac{1}{2}\right), b_x\left(t_n+\frac{1}{2}\right)\right)\to\infty$ $(n\to\infty)$. From (2) we see

$$d_{M'}\left(b_x\left(\frac{1}{2}\right), b_x\left(\frac{1}{2}-\frac{1}{2n}\right)\right) \geqq d_{M'}\left(b_x\left(\frac{1}{2}\right), b_x\left(1-\frac{1}{2n}\right)\right)-$$
$$-d_{M'}\left(b_x\left(1-\frac{1}{2n}\right), b_x\left(\frac{1}{2}-\frac{1}{2n}\right)\right) = d_{M'}\left(b_x\left(\frac{1}{2}\right), b_x\left(1-\frac{1}{2n}\right)\right)-$$
$$-d_{M'}\left(x, b_x\left(\frac{1}{2}\right)\right)\to\infty \qquad (n\to\infty).$$

But this is impossible because the topology of a complex Banach manifold is always finer than that generated by its associated Carathéodory metric (cf. [17]) whence $d_{M'}\left(b_x\left(\frac{1}{2}\right), b_x\left(\frac{1}{2}-\frac{1}{2n}\right)\right)\to 0$ $(n\to\infty)$ since the mapping $t\mapsto b_x(t)$ is differentiable.

The obtained contradiction completes the proof.

2.2. Remark. From step 1) one immediately reads that in general we have

2.2a Lemma. *If* $d^*\colon N\mapsto d_N^*$ *is a metric valued functor on the category of complex Banach manifolds such that for all manifolds* N, N',

(iv) d_N^* *is a metric on* N,

(v) *each holomorphic map* $N'\to N$ *is a* $d_N^*\to d_{N'}^*$ *contraction,*

then $d_M^*|_{M'}=d_{M'}^*$, *whenever* M' *is a submanifold of* M *and there can be found a holomorphic projection of* M *onto* M'.

The proof of Theorem 2.1 can be carried out as well for any metric functor d^* with properties (iv), (v) and

(vi) $\sup\left\{d_{B(F)}^*(f,0)/\|f\|:\|f\|\leq\frac{1}{2}\right\}<\infty$ for any Banach space E.

The Kobayashi invariant metric (def. see [17], [9]) also satisfies these requirements. Hence Theorem 2.1 holds when replacing Carathéodory distances by those of Kobayashi. Moreover we have the following important special case of Lemma 2.2a.

2.2b Lemma. *If* E *denotes a Banach space and* P *is a contractive linear projection* $E\to E$ *then* $d_{B(E)}|_{B(PE)}=d_{B(PE)}$ *and* $d^k_{B(E)}|_{B(PE)}=d^k_{B(PE)}$ *where* d^k *stands for the Kobayashi distance.*

Proof. Since $\|P\|=1$ (otherwise we have the trivial case $P=0$), PE is a closed subspace of E and $PB(E)=B(PE)\subset B(E)$. Thus Lemma 2.2a can be applied to $M\equiv B(E)$ and $M'\equiv B(PE)$.

This latter result can be further specialized as follows: Consider any unit vector $e\in E$. By the Hahn—Banach theorem, there exists $\Phi\in E^*$ with $\|\Phi\|=\langle e,\Phi\rangle=1$. Then the mapping $P\colon f\mapsto\langle f,\Phi\rangle e$ is a contractive linear projection of E onto $\mathbf{C}e$. Thus Lemma 2.2b contains Vesentini's following observation.

2.2c Lemma (Vesentini [16]). *Let* E *be a Banach space,* $e\in E$ *a unit vector and* $\zeta_1,\zeta_2\in\Delta$. *Then we have* $d^k_{B(E)}(\zeta_1 e,\zeta_2 e)=d_{B(\mathbf{C}e)}(\zeta_1 e,\zeta_2 e)=d_\Delta(\zeta_1,\zeta_2)=\operatorname{areath}\left|\frac{\zeta_1-\zeta_2}{1-\zeta_1\zeta_2}\right|$, *i.e. the curve* $[\Delta\ni\zeta\mapsto\zeta e]$ *is a complex geodesic with respect to both the Carathéodory and Kobayashi distances in* $B(E)$.

Later on, we restrict our attention to Banach space unit balls. Recall ([8], [18]) that in a Banach space E, the elements of $\mathrm{Aut}_0 B(E)$ (the connected component of $\mathrm{Aut}\,B(E)$ w.r.t. the topology $\mathscr{T}_a$ defined in [15]) are exactly the exponential images of the second degree polynomial vector fields being complete in $B(E)$ whose Lie-algebra will be denoted by $\log^*\mathrm{Aut}\,B(E)$. Moreover, the orbit $[\mathrm{Aut}\,B(E)]\{0\}\equiv\equiv\{F(0)\colon F\in\mathrm{Aut}\,B(E)\}$ is the intersection of $B(E)$ with a subspace which, in the sequel, we shall denote by E_0 and we have $E_0=[\log^*\mathrm{Aut}\,B(E)]\{0\}$.

2.3. Theorem. *If E is a Banach space and $P\colon E\to E$ is a contractive linear projection then* $P[\log^*\mathrm{Aut}\,B(E)]|_{PE}\subset\log^*\mathrm{Aut}\,B(PE)$.

Proof. Let $u\in\log^*\mathrm{Aut}\,B(E)$ be arbitrarily fixed. We have to show that the vector field $Pu|_{B(PE)}$ is complete in $B(PE)$. As in the proof of Lemma 2.2b, let us consider the manifolds $M\equiv B(E)$, $M'\equiv B(PE)$, the projection $P|_{B(E)}$ of M onto M' and the vector field $v\equiv u|_{B(E)}$ which is by definition complete in M. Take the differential Finsler metric $\delta(x, w)\equiv\|w\|$ $(x\in B(PE),\ w\in PE)$ on M' whose generated intrinsic distance is obviously $d(x,y)\equiv\|x-y\|$ $(x,y\in B(PE))$. To complete the proof, we need only to verify (i), (ii), (iii).

(i): For $x\in B(PE)$ we have $P'(x)v(x)=Pu(x)$ whence by a theorem of Kaup—Upmeier [8],

$$\delta(x, P'v(x))=\|Pu(x)\|\leqq\|u(x)\|=\left\|u(0)+u'(0)x+\frac{1}{2}u''(0)(x,x)\right\|\leqq$$

$$\leqq\|u(0)\|+\|u'(0)\|_{\mathscr{L}(E,E)}+\left\|\frac{1}{2}u''(0)\right\|_{\{\text{bilin } E\times E\to E\}}.$$

(ii): Trivial.

(iii): Assume $x_1, x_2, \ldots$ is a Cauchy sequence with respect to the metric d without a limit in M'. Then for some unit vector $f\in PE$, $\|x_n-f\|\to 0$ $(n\to\infty)$ i.e. $\|x_n\|\to 1$. Therefore, by Lemma 2.2c, $d_{M'}(x_1, x_n)=d_{B(PE)}(x_1, x_n)\geqq d_{B(PE)}(x_n, 0)-$ $-d_{B(PE)}(x_1, 0)=\mathrm{areath}\,\|x_n\|=\mathrm{areath}\,\|x_1\|\to\infty$.

2.4. Corollary. *If E is a Banach space and $P\colon E\to E$ is a contractive linear projection then $P(E_0)\subset(PE)_0$. In particular, if $B(E)$ is a symmetric manifold then so is $B(PE)$, too.*

2.5. Corollary. *Let E be a Banach space. If one can find a family $\mathscr{P}$ of contractive linear projections $E\to E$ such that for every $P\in\mathscr{P}$,* $\mathrm{Aut}\,B(PE)$ *consists only of linear transformations and* $\bigcap_{P\in\mathscr{P}}\ker P=\{0\}$ *then all the elements of* $\mathrm{Aut}\,B(E)$ *are also linear.*

Proof. If $v\in\log^*\mathrm{Aut}\,B(E)$ then $Pv(0)=0$ $\forall P\in\mathscr{P}$ whence $v(0)=0$ i.e. the vector field v is linear. On the other hand $\mathrm{Aut}\,B(E)=\mathrm{Aut}^0B(E)\,\mathrm{Aut}_0B(E)=\mathrm{Aut}^0B(E)\cdot$ $\cdot\exp\log^*\mathrm{Aut}\,B(E)$, where $\mathrm{Aut}^0\equiv\{E\text{-unitarities}\}$.

3. Applications

Let (X,μ) denote a measure space. In [1], [11] it is proved

3.1. Theorem. *The unit ball of $E\equiv L^p(X,\mu)$ admits only linear biholomoprhic automorphisms unless* $\dim E=1$ *or* $p=2,\infty$.

As the first illustration of the projection principle, we show how can this result be reobtained from Thullen's classical 2 dimensional theorem [14].

Proof. Suppose $p\in[1,\infty]\setminus\{2\}$ and $\dim E>1$. If g_1, g_2 are functions in E with norm 1 having disjoint supports then it is easily seen that the mapping P_{g_1,g_2}: $E\ni f\mapsto \sum_{j=1}^{2}\int f\overline{g_j}|g_j|^{p-2}d\mu\cdot g_j$ is a contractive linear projection of E onto the subspace $E_{g_1,g_2}\equiv\sum_{j=1}^{2}\mathbf{C}g_j$. Now $B(E_{g_1,g_2})=\{\zeta_1 g_1+\zeta_2 g_2: |\zeta_1|^p+|\zeta_2|^p<1\}$ is a Reinhardt domain whose biholomorphic automorphisms are all linear by Thullen's theorem. Furthermore we have $\ker P_{g_1,g_2}=\{f\in E: \int f\overline{g_j}|g_j|^{p-2}d\mu=0\ (j=1,2)\}$. Thus $\bigcap_{g_1,g_2}\ker P_{g_1,g_2}=$ $=\{f\in E: \forall g\in E[\exists h\in E\quad \min(|g|,|h|)=0]\Rightarrow \int f\bar{g}|g|^{p-2}d\mu=0\}\subset\{f\in E: \forall X_1\subset X[\exists X_2\subset$ $\subset X\setminus X_1\ 0<\mu(X_1), \mu(X_2)<\infty]\Rightarrow \int_{X_1}df\mu=0\}=\{0\}$. Hence Corollary 2.5 establishes the linearity of $\operatorname{Aut} B(E)$.

To the next application, let $H_1, \dots, H_n$ be arbitrarily fixed Hilbert spaces[1] of at least 2 dimensions and consider the biholomorphic automorphism group of the unit ball $B\equiv B(E)$ of the space $E\equiv H_1\otimes\dots\otimes H_n$, the Banach space of n-linear functionals endowed with the usual norm $\|F\|\equiv\sup\{|F(h_1,\dots,h_n)|: h_j\in H_j, \|h_j\|=1$ $(j=1,\dots,n)\}$ for $F\in E$. For $n=1,2$, the description of $\operatorname{Aut} B$ is completely settled [5], [4]. It is worth to remark that, in the light of the Kaup Vigué theory, the difficulties in this case can be concentraded to the description of linear E-unitary operators: If $n=1, E$ can be identified with H_1 and for any fixed $c\in H_1$, the quadratic vector field $q\equiv[H_1\ni f\mapsto -(f|c)f]$ satisfies [11, (1)] i.e. tangent to the boundary of B.

Similarly, if $n=2$, E can be identified with $\mathcal{L}(H_1, H_2)$ and for fixed $C\in\mathcal{L}(E_1, E_2)$, the vector field $[\mathcal{L}(H_1, H_2)\ni F\mapsto -FC^*F]$ is quadratic and satisfies [11, (1)]. It is easily seen, in both cases that, we have $\{[\exp(tq)](0): t\in\mathbf{R}\}=(-1,1)C$, thus B is symmetric and $\operatorname{Aut} B=(\operatorname{Aut}^0 B)\exp\{q_c: c\in E\}$. Here we turn our attention first of all to the case $n\geqq 3$ which seems heavily treatable with other methods and is not touched by the literature.

3.2. Lemma. Span $\{UC: U \text{ linear}\in\operatorname{Aut}_0 B\}=E$ *whenever* $C\in E\setminus\{0\}$ *and* $\dim H_j<\infty$ $(j=1,\dots,n)$.

Proof. If $C\neq 0$ then we may fix unit vectors $e_j\in H_j$ $(j=1,\dots,n)$ such that $\gamma\equiv C(e_1,\dots,e_n)\neq 0$. Then let P_j denote the orthogonal projection of H_j onto $\mathbf{C}e_j$ and set $U_j^{\vartheta}\equiv\exp(i\vartheta_j P_j)$, $C(\vartheta_1,\dots,\vartheta_n)\equiv(U_j^{\vartheta}\otimes\dots\otimes U_j^{\vartheta})C$ $(\vartheta_j\in\mathbf{R}; j=1,\dots,n)$. Since the operators U_j^{ϑ} are H_j-unitary, $U_1^{\vartheta}\otimes\dots\otimes U_n^{\vartheta}\in\operatorname{Aut}_0 B$, therefore $e_1\otimes\dots\otimes e_n=$

[1] Without danger of confusion, we write simply $(.|.)$ for the inner product in any of $H_1,\dots,H_n$. For $A_j\in\mathcal{L}(H_j, H_j)$ and $e_j\in H_j$ $(j=1,\dots,n)$, we define $A_1\otimes\dots\otimes A_n\equiv[H_1\otimes\dots\otimes H_n\ni$ $\ni F\mapsto F(A_1f_1,\dots,A_nf_n)]$, $e_1\otimes\dots\otimes e_n\equiv[(f_1,\dots,f_n)\mapsto(f_1|e_1)\dots(f_n|e_n)]$ and $\delta_{e_1,\dots,e_n}\equiv[F\mapsto F(e_1,\dots,e_n)$, respectively.

$= -\frac{i}{\gamma} \frac{\partial^n}{\partial \vartheta_1 \dots \partial \vartheta_n}\Big|_0 C \in S \equiv \text{Span}\, \{UC: U \text{ linear} \in \text{Aut}_0 B\}$. Thus for all H_j-unitary operators V_j, $(V_1 e_1) \otimes \dots \otimes (V_n e_n) = (V_1 \otimes \dots \otimes V_n)(e_1 \otimes \dots \otimes e_n) \in S$ i.e. $f_1 \otimes \dots \otimes f_n \in S$ whenever $f_1 \in H_1, \dots, f_n \in H_n$, whence $S = E$ (since $\dim E < \infty$).

3.3. Proposition. *For $n > 2$, all the elements of $\text{Aut}\, B(H_1 \otimes \dots \otimes H_n)$ are linear.*

Proof. Observe that the family $\mathscr{P} \equiv \{P_1 \otimes \dots \otimes P_n$: all P_j-s are orthogonal H_j-projections with $\dim P_j H_j = [2 \text{ if } j \leqq 3 \text{ and } 1 \text{ if } j > 3]\}$ consists of contractive E-projections and $\bigcap_{P \in \mathscr{P}} \ker P = \{0\}$. Since for arbitrary $P \in \mathscr{P}$, the subspace PE is isometrically isomorphic to $\mathbf{C}^2 \otimes \mathbf{C}^2 \otimes \mathbf{C}^2$ ($\mathbf{C}$ is endowed with its usual euclidean norm), by Corollary 2.5 it suffices to see only that the elements of the group $\text{Aut}\, B(\mathbf{C}^2 \otimes \mathbf{C}^2 \otimes \mathbf{C}^2)$ are linear. Thus we may assume $n = 3$ and $H_j = \mathbf{C}$ $(j = 1, 2, 3)$. Assume now that $E_0 \neq 0$. Now Lemma 3.2 establishes $E_0 = E$ i.e. symmetry of B. We show that this is impossible.

Denote by e_1, e_2 the vectors $(1, 0)$ and $(0, 1)$ in $\mathbf{C}^2$, respectively, and consider the elements $C \equiv e_1 \otimes e_1 \otimes e_1$ and $F \equiv e_2 \otimes e_1 \otimes e_1 + e_1 \otimes e_2 \otimes e_1 + e_1 \otimes e_1 \otimes e_2$ of E. Since the space E is finite dimensional, for every $A \in E$ we can find $f_1, f_2, f_3 \in \partial B(\mathbf{C}^2)$ with $\|A\| = A(f_1, f_2, f_3)$. In particular, for arbitrarily given $\lambda \in \left(0, \frac{1}{3}\right)$ we can fix unit vectors $f_j(\lambda)$ such that $\|C + \lambda F\| = \langle C + \lambda F, \delta_{f_1(\lambda), f_2(\lambda), f_3(\lambda)} \rangle$. Since $C, F \geqq 0$ (i.e. $C(g_1, g_2, g_3), F(g_1, g_2, g_3) \geqq 0$ $\forall g_1, g_2, g_3 \geqq 0$) and since $\langle C + \lambda F, \delta_{e_2, e_2, e_2} \rangle = \lambda F(e_2, e_2, e_2) < 1$, for some $r_j(\lambda) \geqq 0$ we can write $f_j(\lambda) = \frac{e_1 + r_j(\lambda) e_2}{(1 + r_j(\lambda))^{1/2}}$ $(j = 1, 2, 3)$. Thus introducing the function $\Phi_\lambda(\varrho_1, \varrho_2, \varrho_3) \equiv \langle C + \lambda F, \delta_{\frac{e_1 + \varrho_1 e_2}{(1+\varrho_1^2)^{1/2}}, \dots, \frac{e_1 + \varrho_3 e_2}{(1+\varrho_3^2)^{1/2}}} \rangle = [1 + \lambda(\varrho_1 + \varrho_2 + \varrho_3)] \sum_{k=1}^{3} (1 + \varrho_k^2)^{-1/2}$, we have $\frac{\partial}{\partial \varrho_j}\Big|_{(r_1(\lambda), r_2(\lambda), r_3(\lambda))} \Phi_\lambda = 0$ $(j = 1, 2, 3)$.

So $\{\lambda(1 + r_j^2) - [1 + \lambda(r_1 + r_2 + r_3)]\} \cdot \sum_{k=1}^{3} (1 + r_k^2)^{-3/2} = 0$ $(j = 1, 2, 3)$ and hence $\lambda = \frac{r_1}{1 - r_1(r_2 + r_3)} = \frac{r_3}{1 - r_2(r_1 + r_3)} = \frac{r_3}{1 - r_3(r_1 + r_2)}$. Therefore $r_j \neq 0$ $(j = 1, 2, 3)$ and $\frac{1}{r_1} + r_1 = \frac{1}{r_2} + r_2 = \frac{1}{r_3} + r_3 \left(= \frac{1}{\lambda} + \sum_{j=1}^{3} r_j \right)$. Observe that from this and from the assumption $\lambda \in \left(0, \frac{1}{3}\right)$ it follows that $r_1 = r_2 = r_3$. (Otherwise there would be $r > 0$ such that two of the numbers r_1, r_2, r_3 coincided with r and the third with $1/r$, respectively. But then $\lambda = \frac{1/r}{1 - (1/r)(r + r)} < 0$.) Thus the relation $\lambda = \frac{r}{1 - 2r}$ holds where $r(\lambda) \equiv r_1(\lambda) = r_2(\lambda) = r_3(\lambda)$. This fact can be so interpreted that for sufficiently small values of $r>0$ $\left(\text{namely for } \lambda>\frac{1}{3} \text{ i.e. } r<\frac{\sqrt{17}-3}{4}\right)$, $F_r\equiv C+\frac{r}{1-2r^2}F$, $\Phi_r\equiv\delta_{e_1+re_2,\, e_1+re_2,\, e_1+re_2}$ fulfill $\|F_r\|\cdot\|\Phi_r\|=\langle F_r, \Phi_r\rangle$. Then by [11, Lemma]

$$\text{(2)}\qquad \|F_r\|^2\overline{\langle C, \Phi_r\rangle}+\langle q(F_r, F_r), \Phi_r\rangle = 0 \qquad \left(0<r<\frac{\sqrt{17}-3}{4}\right),$$

for some symmetric bilinear map $q:E\times E\to E$. Here $\langle C, \Phi_r\rangle=1$, $\|F_r\|=\|\Phi_r\|^{-1}\langle F_r, \Phi_r\rangle=$ $=(1+r^2)^{-3/2}\left(1+3r\frac{r}{1-2r^2}\right)=(1+r^2)^{-1/2}(1-2r^2)^{-1}$ and $\langle q(F_r, F_r), \Phi_r\rangle=\langle q(C, C), \Phi_r\rangle+$ $+2\frac{r}{1-2r^2}\langle q(C, F), \Phi_r\rangle+\left(\frac{r}{1-2r}\right)^2\langle q(F, F), \Phi\rangle$. Taking into consideration that for fixed $V\in E$, the function $r\mapsto\langle V, \Phi_r\rangle$ is a polynomial of 3^{rd} degree in r, from (2) we obtain

$$\text{(2')}\qquad (1+r^2)^{-1}(1-2r^2)^{-2}+p_1(r)+p_2(r)(1-2r^2)^{-1}+p_3(r)(1-2r^2)^{-2}=0$$

for some polynomial-triplet p_1, p_2, p_3. However, (2′) immediately implies the contradictory fact that the function $r\mapsto(1+r^2)^{-1}$ is a polynomial.

3.4. Theorem. *The linear* $H_1\otimes\ldots\otimes H_n$*-unitary operators are exactly those operators* F *for which there exists a permutation* π *of the index set* $\{1, \ldots, n\}$ *and there are surjective linear isometries* $U_k\colon H_k\to H_{\pi(k)}$ $(k=1, \ldots, n)$ *such that*

$$\text{(3)}\qquad F(L)=[(f_1, \ldots, f_n)\mapsto L(U_1^{-1}f_{\pi(1)}, \ldots, U_n^{-1}f_{\pi(n)})].$$

A linear vector field V *belongs to* $\log^*\mathrm{Aut}\, B$ *if and only if it is of the form*

$$\text{(3')}\qquad V=i\cdot\sum_{k=1}^{n}\mathrm{id}_{H_1}\otimes\ldots\otimes\mathrm{id}_{H_{k-1}}\otimes A_k\otimes\mathrm{id}_{H_{k+1}}\otimes\ldots\otimes\mathrm{id}_{H_n}$$

where the A_k*-s are arbitrary self-adjoint* H_k*-operators.*

Proof. Based on some compactness arguments, in the next section we shall establish independently the validity of (3′) if the spaces H_k are all finite dimensional. Our starting point here is (3′) for finite dimensional E. First we extend it to infinite dimensions.

Let V linear$\in\log^*\mathrm{Aut}\, B$ and $e_1^*\in\partial B(H_1), \ldots, e_n^*\in\partial B(H_n)$ be arbitrarily fixed and define the operator $\tilde V\equiv V-\langle V(e_1^*\otimes\ldots\otimes e_n^*), \delta_{e_1^*,\ldots,e_n^*}\rangle\mathrm{id}_E$. Since $i\cdot\mathrm{id}_E\in$ $\in\log^*\mathrm{Aut}\, B$, we have $\tilde V\in\log^*\mathrm{Aut}\, B$. Remark that $\tilde V(e_1^*\otimes\ldots\otimes e_n^*)=0$. Then consider the family of mappings $\mathscr{P}\equiv\{P_1\otimes\ldots\otimes P_n\colon P_k$ is an orthogonal H_k-projection, $\dim P_kH_k<\infty, e_k\in P_kH_k$ $(k=1, \ldots, n)\}$. Any element $P\equiv P_1\otimes\ldots\otimes P_n$ of $\mathscr{P}$ is a contractive linear projection of the space E onto its subspace $(P_1H_1)\otimes\ldots\otimes(P_nH_n)$. Thus by the projection principle, $P\tilde V|_{PE}\in\log^*\mathrm{Aut}\, B(PE)$ $\forall P\in\mathscr{P}$. Hence (applying (3′) to the *finite dimensional* $(P_1H_1)\otimes\ldots\otimes(P_nH_n)$) for each $P\in\mathscr{P}$, there exists a unique choice of $A_1^P \in \{\text{self-adj. } H_1\text{-op.-s}\}, \ldots, A_n^P \in \{\text{self-adj. } H_n\text{-op. -s}\}$ such that

$$A_k^P H_k \subset P_k H_k (\text{i.e. } P_k A_k^P P_k = A_k^P) \quad \text{and} \quad (A_k^P e_k^* | e_k^*) = 0 \quad (k = 1, \ldots, n),$$

$$P\tilde{V}P = \sum_{k=1}^{n} i \cdot \mathrm{id}_{H_1} \otimes \ldots \otimes \mathrm{id}_{H_{k-1}} \otimes A_k \otimes \mathrm{id}_{H_{k+1}} \otimes \ldots \otimes \mathrm{id}_{H_n}.$$

Introduce the following partial ordering $\leqq$ in $\mathscr{P}$: If $P = P_1 \otimes \ldots \otimes P_n$ and $Q = Q_1 \otimes \ldots \otimes Q_n$ then let $P \leqq Q \overset{\text{def}}{\Longleftrightarrow} P_k H_k \subset Q_k H_k$ (i.e. $P_k \leqq Q_k$) $k = 1, \ldots, n$. From the relation $P \leqq Q \Rightarrow P\tilde{V}P = PQ\tilde{V}QP$ we immediately see

$$A_k^P = P_k A_k^Q P_k \quad (k = 1, \ldots, n) \quad \text{whenever} \quad P \leqq Q. \tag{4}$$

Observe that for any fixed $P \in \mathscr{P}$ and index k,

$$|(A_k^P e|f)| = |\langle (P\tilde{V})(e_1^* \otimes \ldots \otimes e_{k-1}^* \otimes e \otimes e_{k+1}^* \otimes \ldots \otimes e_n^*), \delta_{e_1^*, \ldots, e_{k-1}^*, \ldots, f, e_{k+1}^*, \ldots, e_n^*} \rangle| \leqq$$

$$\leqq \|P\tilde{V}\| \cdot \|e_1^* \otimes \ldots \otimes e \otimes \ldots \otimes e_n^*\| \cdot \|\delta_{e_1^*, \ldots, f, \ldots, e_n^*}\| = \|P\tilde{V}\| \leqq \|\tilde{V}\| \qquad \forall e, f \in \partial B(H_k),$$

that is

$$\|A_k^P\| \leqq \|\tilde{V}\| \quad (k = 1, \ldots, n) \quad \forall P \in \mathscr{P}. \tag{5}$$

Since obviously $\forall P, Q \in \mathscr{P}\, \exists R \in \mathscr{P}\; P, Q \leqq R$ and since by (4), (5) the relation $P \leqq Q$ entails $|(A_k^Q e|f) - (A_k^P e|f)| = |(A_k^Q (e - P_k e)|f) + (A_k^Q P_k e|f - P_k f)| \leqq \|\tilde{V}\| (\|e - P_k e\| + \|f - P_k f\|)$ $\forall e, f \in \partial B(H_k)$, $k = 1, \ldots, n$, the definitions

$$a_k(e, f) \equiv \lim_{P \in \mathscr{P}} (A_k^P e|f) \quad (e, f \in H_k, \quad k = 1, \ldots, n)$$

make sense and determine bounded sesquilinear functionals. Therefore there exist self-adjoint operators $A_1 \colon H_1 \to H_1, \ldots, A_n \colon H_k \to H_n$ such that $a_k(e, f) = (A_k e|f)$ and hence $(A_k^P e|f) = (A_k^P (P_k e)|P_k f) = (A_k P_k e|P_k f) = (A_k P_k e|P_k f) = (P_k A_k P_k e|f)$ $\forall e, f \in H_k$ i.e. $A_k^P = P_k A_k P_k$ $(P \in \mathscr{P}, k = 1, \ldots, n)$. Now for arbitrary $L \in E$, $e_1 \in H_1, \ldots, e_n \in H_n$ the projections $P_k \equiv \mathrm{proj}_{\mathrm{Span}\{e_k, A_k e_k, e_k^*\}}$ $(k = 1, \ldots, n)$ satisfy

$$[\tilde{V}L](e_1, \ldots, e_n) = [\tilde{V}L](P_1 e_1, \ldots, P_n e_n) = [P\tilde{V}L](e_1, \ldots, e_n) =$$

$$= \sum_{k=1}^{n} L(e_1, \ldots, P_k A_k e_k, \ldots, e_n) = \sum_{k=1}^{n} L(e_1, \ldots, A_k e_k, \ldots, e_n).$$

Thus we can write $VL(e_1, \ldots, e_n) = \sum_{k=1}^{n} L(e_1, \ldots, B_k e_k, \ldots, e_n)$ where $B_j \equiv A_j$ for $j = 1, \ldots, n-1$ and $B_n \equiv A_n + \langle V(e_1^*, \ldots, e_n^*), \delta_{e_1^*, \ldots, e_n^*} \rangle \mathrm{id}_E$, proving (3′) in general.

To prove (3), let F be an arbitrarily given linear E-unitary operator and introduce the families $\mathscr{P}_k \equiv \{P_1 \otimes \ldots \otimes P_n \colon P_k$ is an orthogonal H_k-projection, $P_j = \mathrm{id}_{H_j}$ for $j \neq k\}$ $(k = 1, \ldots, n)$. From (3′) we see $i\mathscr{P}_k \subset \log^* \mathrm{Aut}\, B$ and hence for every $P \in \mathscr{P}_k$, the mapping $Q \equiv FPF^{-1}$ also has the properties $iQ \in \log^* \mathrm{Aut}\, B$ and $Q^2 = Q$

(since $P^2=P$) which is possible (by (3′)) only if $Q\in\mathscr{P}_{\ell_k(P)}$ for some index $\ell_k(P)$ $(k=1, \ldots, n)$.

Let $k\in\{1, \ldots; n\}$ be fixed. We show that $\ell_k(P_1)=\ell_k(P_2)$ $\forall P_1, P_2\in\mathscr{P}_k\setminus\{\mathrm{id}_E\}$. Indeed, if $\ell_k(R_1)\neq\ell_k(R_2)$ then the operators $Q_j\equiv FR_jF^{-1}\,(j=1, 2)$ commute (i.e. $[Q_1, Q_2]\equiv Q_1Q_2-Q_2Q_1=0$) whence we would have $[R_1, R_2]=0$. Observe that $\forall P_1, P_2\in\mathscr{P}_k\setminus\{\mathrm{id}_E\}\,\exists P_3\in\mathscr{P}_k$ $[P_1, P_3], [P_2, P_3]\neq 0$, thus (by taking $R_1\equiv P_j$ and $R_2\equiv P_3$ $j=1, 2$) $\ell_k(P_j)=\ell_k(P_3)$ holds for $j=1, 2$.

Therefore there exists a permutation π with

$$F\mathscr{P}_kF^{-1} = \mathscr{P}_{\pi(k)} \quad (k = 1, \ldots, n). \tag{6}$$

Since the finite linear combinations of orthogonal projections form a dense submanifold of the algebra of linear operators in any Hilbert space, it directly follows the existence of surjective linear isometries $S_k\colon \mathscr{L}(H_k, H_k)\to\mathscr{L}(H_{\pi(k)}, H_{\pi(k)})$ such that

$$F(\mathrm{id}_{H_1}\otimes\ldots\otimes\mathrm{id}_{H_{k-1}}\otimes A_k\otimes\mathrm{id}_{H_{k+1}}\otimes\ldots\otimes\mathrm{id}_{H_n})F^{-1}=$$
$$=\mathrm{id}_{H_1}\otimes\ldots\otimes\mathrm{id}_{H_{\pi(k)-1}}\otimes S_k(A_k)\otimes\mathrm{id}_{H_{\pi(k)+1}}\otimes\ldots\otimes\mathrm{id}_{H_n}$$
$$(A_k\in\mathscr{L}(H_k, H_k);\ k = 1, \ldots, n).$$

As a consequence of the relations (6), the mappings S_k send orthogonal projections into orthogonal projections and therefore they constitute *-isomorphisms between the C*-algebras $\mathscr{L}(H_k, H_k)$ and $\mathscr{L}(H_{\pi(k)}, H_{\pi(k)})$. It is well-known that now we can write

$$S_k\colon A_k\mapsto U_kA_kU_k^{-1} \quad (k = 1, \ldots, n)$$

for some surjective linear isometries $U_k\colon H_k\mapsto H_{\pi(k)}$. Thus if we denote by σ the inverse of the permutation π, for any linear E-operator A of the form $A\equiv A_1\otimes\ldots\otimes A_n$ (where $A_k\in\mathscr{L}(H_k, H_k)$ $k=1, \ldots, n$) we have

$$(FAF^{-1})L = [(f_1, \ldots, f_n)\mapsto L(U_{\sigma(1)}A_{\sigma(1)}U_{\sigma(1)}^{-1}f, \ldots, U_{\sigma(n)}A_{\sigma(n)}U_{\sigma(n)}^{-1}f_n)] \quad \forall L\in E.$$

This means that $FAF^{-1}=UAU^{-1}$ $\forall A\in\mathscr{L}(E, E)$ holds for the E-unitary operator U defined by

$$U(L)\equiv[(f_1, \ldots, f_n)\mapsto L(U_1^{-1}f_{\pi(1)}, \ldots, U_n^{-1}f_{\pi(n)})] \quad (L\in E).$$

It is easily seen that this is possible only if $F=e^{i\vartheta}U$ for some $\vartheta\in\mathbf{R}$ which completes the proof.

In the remainder part of this section, by making use of the projection principle, we shall examine the structure of biholomorphic unit ball automorphisms in case of minimal atomic Banach lattices (abbr. by min. B-lattices).

A Banach lattice E is called a min. B-lattice if it is norm-spanned by its 1 dimensional ideals. Henceforth we reserve the symbol E to designate a fixed min. B-lattice.

According to a well-known representation lemma [10. p. 143, Ex. 7 (b)], we may assume that for a fixed set X, E is a sublattice of $\{X \to \mathbf{C}$ functions$\}$ such that

(7) $$1_x \in E \quad \text{and} \quad \|1_x\| = 1 \quad \forall x \in X,$$

(8) Span $\{1_x: x \in X\} = E$. (1_x stand for $[X \ni y \mapsto 1$ if $y = x$ and 0 elsewhere]).

Remark that then

(8′) $$wf \in E \quad \text{and} \quad wf = \lim_{Y \text{ finite} \subset X} w 1_Y f \quad \text{whenever} \quad f \in E, \quad \sup_{x \in X} |w(x)| \leqq 1.$$ [2]

For the sake of simplicity we write $B \equiv B(E)$ and the functional $[E \ni f \mapsto f(x)]$ will be denoted by 1_x^*.

First we describe the linear part of Aut B.

3.5 Definition. For $x, y \in X$, let $x \sim y$ if $\langle \ell(1_x), 1_y \rangle \neq 0$ for some linear element ℓ of $\log^*$Aut B.

3.6. Lemma. (i) *$x \sim y$ if and only if for all $f, g \in E$, $f - g \in 1_{\{x,y\}} E$ and $\sum_{z=x,y} |f(z)|^2 = \sum_{z=x,y} |g(z)|^2$ entail $\|f\| = \|g\|$.*

(ii) *The relation $\sim$ is an equivalence. Moreover, in case of $x_1 \sim \ldots \sim x_n$,*

$$f - g \in 1_{\{x, \ldots, x_n\}} \quad \textit{and} \quad \sum_{j=1}^{n} |f(x_j)|^2 = \sum_{j=1}^{n} |g(x_j)|^2 \quad \textit{imply} \quad \|f\| = \|g\|$$

for all $f, g \in E$ whenever $x_1, \ldots, x_n$ are distinct points.

Proof. (i) Let $Y \equiv \{y_1, \ldots, y_n\}$ be an arbitrary finite subset of X and ℓ linear $\in \log^*$ Aut B. Set $\alpha_{jk} \equiv \langle \ell(l_{y_j}), 1_{y_k} \rangle$ and assume $\alpha_{12} \neq 0$ (i.e. $y_1 \sim y_2$). Since the mapping $P: f \mapsto 1_Y f$ is a band projection of E onto $\sum_{j=1}^{n} \mathbf{C} 1_{y_j}$, the projection principle establishes $\tilde{\ell} \in \log^*$Aut PB where $\tilde{\ell} \equiv P\ell|_{PE}$. Thus by [11, Lemma][3]

(9) $$\operatorname{Re} \langle \tilde{\ell}(f), \Phi \rangle = 0 \Leftarrow \langle f, \Phi \rangle = \|f\| \|\Phi\| \quad \forall f \in PE, \Phi \in (PE)^*.$$

[2] Proof: Given $\varepsilon > 0$, by (8), there are Z finite $\subset X, g \in 1_Z f$ with $\|f - g\| < \varepsilon/2$. Now $Z \subset Y_1, Y_2$ finite $\subset X$ implies $\|f - g\| \geqq |f - 1_Z f| \geqq w|(f - 1_Z f)| \geqq |w(1_{Y_1 \cup Y_2} f - 1_{Y_j} f)|$ $(j = 1, 2)$ i.e. by triangle inequality $\varepsilon \geqq \|w 1_{Y_1} f - w 1_{Y_2} f\|$. Thus $\{w 1_Y f\}_{Y \text{ finite}}$ is a Cauchy net in E. Hence for some $h \in E^2$, $w 1_Y f \to h$. But $h(x) = \langle h, 1_x^* \rangle = \lim_Y \langle w 1_Y f, 1_x \rangle = w(x) f(x)$ $\forall x$.

[3] In the same way as in [11, Lemma], one can see that if a linear vector field ℓ on Banach space F belongs to $\log^*$Aut $B(F)$ then $\operatorname{Re} \langle \ell(f), \Phi \rangle = 0 \Leftarrow \langle f, \Phi \rangle = \|f\| \|\Phi\|$ $\forall f \in F, \Phi \in F^*$.

Proof: Since ℓ is tangent to $\partial B(F)$, we have $\ell(f) \in (H - f)$ whenever $\|f\| = 1$ and H is a real hyperplane in F supporting $B(f)$ at f. But the general form of such a supporting hyperplane is $H = \{h \in F: \operatorname{Re} \langle h, \Phi \rangle = 1\}$ where $\Phi \in F^*$ with $\|\Phi\| = \langle f, \Phi \rangle = 1$.

Introduce the function $p(\varrho_1, \ldots, \varrho_n) \equiv \sum_{j=1}^{n} \varrho_j 1_{y_j}$ on $\mathbf{R}_+^n$ and set $C \equiv \{\varrho \in \mathbf{R}_+^n:$ $\operatorname{grad}|_\varrho p$ does not exist$\}$. Since p is an increasing positively homogenenous convex function, C is a cone of Lebesgue measure 0. Let us fix arbitrary vectors $\varrho \in \mathbf{R}_+^n \setminus C$, $\vartheta \in \mathbf{R}^n$ and set $\pi \equiv \operatorname{grad}|_\varrho p$, $f_0 \equiv \sum_{j=1}^{n} \varrho_j e^{i\vartheta_j} 1_{y_j}$, $\Phi \equiv \sum_{j=1}^{n} \varrho_j e^{-i\vartheta_j} 1_{y_j}^*$. Since the function p is increasing, $\pi, \ldots, \pi_n \geqq 0$. Since π is positive homogeneous and convex, $\sum_{j=1}^{n} \pi_j \varrho_j = p(\varrho_1, \ldots, \varrho_n)$ i.e. $\langle f_0, \Phi \rangle = \|f_0\|$. On the other hand, for any $f \in PE$

$$|\langle f, \Phi \rangle| = \left| \sum_{j=1}^{n} \pi_j e^{-i\vartheta_j} f(y_j) \right| \leqq \sum_{j=1}^{n} \pi_j |f(y_j)| \leqq p(|f(y_1)|, \ldots, |f(y_n)|) = \|f\|$$

i.e. $\|\Phi\| = 1$. Hence (9) can be applied to f_0 and Φ. Thus

$$\operatorname{Re} \left\langle \ell \left(\sum_{j=1}^{n} \varrho_j e^{i\vartheta_j} 1_{y_j} \right), \sum_{j=1}^{n} \pi_j e^{-i\vartheta_j} 1_{y_j}^* \right\rangle = 0. \tag{9'}$$

By the arbitrary choice of $\vartheta \in \mathbf{R}^n$, an equivalent form to (9′) is

$$\operatorname{Re} \left[\sum_j \varrho_j \pi_j \alpha_{jj} + \sum_{j \neq k} (\varrho_j \pi_k \alpha_{jk} + \varrho_k \pi_j \overline{\alpha_{kj}}) z_j z_k^{-1} \right] = 0 \quad \text{whenever} \quad |z_1| = \ldots = |z_n| = 1. \tag{9''}$$

This is possible only if the rational expression (w.r.t. $z_1, \ldots, z_n$) in the argument of the Re operation vanishes. Thus in particular $\varrho_1 \pi_2 \alpha_{12} + \varrho_2 \pi_1 \overline{\alpha_{21}} = 0$. I.e. we obtained the following partial differential equation

$$\varrho_1 \frac{\partial p}{\partial \varrho_2} \alpha_{12} + \varrho_2 \frac{\partial p}{\partial \varrho_1} \overline{\alpha_{21}} = 0 \quad (\varrho \in \mathbf{R}_+^n \setminus C). \tag{10}$$

Since $\varrho_2 = \|\varrho_2 1_{y_2}\| \leqq \left\| \sum_j \varrho_j 1_{y_j} \right\| = p(\varrho)$ $\forall \varrho \in \mathbf{R}_+^n$, there exists $\varrho \in \mathbf{R}_+^n \setminus C$ with $\dfrac{\partial \varrho}{\partial p_2} > 0$. Therefore $\alpha_{21} \neq 0$, moreover $\overline{\alpha_{21}}/\alpha_{12} < 0$, i.e. $\overline{\alpha_{21}}/\alpha_{12} = -|\alpha_{21}|/|\alpha_{12}|$.

For $(\varrho_3, \ldots, \varrho_n) \in \mathbf{R}_+^{n-2}$, define $\varphi_{\varrho_3, \ldots, \varrho_n}: \mathbf{R} \to \mathbf{R}$ by $\varphi_{\varrho_3, \ldots, \varrho_n}(t) \equiv p(|\alpha_{12}| \cos t, |\alpha_{21}| \sin t, \varrho_3, \ldots, \varrho_n)$. Since C is a cone of measure 0 in $\mathbf{R}_+^n$, (10) implies

$$\varphi_{\varrho_3, \ldots, \varrho_n}(t) = 0 \quad \text{for almost every} \quad t \in (0, \pi/2) \quad \text{and} \quad (\varrho_3, \ldots, \varrho_n) \in \mathbf{R}_n^{n-2}. \tag{11}$$

From the convexity of p it follows that it is locally Lipschitzian in the interior of $\mathbf{R}_+^n$. Hence, by (11),

$$\varphi_{\varrho_3, \ldots, \varrho_n}(t) = \varphi_{\varrho_3, \ldots, \varrho_n}(0) \quad \forall t \in [0, \pi/2], (\varrho_3, \ldots, \varrho_n) \in \mathbf{R}_+^{n-2}. \tag{11'}$$

But then $|\alpha_{12}|=\varphi_{0,\dots,0}(\pi/2)=|\alpha_{21}|$ whence

$$p|\alpha_{12}|^{-1}(\varrho_1^2+\varrho_2^2)^{1/2}\cdot\varphi_{\varrho_3,\dots,\varrho_n}\left(\arccos\frac{\varrho_1}{(\varrho_1+\varrho_2)^{1/2}}\right)=|\alpha_{12}|^{-1}(\varrho_1^2+\varrho_2^2)^{1/2}\varphi_{\varrho_3,\dots,\varrho_n}(0)=$$
$$=p(\sqrt{\varrho_1^2+\varrho_2^2},0,\varrho_3,\dots,\varrho_n).$$

Let now $f,g\in E$ be functions such that $f-g\in 1_{\{y_1,y_2\}}E$ and $\sum_{j=1}^{2}|f(y_j)|^2=\sum_{j=1}^{2}|g(y_j)|^2$. Then $\|1_Yf\|=p\left(\left(\sum_{j=1}^{2}|f(y_j)|^2\right)^{1/2},0,|f(y_3)|,\dots,|f(y_n)|\right)=\|1_Yg\|$. Taking into consideration the fact that Y may be any finite subset of X, from (8′) we obtain $\|f\|=\|g\|$.

Conversely: Assume that $f-g\in 1_{\{y_1,y_2\}}E$ and $\sum_{j=1}^{2}|f(y_j)|^2=\sum_{j=1}^{2}|g(y_j)|^2$ imply $\|f\|=\|g\|$ for all $f,g\in E$. Then the mappings $U^t\equiv[f\mapsto 1_{X\setminus\{y_1,y_2\}}f+((\cos t)\cdot f(y_1)+(\sin t)\cdot f(y_2))1_y+((-\sin t)\cdot f(y_1)+c+(\cos t)\cdot f(y_2))1_y]$ $(t\in\mathbf{R})$ form a one-parameter E-unitary operator group. Hence the linear field $\frac{d}{dt}\Big|_0 U^t=[f\mapsto f(y_2)1_{y_1}-f(y_1)1_{y_2}]$ belongs to $\log^*\mathrm{Aut}\,B$.

Proof of (ii): Say that $f\sim^Y g$ if Y finite$\subset X$, $f,g\in E$, $f-g\in 1_YE$ and $\sum_{y\in Y}|f(y)|^2=\sum_{y\in Y}|g(y)|^2$. Obviously, the relations $\sim^Y$ are all equivalences. Consider the set $N\equiv\{m:\exists x_1\sim\dots\sim x_m\ \exists f,g\in E\ f\sim^{\{x_1,\dots,x_m\}}g,\ \|f\|\neq\|g\|\}$. Suppose $N\neq 0$ and set $n\equiv\min N$. From (i) it follows $n>2$. Fix a set $Y\equiv\{y_1,\dots,y_n\}$ and functions $f_1,f_2\in E$ such that $f_1\sim^Y f_2$, $y_1\sim\dots\sim y_n$ but $\|f_1\|\neq\|f_2\|$. Consider the functions $g_j\equiv 1_{(X\setminus Y)\cup\{y_1\}}f_j+\left(\sum_{k=2}^{n}f_j(y_k)^2\right)^{1/2}1_{y_2}$ $(j=1,2)$. Observe that $f_j\sim^{\{y_2,\dots,y_n\}}g_j$ whence $\|f_j\|=\|g_j\|$ $(j=1,2)$. However, $g_1\sim^{\{y_1,y_2\}}g$ and therefore by (i) we have $\|g_1\|=\|g_2\|$ contradicting the assumption $\|f_1\|\neq\|f_2\|$. Thus $N=\emptyset$. Hence if $y_1\sim y_2\sim y_3$ then $\forall f,g\in E$ $f\sim^{\{y_1,y_2,y_3\}}g\Rightarrow f\sim^{\{y_1,y_3\}}g$ i.e. by (i), $y_1\sim y_3$ holds.

3.7. Corollary. *The proof of* (i) *shows that* $\langle\ell(1_{y_1}),1^*_{y_2}\rangle=-\langle\ell(1_{y_2}),1^*_{y_1}\rangle$ *whenever* $y_1,y_2\in X$ *and* ℓ linear$\in\log^*\mathrm{Aut}\,B$.

3.8. Definition. From now on we reserve the notation $\{S_i: i\in\mathcal{I}\}$ to denote the partition of X formed by the equivalence classes of the relation $\sim$. For each $i\in\mathcal{I}$, we shall denote the projection band $1_{S_i}E$ of E by H_i.

3.9. Proposition. (i) *If* $f,g\in E$ *are functions with finite support and* $\|f|_{S_i}\|_{\ell^2}=\|g|_{S_i}\|_{\ell^2}$ $(\equiv(\sum_{x\in S_i}|g(x)|^2)^{1/2})$ $\forall i\in\mathcal{I}$ *then* $\|f\|=\|g\|$.

(ii) *For any* $i\in\mathcal{I}$, H_i *is a Hilbert space (i.e. the norm* $\|\cdot\|$ *restricted to* H_i *satisfies parallelogram identity). Namely, a function* $h: X\to\mathbf{C}$ *belongs to* H_i *iff* $\operatorname{supp}(h)\subset S^i$, $\sum_{x\in S_i}|h(x)|^2<\infty$, *furthermore we have* $\|f\|=\|f\|_{\ell^2}$ $\forall f\in H_i$.

(iii) *If* $f,g\in E$ *and* $\|f|_{S_i}\|=\|g|_{S_i}\|$ $\forall i\in\mathcal{I}$ *then* $\|f\|=\|g\|$.

(iv) *If* $g: X\to\mathbf{C}, f\in E$ *and* $\|f|_{S_i}\|_{\ell^2}=\|g|_{S_i}\|_{\ell^2}$ $\forall i\in\mathcal{I}$ *then* $g\in E$.

(v) *Assume* $\ell\in\mathcal{L}(E,E)$. *Then* $\ell\in\log^*\operatorname{Aut}B$ *if and only if there exists a family of linear mappings* $\{\ell_j: j\in\mathcal{I}\}$ *such that* $i\cdot\ell_j$ *is a self-adjoint* H_j*-operator for each* $j\in\mathcal{I}$, $\sup_{j\in\mathcal{I}}\|\ell_j\|<\infty$ *and* $\ell=\bigotimes_{j\in\mathcal{I}}\ell_j$.

Proof. (i) is a directe consequence of Lemma 3.6 (i).

(ii): Let $f\in H$ and $x_0\in E$ be arbitrarily fixed. By (i), $\|1_Y f\|=\|(\sum_{y\in Y}|f(y)|^2)^{1/2}1_{x_0}\| =(\sum_{y\in Y}|f(y)|^2)^{1/2}$ for all Y finite$\subset X$. Hence by (8′), $\infty>\|f\|=\|f\|_{\ell^2}$. Furthermore, if g is a function $X\to\mathbf{C}$ having support in S_i and $\|g\|_{\ell^2}<\infty$ then (i) ensures $\forall Y_1, Y_2$ finite$\subset X$, $\|1_{Y_1}f-1_{Y_2}f\|=\|1_{Y_1}f-1_{Y_2}f\|_{\ell_2}=\|1_{Y_1\Delta Y_2}f\|$ i.e. the net $\{1_Y f\}_Y$ is a Cauchy net whence $f\in E$.

(iii): Let $\varepsilon>0$ be fixed. According to (8′), one can find Y finite$\subset X$ with $\|f-1_Z f\|, \|g-1_Z g\|<\varepsilon$ $\forall Z\subset Y$. Since the index set $J\equiv\{i\in\mathcal{I}: Y\cap S_i=\emptyset\}$ is finite, there exists a family of sets $\{Z_i: i\in J\}$ such that $Y\cap S_i\subset Z_i$ finite$\subset S_i$ $(i\in J)$ and $\sum_{i\in J}\|1_{S_i}f-1_{Z_i}f\|_{\ell_2}<\varepsilon$. Consider now the functions $f_\varepsilon\equiv\sum_{i\in J}\|1_{Z_i}f\|_{\ell^2}\cdot 1_{x_i}$ and $g_\varepsilon\equiv \equiv\sum_{i\in J}\|1_{Z_i}g\|_{\ell^2}\cdot 1_{x_i}$ where x_i denotes an arbitrarily fixed point of S_i $(i\in J)$. By writing $Z\equiv\bigcup_{i\in J}Z_i$, we can see $\|f_\varepsilon\|=\|1_Z f\|$, $\|g_\varepsilon\|=\|1_Z g\|$ and $\|f-1_Z f\|, \|g-1_Z g\|<\varepsilon$. Using the triangle inequality, $\|f_\varepsilon-g_\varepsilon\|\leqq\sum_{i\in J}|\|1_{Z_i}f\|_{\ell^2}-\|1_{Z_i}g\|_{\ell^2}|=$(since $\|1_{S_i}f\|_{\ell^2}=\|1_{S_i}g\|_{\ell^2}$ for all i)$=\sum_{i\in J}|\|1_{Z_i}f\|_{\ell^2}-\|1_{S_i}f\|_{\ell^2}+\|1_{S_i}g\|_{\ell^2}-\|1_{Z_i}g\|_{\ell^2}|\leqq(\sum_{i\in J}(\|1_{S_i}f-1_{Z_i}f\|_{\ell^2}=\|1_{S_i}g- -1_{Z_i}g\|_{\ell^2})<2\varepsilon$. Thus $|\|f\|=\|g\||\leqq\|f-1_Z f\|+|\|1_Z f\|=\|1_Z g\||+\|g-1_Z g\|\leqq 4\varepsilon$.

(iv): By (8′), to every number $n\in\mathbf{N}$, we can choose Z_n finite$\subset X$ such that $\|f-1_{Z_n}f\|<\frac{1}{n}$. We may assume without loss of generality $Z_1\subset Z_2\subset\dots$. Then set $\mathcal{I}_n\equiv\{i\in\mathcal{I}: Z_n\cap S_i\neq\emptyset\}$, $g_n\equiv\sum_{i\in\mathcal{I}_n}1_{S_i}g$. By (ii) and the finiteness of the sets $\mathcal{I}_n$, $g_n\in E$ $\forall n\in\mathbf{N}$. If $n>m$ then $\|g_n-g_m\|=\|\sum_{i\in\mathcal{I}_n}1_{S_i}g\|=$(by (iii))$=\|\sum_{i\in\mathcal{I}_n\setminus\mathcal{I}_m}1_{S_i}f\|\leqq$(since $|\sum_{i\in\mathcal{I}_n\setminus\mathcal{I}_m}1_{S_i}f|\leqq|f-1_{Z_m}f|$)$\leqq\|f-1_{Z_m}f\|<\frac{1}{m}$. Thus $\{g_n\}_n$ is a Cauchy sequence in E. For all $x\in X$, $\lim_{n\to\infty}g_n(x)=g(x)$ whence $g=\lim_{n\to\infty}g_n$.

(v) First let $\ell\in\log^*\operatorname{Aut}B$. If $j,k\in\mathcal{I}, j\neq k, x\in S_j, y\in S_k$ then by the definition of the classes S_i and by Lemma 3.6 (i), $\langle\ell(1_x), 1_y^*\rangle=0$. This fact shows $\ell(H_j)\subset H_j$

$\forall j\in\mathscr{I}$. Thus by setting $\ell_j\equiv\ell|_{H_j}$ we obviously have $\|\ell_j\|\leqq\|\ell\|$ and $\ell=\bigoplus_{j\in\mathscr{I}}\ell_j$. Furthermore, [11, Lemma] establishes $\ell_j\in i\cdot\{\text{self-adj. } H_j\text{-op.-s}\}$ $\forall j\in\mathscr{I}$.

The converse statement is immediate from (ii) since then we have $\exp(\ell)=\bigoplus_{j\in\mathscr{I}}\exp(\ell_j)$ and, by assumption, all the operators $\exp(\ell_j)$ are H_j-unitary here.

3.10. Corollary. *For some subset $\mathscr{I}_0\subset\mathscr{I}$, by writing $X_0\equiv\bigcup_{i\in\mathscr{I}_0}S_i$, we have $E_0=1_{X_0}E$ (where $E_0\equiv\mathbf{C}\cdot[\operatorname{Aut}B]\{0\}$ cf. Introduction).*

Proof. Set $Z\equiv\{x\in X\colon \exists c\in E_0\ c(x)\neq 0\}$. Clearly $E_0\subset 1_Z E$. On the other hand, if $x\in Z, c\in E_0$ and $c(x)\neq 0$ then, by (v), the linear field $\ell\equiv[f\mapsto i\cdot f(x)1_x]$ satisfies $1_{X\setminus\{x\}}c+e^{it}c(x)1_x=\exp(t\ell)\in E_0$ $\forall t\in\mathbf{R}$ whence $E_0\supset\operatorname{Span}\{1_x\colon x\in Z\}=1_Z E$ i.e. $E_0=1_Z E$. Suppose now $x\in Z, c\in E_0, c(x)\neq 0$ and $x\in S_i$. Let $y\in S_i\setminus\{x\}$ and $\ell_1\equiv[f\mapsto if(x)1_y+if(y)1_x]$. As in the previous case, $c_1\equiv\ell_1(c)=\frac{d}{dt}\Big|_0\exp(t\ell_1)c\in E_0$ since by (v), $\ell_1\in\log^*\operatorname{Aut}B$. However, $c_1(y)=ic(x)\neq 0$ i.e. $y\in S_i$. Thus $S_i\subset Z$.

Next we turn our attention to the quadratic part of $\log^*\operatorname{Aut} B$.

In the sequel we shall use the notations $\mathscr{I}_0, X_0$ introduced in Corollary 3.10. Recall that for any $c\in E_0$, there is a unique symmetric bilinear form $q_c\colon E\times E\to E$ with $[f\mapsto c+q_c(f,f)]\in\log^*\operatorname{Aut}B$ and that the mapping $c\mapsto q_c$ is conjugate-linear and continuous. Since the finitely supported functions are dense in E, to get the complete description of $\log^*\operatorname{Aut} B$ it is enough to determine only the values $\langle q_{1_{x_1}}(1_{x_2},1_{x_3}),1_{x_4}\rangle$ $(x_1\in X_0, x_2,x_3,x_4\in X)$. To this task, the projection principle provides an essential reduction.

3.11. Lemma. *Let $x_1,\dots,x_n\in X, x_1\in X_0$ and $\beta^l_{jk}\equiv\langle q_{1_{x_l}}(1_{x_j},1_{x_k}),1^*_{x_l}\rangle$. Then*

(i) *$\beta^l_{jk}=0$ if $\{1,\ell\}\neq\{j,k\}$,*

(ii) *$\beta^1_{11}=-1$,*

(iii) *$\beta^2_{12}\in[-1,0]$ and $1_{\{x_1,x_2\}}B=\{\zeta_1 1_{x_1}+\zeta_2 1_{x_2}\colon |\zeta_1|^2+|\zeta_2|^{-1/\beta}<1\}$ if $\beta^2_{12}=0$ or $1_{\{x_1,x_2\}}B=\{\zeta_1 1_{x_1}+\zeta_2 1_{x_2}\colon \max(|\zeta_1|,|\zeta_2|)<1\}$ in case of $\beta^2_{12}=0$,*

(iv) *$\beta^2_{12}=-1/2$ if $x_1\sim x_2\neq x_1$ and $\beta^2_{12}=0$ if $x_1\nsim x_2\in X_0$;*

(v) *if $x_1,\dots,x_n\in X_0$ and $x_i\nsim x_j$ for $i\neq j$ then $\|\zeta_1 1_{x_1}+\dots+\zeta_n 1_{x_n}\|=\max(|\zeta_1|,\dots,|\zeta_n|)$ for all $\zeta_1,\dots,\zeta_n\in\mathbf{C}$.*

Proof. (i) Consider the band projection $P\colon f\mapsto 1_{\{x_1,\dots,x_n\}}f$. By the projection principle, $[f\mapsto 1_{x_1}+Pq_{1_{x_1}}(f,f)]\in\log^*\operatorname{Aut}PB$. Applying [11, Lemma] to PB, we obtain

$$0=\|f\|^2\overline{\langle 1_{x_1},\Phi\rangle}+\langle Pq_{1_x}(f,f),\Phi\rangle \Leftarrow \|f\|\cdot\|\Phi\|=\langle f,\Phi\rangle\quad \forall f\in PE, \Phi\in(PE)^*.$$

Introducing the same function $p:\mathbf{R}_+^n\to\mathbf{R}_+$ and set $C\subset\mathbf{R}_+^n$ as in the proof of Lemma 3.6,

$$(12)\qquad 0=p(\varrho_1,\dots,\varrho_n)^2\left\langle 1_{x_1},\ \sum_{j=1}^n\frac{\partial p}{\partial\varrho_j}e^{-i\vartheta_j}1_{x_j}^*\right\rangle+$$
$$+\left\langle q_{1_x}\left(\sum_{j=1}^n\varrho_je^{i\vartheta_j}1_{x_j},\ \sum_{k=1}^n\varrho_ke^{i\vartheta_k}1_{x_k}\right),\ \sum_{\ell=1}^n\frac{\partial p}{\partial\varrho_\ell}e^{-i\vartheta_\ell}1_{x_\ell}^*\right\rangle$$

for all $\varrho\in\mathbf{R}_+^n\setminus C$ and $\vartheta\in\mathbf{R}^n$. Thus

$$(12')\qquad p^2\frac{\partial p}{\partial\varrho_1}e^{i\vartheta_1}+\left(\sum_{j,k,\ell=1}^n\beta_{jk}^\ell\varrho_j\varrho_k\frac{\partial p}{\partial\varrho_\ell}e^{i(\vartheta_j+\vartheta_k-\vartheta_\ell)}\right)=0\qquad(\varrho\notin C,\ \vartheta\in\mathbf{R}^n).$$

Therefore (for fixed $\varrho\in\mathbf{R}_+^n\setminus C$) the rational expression $p^2\frac{\partial p}{\partial\varrho_1}z_1+\sum_{j,k,\ell=1}^n\beta_{jk}\varrho_j\varrho_k\cdot\frac{\partial p}{\partial\varrho_\ell}z_jz_kz_\ell^{-1}$ vanishes on $\partial_0\Delta^n$ i.e. its homogeneous parts are 0-s. Hence only the coefficients of the form $\beta_{1k}^1(=\beta_{k1}^1)$ may differ from 0.

(ii) is immediate from (12′) if we take $n=1$ because then $p(\varrho_1)=\varrho_1$.

For the proof of (iii) and (iv), consider the case $n=2$. From (12′) and (ii) we then see

$$(12'')\qquad (p^2-\varrho_1^2)\frac{\partial p}{\partial\varrho_1}+2\varrho_1\varrho_2\frac{\partial p}{\partial\varrho_2}\beta_{12}^2=0\quad(\varrho\in\mathbf{R}_+^n\setminus C).$$

Since $p(0,\varrho)=p(\varrho,0)$ and since the function p is increasing and convex, $\forall\varrho\in[0,1)\ \exists!t\geqq0\ \ p(\varrho,t)=1$. Thus the function $t:[0,1)\to\mathbf{R}_+$ is welldefined by $p(\varrho,t(\varrho))=1$. Observe that now t is a decreasing concave function and $t(0)=0$. By the implicite function theorem, $t'(\varrho_1)=-\frac{\partial p/\partial\varrho_1}{\partial p/\partial\varrho_2}$ whenever $(\varrho_1,t(\varrho_1))\notin C$. Thus, since C is a cone with measure 0 in $\mathbf{R}_+^2$, (12″) implies

$$(12''')\qquad t'(\varrho)(1-\varrho^2)=2\varrho t(\varrho)\beta_{12}^2\quad\text{for almost every}\quad\varrho\in(0,1).$$

Since $t'\leqq0$, we have $\beta_{12}^2\leqq0$. If $\beta_{12}^2=0$ then $t(\varrho)=t(0)=1\ \forall\varrho\in[0,1)$. In this case, $p(\varrho_1,\varrho_2)\leqq1$ if $\varrho_1<1$ and $\varrho_2\leqq t(\varrho_1)=1$ or $\varrho_1=1$ and $\varrho_2\leqq1$, i.e. $p(\varrho_1,\varrho_2)=\max(\varrho_1,\varrho_2)$. If $\beta_{12}^2<0$ then the solution of (12‴) with initial value $t(0)=1$ is $t(\varrho)=(1-\varrho^2)^{-\beta_{12}^2}$. Thus by setting $K\equiv\{(\varrho_1,\varrho_2):p(\varrho_1,\varrho_2)\leqq1\}$,

$$(13)\qquad K=\{(\varrho_1,\varrho_2):\varrho_1^2+\varrho_2^{-1/\beta_{12}^2}\leqq1\}.$$

The convexity of the function p entails that K is convex whence $\beta_{12}^2\geqq-1$ yielding (iii).

(iv): If $x_1 \sim x_2 \neq x_1$ then $p(\varrho_1, \varrho_2)=(\varrho_1^2+\varrho_2^2)^{1/2}$ (cf. Proposition 3.9 (ii)), that is, by (13), we have $\beta_{12}^2=-\frac{1}{2}$.

On the other hand, suppose $x_1 \not\sim x_2 \in X_0$ and $\beta_{12}^2 \neq 0$. Since $x_2 \in X_0$, all the previous considerations can be carried out by interchanging x_1 and x_2. Thus by (iii),

$$1_{\{x_1,x_2\}}B = \{\zeta_1 1_{x_1}+\zeta_2 1_{x_2}: |\zeta_1|^2+|\zeta_2|^{-1/\langle q_{1_{x_1}}(1_{x_1},1_{x_2}),1_{x_2}^*\rangle} \leq 1\} =$$
$$= \{\zeta_1 1_{x_1}+\zeta_2 1_{x_2}: |\zeta_2|^2+|\zeta_1|^{-1/\langle q_{1_{x_1}}(1_{x_2},1_{x_1}),1_{x_2}\rangle} \leq 1\}.$$

This is possible only if $\beta_{12}^2=-\frac{1}{2}=\langle q_{1_{x_1}}(1_{x_2}, 1_{x_1}), 1_{x_1}^*\rangle$ thus $p(\varrho_1, \varrho_2)=(\varrho_1^2+\varrho_2^2)^{-1/2}$. If S_i denotes the equivalence class (w.r.t. $\sim$) of x_1 then by Proposition 3.9 (iii), $\|f+1_{x_2}\|=\|\|f\|_{\ell^2}\cdot 1_{x_1}+\varrho 1_{x_2}\|=p(\|f\|_{\ell^2}, \varrho)=\|f+\varrho 1_{x_2}\|_{\ell^2}$ for arbitrary $f\in H_i$ whence it follows $x_2 \in S_i$ i.e. $x_1 \sim x_2$. The obtained contradiction proves (iv).

(v): Let $y_1, \dots, y_n \in X_0$ be pairwise non-$\sim$-equivalent. Now for arbitrarily fixed $f, c \in 1_{\{y_1,\dots,y_n\}}E$,

$$q_c(f,f) = \sum_{m=1}^{n} \overline{c(y_m)}\, q_{1_{y_m}}(f,f) = \sum_{m=1}^{n} \overline{c(y_m)} \sum_{j,k,l=1}^{n} f(y_j)f(y_k)\langle q_{1_{y_m}}(1_{y_j}, 1_{y_j}), 1_{y_\ell}^*\rangle\, 1_{y_\ell}.$$

Applying (i) and (iii) to $x_1 \equiv y_m, x_k \equiv y_k$ and $x_j \equiv y_j$, hence we obtain

$$q_c(f,f) = -\sum_{m=1}^{n} \overline{c(y_m)}\, f(y_m)^2 1_{y_m} = -\bar{c}\cdot f^2.$$

Therefore the solution of the initial value problem $\left\{\frac{d}{dt} f_t = c - q_c(f_t, f_t), f_0 = 0\right\}$ is $f_t = \tanh(tc)$. Hence $\left\{\sum_{m=1}^{n} \varrho_m 1_{y_m}: \varrho_1, \dots, \varrho_n \in [0,1)\right\} \subset \{\exp[f \mapsto c + q_c(f,f)](0): c \in 1_{\{y_1,\dots,y_n\}}E\} \subset [\operatorname{Aut} B]\{0\} \subset B$. Then $\max_{m=1}^{n} \varrho_m \leq \left\|\sum_{m=1}^{n} \varrho_m 1_{y_m}\right\| \leq 1$ whenever $\varrho_1, \dots, \varrho_n \in [0,1]$. Consequently $\left\|\sum_{m=1}^{n} \varrho_m 1_{y_m}\right\| = 1$ whenever $\max_{m=1}^{n} |\varrho_m| = 1$ whence $\left\|\sum_{j=1}^{n} \zeta_j 1_{y_j}\right\| = \max_{m=1}^{n} |\zeta_m|$. The proof is complete.

From Lemma 3.11 (i) and the symmetry of the bilinear mappings q_c follows directly that introducing the functions

$$w_{x_1}(x_2) \equiv \begin{cases} -1/2 & \text{if } x_1 = x_2 \\ \langle q_{1_x}(1_{x_1}, 1_{x_2}), 1_{x_2}^{*1}\rangle & \text{if } x_1 \neq x_2 \end{cases} \qquad (x_1 \in X_0, x_2 \in X),$$

we have

$$q_{1_x}(1_x, 1_x) = 2w_x(x)1_x \quad \text{for all} \quad x\in X_0,$$
$$q_{1_x}(1_x, 1_y) = w_x(y)1_y \quad \text{if} \quad x\in X_0,\ y\in X\setminus\{x\},$$
$$q_{1_x}(1_y, 1_z) = 0 \quad \text{if} \quad x\notin\{y, z\},\ x\in X_0.$$

Hence

(14) $$q_{1_x}(f, g) = f(x)\, w_x g + g(x)\, w_x f \quad (x\in X_0)$$

whenever the function $f\in E$ is finitely supported. Moreover by (8′) and Lemma 3.11 (iii), (14) holds for every $f\in E$.

For sake of brevity, in what follows we shall write $f^{(i)}$ instead of the function $1_{S_i} f$.

3.12. Lemma. (i) $w_x^{(i)} = -\frac{1}{2} 1_{S_i}$ *whenever* $x\in S_i$ $(i\in\mathcal{I}_0)$,

(ii) $w_x^{(i)}=0$ *whenever* $x\notin S_i$ $(i\in\mathcal{I}_0)$,

(iii) *There exists a unique matrix* $(\gamma_{ij})_{i\in\mathcal{I}_0, j\in\mathcal{I}\setminus\mathcal{I}_0}$ *consisting of numbers belonging to* $[0, 1]$ *such that* $w_x^{(j)} = -\gamma_{ij} 1_{S_j}$ *whenever* $x\in S_i\subset X_0$ *and* $j\in\mathcal{I}\setminus\mathcal{I}_0$.

Proof. (i) and (ii) are contained in Lemma 3.11 (iv).

(iii): Let $x, x'\in S_i$ and $y, y'\in S_j$ where $i\in\mathcal{I}_0$, $j\notin\mathcal{I}_0$. From Proposition 3.9 (v) it follows the existence of an E-unitary operator U such that $1_{x'}=U1_x$ and $1_{y'}=U1_{y_1}$. From the elementary theory of Lie-groups it is well-known that $UvU^{-1}\in\log^*\mathrm{Aut}\, B$ for every $v\in\log^*\mathrm{Aut}\, B$. In particular, $[f\mapsto U(1_x+q_{1_x}(U^{-1}f, U^{-1}f))]\in\log^*\mathrm{Aut}\, B$ whence $q_{1_{x'}}(f,f)=q_{U1_x}(f,f)=q_{1_x}(U^{-1}f, U^{-1}f)$. Therefore $\langle q_{1_{x'}}(1_{x'}, 1_{y'}), 1_{y'}^*\rangle = \langle Uq_{1_x}(U^{-1}1_{x'}, U^{-1}1_{y'}), 1_{y'}^*\rangle = \langle Uq_{1_x}(1_x, 1_y), 1_{y'}^*)\rangle = \langle q_{1_x}(1_x, 1_y), 1_y^*\rangle$ since if $U=\bigoplus_{i\in\mathcal{I}} U_i$ is the directe decomposition of U provided by Proposition 3.9 (v) and $f\in E$ then $\langle Uf, 1_{x'}^*\rangle = (U_i f^{(i)}|1_x) = (f^{(i)}|U_i^{-1}1_x) = (f^{(i)}|U_i^{-1}1_{x'}) = (f^{(i)}|1_x)$.

Henceforth we reserve the notation $(\gamma_{ij})_{i\in\mathcal{I}_0, j\in\mathcal{I}\setminus\mathcal{I}_0}$ for the matrix introduced in Lemma 3.12 (iii).

3.13. Corollary. *For arbitrary finitely supported* $c\in E_0$ *and* $f\in E$,

(15) $$q_c(f,f) = -\sum_{i\in\mathcal{I}_0} (f^{(i)}|c^{(i)}) f^{(i)} - 2 \sum_{j\in\mathcal{I}\setminus\mathcal{I}_0} \Big[\sum_{i\in\mathcal{I}_0} \gamma_{ij}(f^{(i)}|c^{(i)})\Big] f^{(j)}.$$

Proof. Applying Lemma 3.12. and (14), we can see that if $c\in E_0$ and $f\in E$ have finite supports then $q_c(f,f) = -\sum_{x\in X_0} \overline{c(x)}\, q_{1_x}(f,f) \sum_{i\in\mathcal{I}_0} \sum_{x\in S_i} 2\overline{c(x)} f(x)\cdot$ $\cdot\Big[-\frac{1}{2} f^{(i)} - \sum_{j\notin\mathcal{I}_0} \gamma_{ij} f^{(j)}\Big]$.

In order to extend (15) to every $c\in E_0$ and $f\in E$, we need the following observations.

3.14. Lemma. (i) $E_0=\bigoplus_{i\in\mathscr{I}_0}^{c_0}H_i$ *i.e. a function* $c\colon X\to\mathbf{C}$ *belongs to* E_0 *if and only if* $\forall i\in\mathscr{I}$ $\|c^{(i)}\|_{\ell^2}<\infty$ *and* $\forall\varepsilon>0$ $\{i\in\mathscr{I}_0\colon\|c^{(i)}\|_{\ell^2}\geqq\varepsilon\}$ finite$\subset\mathscr{I}_0$ (*in the latter case* $\|c\|=\sup_{i\in\mathscr{I}_0}\|c^{(i)}\|_{\ell^2}$).

(ii) $\sup_{j\in\mathscr{I}\setminus\mathscr{I}_0}\sum_{i\in\mathscr{I}_0}\gamma_{ij}\leqq 4\|q\|(\equiv 4\sup_{c\in B\cap E_0}\|q_c\|=4\sup_{\substack{c\in B\cap E_0\\ f,g\in B}}\|q_c(f,g)\|)$.

Proof. (i): Trivial from Proposition 3.9 (v), Lemma 3.11 (v) and the fact that the finitely supported functions are dense in E.

(ii): Let $j\in\mathscr{I}\setminus\mathscr{I}_0$, $i_1,\dots,i_n\in\mathscr{I}_0$, $y\in S_j$ and $x_1\in S_{i_1},\dots,x_n\in S_{i_n}$. Consider the functions $c\equiv\sum_{m=1}^{n}1_{x_m}$ and $f\equiv 1_y+\sum_{m=1}^{n}1_{x_m}$. By (i) we have $\|c\|=1$ and $\|f\|\leqq 2$. By (15), $\langle q_c(f,f),1_y^*\rangle=\sum_{m=1}^{n}\gamma_{i_m j}$. At the same time, $|\langle q_c(f,f),1_y^*\rangle|\leqq\|q\|\cdot\|c\|\cdot\|f\|^2\cdot\|1_y^*\|\leqq$ $\leqq 4\|q\|$.

3.15. Corollary. (15) *holds for each* $c\in E_0$ *and* $f\in E$.

Proof. The previous lemma shows that the right hand side of (15) makes always sense. Observe that the mapping $Q\colon E_0\times E\ni(c,f)\mapsto$ {right hand side of (15)} is real-linear in c and real-quadratic in f. For $\|c\|,\|f\|\leqq 1$ we have $\|Q(c,f)\|\leqq$ $\|\sum_{i\in\mathscr{I}_0}(f^{(i)}|c^{(i)})f^{(i)}\|+2\|\sum_{j\notin\mathscr{I}_0}(\sup_{k\notin\mathscr{I}_0}\sum_{i\in\mathscr{I}_0}\gamma_{ik}\|f^{(i)}\|_{\ell_2}\cdot\|c^{(i)}\|_{\ell_2})f^{(j)}\|\leqq\|f\|^2\cdot\|c\|+4\|q\|\cdot\|c\|\cdot$ $\cdot\|f\|^2$. Thus Q is a continuous map. On the other hand, the relation $Q(c,f)=$ $=+q_c(f,f)$ is already established for a dense submanifold of $E_0\times E$ by Corollary 3.13.

In this way we completely know log*Aut B. The mappings $\exp[B\ni f\mapsto$ $\mapsto c+q_c(f,f)]$ are easy to describe: By (15), the equation $\frac{d}{dt}f_t=c+q_c(f_t,f_t)$ is equivalent with

$$\frac{d}{dt}f_t^{(i)}=c^{(i)}-(f_t^{(i)}|c^{(i)})f_t^{(i)}\quad(i\in\mathscr{I}_0) \tag{16'}$$

$$\frac{d}{dt}f_t^{(j)}=-2\sum_{i\in\mathscr{I}_0}\gamma_{ij}(f_t^{(i)}|c^{(i)})f_t^{(j)}\quad(j\in\mathscr{I}\setminus\mathscr{I}_0). \tag{16''}$$

If we represent $c^{(i)}$ in the form $c^{(i)}\equiv\varrho_i c_0^{(i)}$ where $\varrho_i\geqq 0$, $\|c_0^{(i)}\|=1$ and if $f_0^{(i)}=$ $=\zeta_i c_0^{(i)}+f_\perp^{(i)}$ where $f_\perp^{(i)}$ lying orthogonally to $c_0^{(i)}$, one then cheks immediately that for arbitrarily given $f_0\in B$, the solution of (16′) is

$$f_t^{(i)}=M_{\varrho_i t}(\zeta_i)c_0^{(i)}+M_{\varrho_i t}^{\perp}(\zeta_i)f_\perp^{(i)}\quad(i\in\mathscr{I}_0) \tag{17'}$$

where M_τ and $M_\tau^\perp$ are the Moebius- and co-Moebius transformations

(18) $$M_\tau(\zeta) \equiv \frac{\zeta+\tanh(\tau)}{1+\zeta\tanh(\tau)},\ M_\tau^\perp(\zeta) \equiv \frac{\{1-(\tanh(\tau))^2\}^{1/2}}{1+\zeta\tanh(\tau)} \qquad (\tau\in\mathbf{R},\ |\zeta|<1).$$

Substituting (17′) into (16″), we obtain

$$\frac{d}{dt}f_t^{(j)} = \Big[-2\sum_{i\in\mathscr{I}_0}\gamma_{ij}\varrho_i M_{\varrho_i t}(\zeta_i)\Big]f_t^{(j)} \quad (j\in\mathscr{I}\setminus\mathscr{I}_0)$$

whose solution is given by

(17″) $$f_t^{(j)} = \exp\Big[-2\sum_{i\in\mathscr{I}_0}\gamma_{ij}\varrho_i\int_0^1 M_{\varrho_i\tau}(\zeta_i)\,d\tau\Big]f_0^{(j)} =$$

$$= \Big[\prod_{i\in\mathscr{I}_0} M_{\varrho_i t}^\perp(\zeta_i)^{2\gamma_{ij}}\Big]f_0^{(j)} \quad (j\in\mathscr{I}\setminus\mathscr{I}_0).$$

The fact that the right hand side in (17″) makes sense, is guaranteed by Lemma 3.14 (ii). Fortunately, by Lemma 3.14 (i) and (17′),

$$[\mathrm{Aut}\,B]\{0\} = B\cap E_0 = \Big\{\sum_{i\in\mathscr{I}_0}\lambda_i c_i:\ 0\leqq\lambda_i\leqq 1,\ c_i\in\partial B(H_i)\quad i\in\mathscr{I}_0 \text{ and}$$

$$[i\mapsto\lambda_i]\in c_0(\mathscr{I}_0)\Big\} = \Big\{\sum_{i\in\mathscr{I}_0}M_{\varrho_i}(0)c_i:\ \varrho_i\in\mathbf{R}_+,\ c_i\in\partial B(H_i)\quad \forall i\in\mathscr{I}_0 \text{ and}$$

$$[i\mapsto\lambda_i]\in c_0(\mathscr{I}_0)\Big\} = \{\exp[f\mapsto c+q_c(f,f)](0): c\in E_0\}$$

where $c_0(\mathscr{I}_0)\equiv\{\mathscr{I}_0\to\mathbf{C}$ functions vanishing at infinity$\}$. A classical theorem of Cartan asserts that the relations $U\in\mathrm{Aut}\,B$ and $U(0)=0$ entail the linearity of U. Thus given $F\in\mathrm{Aut}\,B$, if we choose the vector $c\in E_0$ so that the automorphism $G\equiv\exp[B\ni f\mapsto -c+q_{(-c)}(f,f)]$ satisfies $G(0)=F^{-1}(0)$ then the automorphism $U\equiv F\circ G$ is necessarily linear, i.e. we have $F\in U\cdot\exp[f\mapsto c+q_c(f,f)]$ for suitable $c\in E_0$ and linear E-unitary U. Hence we arrive at the following characterization of $\mathrm{Aut}\,B$:

3.16. Theorem. *Let E denote a minimal atomic Banach lattice. The space E is spanned by a family $\{H_i: i\in\mathscr{I}\}$ of its pairwise lattice-orthogonal Hilbertian projection bands such that*

(i) *the linear members of* $\mathrm{Aut}_0 B(E)$ *map* $B(H_i)$ *onto themselve* $(\forall i\in\mathscr{I})$,

(ii) *conversely, if for any index $i\in\mathscr{I}$, U_i is an H_i-unitary operator then* $\bigoplus_{i\in\mathscr{I}} U_i|_{B(E)}\in$ $\in\mathrm{Aut}_0 B(E)$.

Furthermore there exists a matrix $(\gamma_{ij})_{i,j}\in\mathscr{I}$ and an index subfamily $\mathscr{I}_0\subset\mathscr{I}$ such that

(iii) $E_0(\equiv\mathbf{C}[\mathrm{Aut}\,B(E)]\{0\}) = \bigoplus_{i\in\mathscr{I}_0}^{c_0} H_i$,

(iv) $0 \leqq \gamma_{ij} \leqq 1$ *for all* $i,j \in \mathscr{I}$; $\gamma_{ii} = \frac{1}{2}$ *for all* $i \in \mathscr{I}_0$; $\gamma_{ij} = 0$ *whenever* $i,j \notin \mathscr{I}_0$ *or* i *and* j *are distinct elements of* $\mathscr{I}_0$.

(v) *A mapping* $F: B(E) \to E$ *belongs to* $\mathrm{Aut}_0 B(E)$ *if and only if, by denoting the band projection onto* H_i *by* P_i, *we have*

$$P_i F(f) = U_i\{M_{\varrho_i}((P_i f|c_i^0))\, c_i^0 + M_{\varrho_i}^{\perp}((P_i f|c_i^0))[P_i f - (P_i f|c_i^0)\, c_i^0]\} \quad (i \in \mathscr{I}_0),$$

$$P_j F(f) = \left\{\exp \int_0^1 \sum_{i \in \mathscr{I}_0} \gamma_{ij}\, \varrho_i M_{\varrho_i \tau}((P_i f|c_i^0))\, d\tau\right\} U_j P_j f \quad (j \in \mathscr{I} \setminus \mathscr{I}_0)$$

for suitable H_j*-unitary operators* U_j $(j \in \mathscr{I})$, *unit vectors* $c_i^0 \in H_i$ $(i \in \mathscr{I}_0)$ *and a function* $[\mathscr{I}_0 \ni i \mapsto \varrho_i]$ *assuming values in* $\mathbf{R}_+$ *and vanishing at infinity, respectively (the transformations* M_{ϱ_i}, $M_{\varrho_i}^{\top}$ *are those defined in* (18)*).*

4. Appendix
Linear finite dimensional tensor unit ball automorphisms

Throughout this section $H_1, \ldots, H_n$ are fixed finite dimensional Hilbert spaces. We are aimed to describe the structure of the linear unitary operators in the space $E \equiv H_1 \otimes \ldots \otimes H_n$.

We shall use the notations $B \equiv B(E)$, $B^* \equiv B(E^*)$,

$$K \equiv \{F \in \partial B\colon \exists! \Phi \in \partial B^* \quad \langle F, \Phi \rangle = 1\},$$

$$K^* \equiv \{\Phi \in \partial B^*\colon \exists F \in K \quad \langle F, \Phi \rangle = 1\}.$$

4.1. Lemma. $K^* = \{\delta_{e_1,\ldots,e_n}\colon e_1 \in \partial B(H_1), \ldots, e_n \in \partial B(H_n)\}$.

Proof. Since $\dim E < \infty$, $\bar{B}$ is compact, thus for any n-linear functional $F \in \partial B$, one can find $e_1 \in \partial B(H_1), \ldots, e_n \in \partial B(H_n)$ with $F(e_1, \ldots, e_n) = 1$. Hence $K^* \subset \{\delta_{e_1,\ldots,e_n}\colon e_j \in \partial B(H_j)\}$. On the other hand, every E-unitary operator maps K onto itself and therefore also

(19) $$U^* K^* = K^* \quad \text{for all } E\text{-unitary operators.}$$

From the compactness of B it follows $K \neq \emptyset$ (indeed: for any smooth norm $\|\cdot\|_1$ on E, $\emptyset \neq \{F \in \partial B\colon \|F\|_1 \leqq \|G\|_1\ \forall G \in \partial B\} \subset K$) whence $K^* \neq \emptyset$. That is, for some unit vectors $e_1^0 \in H_1, \ldots, e_n^0 \in H_n$ we have $\delta_{e_1^0,\ldots,e_n^0} \in K^*$. Now from (19) we obtain $\delta_{U_1 e_1^0,\ldots,U_n e_n^0} = (U_1 \otimes \ldots \otimes U_n)^* \delta_{e_1^0,\ldots,e_n^0} \in K^*$ whenever the U_j-s are H_j-unitary operators. Thus $\{\delta_{e_1,\ldots,e_n}\colon e_j \in \partial B(H_j)\} \supset K^*$.

4.2. Lemma. *Let* $\Phi \equiv \delta_{f_1,\ldots,f_n}$, $\psi \equiv \delta_{g_1,\ldots,g_n}$ *and* $\Theta \equiv \delta_{h_1,\ldots,h_n}$ *where* $0 \neq f_j, g_j, h_j \in H_j$ $(j=1, \ldots, n)$ *and assume* $\Phi + \Psi = \Theta$. *Then there exists* k *such that for each* $j \neq k$ *we have* $f_j \| g_j$ *(i.e.* f_j *and* g_j *are linearly dependent).*

Proof. The statement holds obviously if for some index m, $f_j \| h_j$ for all $j \neq m$ or $f_j \| g_j$ for all $j \neq m$. In the contrary case $f_k \nparallel g_k$ and $f_m \nparallel h_m$ for some pair of indices $k \neq m$. We may then suppose $k=1$ and $m=2$. First we show that in this case we have $h_1 \nparallel f_1$. Indeed: from $h_1 \nparallel f_1$ it follows that introducing the tensor $\tilde{E} \equiv \tilde{g}_1 \otimes g_2 \otimes \ldots \otimes g_n$ where $\tilde{g}_1 \equiv g_1 - \|f_1\|^{-2}(g_1|f_1)f_1$ the relations $\langle \tilde{E}, \Phi \rangle = \langle \tilde{E}, \Theta \rangle = 0 \neq \langle \tilde{E}, \Psi \rangle$ hold. One can see in the same manner that $h_2 \nparallel g_2$. Since $h_1 \nparallel f_1$, there exists $u_1 \in H_1$ with $f_1 \perp u_1 \not\perp h_1$ and since $h_2 \nparallel g_2$ one can find $u_2 \in H_2$ with $g_2 \perp u_2 \not\perp h_2$. But then the tensor $T \equiv u_1 \otimes u_2 \otimes h_3 \otimes \ldots \otimes h_n$ satisfies $\langle T, \Phi \rangle = \langle T, \Psi \rangle = 0 \neq \langle T, \Theta \rangle$ which is impossible.

4.3. Proposition. *Set $r_j \equiv \dim H_j$ $(j=1, \ldots, n)$ and let $U \in \mathscr{L}(E, E)$ be fixed so that $U|_B \in \mathrm{Aut}_0 B$. Then one can choose H_j-unitary operators U_j such that $U = U_1 \otimes \ldots \otimes U_n$.*

Proof. It is enough to prove the statement only for E-unitary operators lying in a suitable neighbourhood of id_E as it is well-known (see e.g. [6]).

To do this, fix $\varepsilon > 0$ such that the functionals $\Phi \equiv \delta_{e_1, \ldots, e_n}$, $\tilde{\varphi} \equiv \delta_{\tilde{e}_1, \ldots, \tilde{e}_n}$, $\Psi \equiv \delta_{f_1, \ldots, f_n}$, $\tilde{\Psi} \equiv \delta_{\tilde{f}_1, \ldots, \tilde{f}_n}$ $(\in E^*)$ fulfil

$$\text{(20)} \qquad \exists k \quad e_k \perp \tilde{e}_k, f_k \perp \tilde{f}_k \quad \text{and} \quad \forall j \neq k \quad e_j \| \tilde{e}_j, \ f_j \| \tilde{f}_j$$

whenever we have

$$\text{(21)} \qquad \Phi - \tilde{\Phi},\ \Psi - \tilde{\Psi} \in K^*,\ \|\Phi - \tilde{\Phi}\| = \|\Psi - \tilde{\Psi}\| = \sqrt{2} \quad \text{and} \quad \|\Phi - \Psi\|,\ \|\tilde{\Phi} - \tilde{\Psi}\| < \varepsilon,$$

$$\text{(22)} \qquad \|e_j\| = \|\tilde{e}_j\| = \|f_j\| = \|\tilde{f}_j\| = 1 \quad (j = 1, \ldots, n).$$

A value $\varepsilon > 0$ with the above properties in fact exists: Otherwise there would be a sequence $\Phi_m \equiv \delta_{e_1^m, \ldots, e_n^m}$, $\tilde{\Phi}_m \equiv \delta_{\tilde{e}_1^m, \ldots, \tilde{e}_n^m}$, $\Psi_m \equiv \delta_{f_1^m, \ldots, f_n^m}$, $\tilde{\Psi}_m \equiv \delta_{\tilde{f}_n^m, \ldots, \tilde{f}_n^m}$ $(m = 1, 2, \ldots)$ satisfying (21), (22) for $\varepsilon = \dfrac{1}{m}$ but without property (20). For a suitable index subsequence $\{m_s\}_s$ and for some unit vectors $e_j, \tilde{e}_j, f_j, \tilde{f}_j$ we have $e_j^{m_s} \to e_j$, $\tilde{e}_j^{m_s} \to e_j$, $f_j^{m_s} \to f_j$, $\tilde{f}_j^{m_s} \to f_j$ $(s \to \infty, j=1, \ldots, n)$. Then the limits $\Phi, \tilde{\Phi}, \Psi, \tilde{\Psi}$ satisfy $\Phi = \Psi$, $\tilde{\Phi} = \tilde{\Psi}$, $\|\Phi - \tilde{\Phi}\| = \|\Psi - \tilde{\Psi}\| = \sqrt{2}$ and the contrary of (20). At the same time we also have $\Phi - \tilde{\Phi}, \Psi - \tilde{\Psi} \in K^*$ because of the closedness of K^*. Thus by Lemma 4.2, $\exists! k_0\ \forall j \neq k_0\ e_j \| \tilde{e}_j$. Since $\|\Phi - \tilde{\Phi}\| = \sqrt{2}$, hence $\|e_{k_0} - \tilde{e}_{k_0}\| = \sqrt{2}$ i.e. $e_{k_0} \perp \tilde{e}_{k_0}$. Similarly $\exists! \ell_0\ f_{\ell_0} \perp \tilde{f}_{\ell_0}$ and $\forall j \neq \ell_0\ f_j \| \tilde{f}_j$. Since (20) does not hold, necessarily $k_0 \neq \ell_0$. However the relations $\Phi = \Psi$, $\tilde{\Phi} = \tilde{\Psi}$ entail $k_0 = \ell_0$.

Now assume $\|U - \mathrm{id}_E\| < \varepsilon$. Fix an orthonormed basis $\{e_j^k : j=1, \ldots, r_k\}$ in H_k $(k=1, \ldots, n)$, respectively and let us write the functional $U^* \delta_{e_1^1, \ldots, e_1^n}$ in the form $U^* \delta_{e_1^1, \ldots, e_1^n} = \delta_{f_1^1, \ldots, f_1^n}$ (cf. Lemma 4.1.) where f_1^k is a fixed unit vector in H_k $(k = 1, \ldots, n)$. It follows from the choice of ε that for arbitrary index k, the singleton $\{f_1^k\}$ can be continued to an orthonormed basis $\{f_j^k : j=1, \ldots, r_k\}$ of H_k in a unique way so that we have

$$U^*\delta_{e_1^1,\ldots,e_1^{k-1}e_j^k,e_1^{k+1},\ldots,e_1^n} = \delta_{f_1^1,\ldots,f_1^{k-1}f_j^k,f_1^{k+1},\ldots,f_1^n} \quad (j=1,\ldots,r_k).$$

Set $I_0 \equiv \{(1,\ldots,1,\overset{k}{j},1,\ldots,1)\colon k=1,\ldots,n; j=1,\ldots,r_k\}$, $I_1 \equiv \bigtimes_{k=1}^{n} \{1,\ldots,r_k\}$ and let a family $I\subset I_1$ of multiindices be called *thick* if $\forall i\in I, \forall i'\in I_1 \quad i'\leqq i \Rightarrow i'\in I$.

Observe that for any multiindex $i\equiv(i_1,\ldots,i_n)\in I_1$ there exists a unique complex number which we shall denote by $\varkappa_i$ such that $|\varkappa_i|=1$ and

(23) $$U^*\delta_{e_{i_1}^1,\ldots,e_{i_n}^n} = \varkappa_i\delta_{f_{i_1}^1,\ldots,f_{i_n}^n}.$$

Indeed: If not, we can find a minimal (w.r.t. $\leqq$) $i\in I_1$ not satisfying (23). Now $U^*\delta_{e_{i_1}^1,\ldots,e_{i_n}^n}=\delta_{h_1,\ldots,h_n}$ for some vectors $h_k\in\partial B(H_k)$ $(k=1,\ldots,n)$. Since obviously $i\notin I_0$, for arbitrarily fixed k, there is $\tilde{k}\neq k$ with $i_{\tilde{k}}\neq 1$. Consider the multiindex j defined by $j_\ell\equiv[i_\ell$ if $\ell\neq k$, 1 if $\ell=k]$ $(\ell=1,\ldots,n)$. By the minimality of i, $U^*\delta_{e_{j_1}^1,\ldots,e_{j_n}^n} = \varkappa_j\delta_{f_{j_1}^1,\ldots,f_{j_n}^n}$. Since $U^*\left(\frac{1}{\sqrt{2}}\delta_{e_{i_1}^1,\ldots,e_{i_n}^n}+\frac{1}{\sqrt{2}}\delta_{e_{j_1}^1,\ldots,e_{j_n}^n}\right)\in K^*$, using Lemma 4.2 we can see $h_k\| f_{i_k}^k$ i.e. $h_k=\alpha_k f_{i_k}^k$ for suitable $\alpha_j\in\partial\Delta$ $(k=1,\ldots,n)$.

Then let I be a maximal thick subset of I_1 such that $I_1\supset I_0$ and $\varkappa_i=1$ $\forall i\in I$. (Remark: $\varkappa_i=1$ $\forall i\in I_0$.) We shall show that necessarily $I=I_1$. Hence and from the linearity of the mapping U, (23) immediately yields the statement of the lemma.

Assume $I_1\setminus I\neq\emptyset$. Let j be a minimal element of $I_1\setminus I$. Observation: $\forall i\in I_1$ $j\neq i\leqq j\Rightarrow i\in I$. I.e. the family $I'\equiv I\cup\{j\}$ is thick. Therefore it suffices to prove $\varkappa_j=1$ (which contradicts our assumption). By writing $J\equiv\{1,j_1\}\times\ldots\times\{1,j_n\}$,

$$U^*\delta_{e_1^1+e_{j_1}^1,\ldots,e_1^n+e_{j_n}^n} = \sum_{i\in J} U^*\delta_{e_{i_1}^1,\ldots,e_{i_n}^n} = \sum_{i\in J}\varkappa_i\delta_{f_{i_1}^1,\ldots,f_{i_n}^n} =$$

$$= \varkappa_j\delta_{f_{j_1}^1,\ldots,f_{j_n}^n} + \sum_{i\in J\setminus\{j\}}\delta_{f_{i_1}^1,\ldots,f_{i_n}^n} = (\varkappa_j-1)\delta_{f_{j_1}^1,\ldots,f_{j_n}^n} + \delta_{f_1^1+f_{j_1}^1,\ldots,f_1^n+f_{j_n}^n}.$$

However, the function $U^*\delta_{e_1^1+e_{j_1}^1,\ldots,e_1^n+e_{j_n}^n}$ has the form $\delta_{h_1,\ldots,h_n}$ whence directly $\varkappa_j=1$.

4.4. Corollary. *The vector fields V being tangent to $\partial B(E)$ are exactly those of the form*

$$V = i\cdot\sum_{j=1}^{n}\mathrm{id}_{H_1}\otimes\ldots\otimes\mathrm{id}_{H_{j-1}}\otimes A_j\otimes\mathrm{id}_{H_{j+1}}\otimes\ldots\otimes\mathrm{id}_{H_n}$$

where each A_j is a self-adjoint H_j-operator.

Proof. For every H_j-operator U_j there is a self-adjoint A_j with $U_j=\exp(i\cdot A_j)$. Thus by Proposition 4.3, V has the form $V=\frac{d}{dt}\Big|_0 \exp(it\cdot A_1)\otimes\ldots\otimes\exp(it\cdot A_n)$.

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