A projection principle concerning biholomorphic automorphisms

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1. Introduction

Let $E$ denote a Banach space and $D$ be a bounded domain in $E$. A mapping $F$ of $D$ onto itself is called a biholomorphic automorphism of $D$ if the Fréchet derivative of $F$ exists at each point $x \in D$ and is a bounded invertible linear $E$-operator. Our basic motivation in this article is the problem of describing $\text{Aut} B(E)$ the group of all biholomorphic automorphisms of the unit ball $B(E)$ of $E$. By recent results of W. KAUP [7] and J.-P. Vigué [18], this problem stands in a close relationship with that of the classification of symmetric complex Banach manifolds which is solved since a long time in the finite dimensional case [2] but fairly not settled for infinite dimensions.

In 1979, E. Vesentini [16] has shown that the unit ball of a nontrivial $L^1$-space admits only linear biholomorphic automorphisms. His proof goes back to investigations on Aut-invariant distances and a classical two dimensional result of M. KRITIKOS [9]. Using a characterization of polynomial vector fields tangent to $\partial B(E)$ (the boundary of $B(E)$) we found [11] an essentially two dimensional argument that enabled us to establish the sufficient and necessary condition for an $L^p$-space to have only linear unit ball automorphisms (for different approaches cf. also [1], [16]).

The purpose of Section 2 the general abstract part of this work is to clear up the deeper geometric background and connections of the seemingly different methods in treating $L^p$-spaces that occur in [16] and [11], respectively. Our main theorem provides a sufficient condition in terms of the Carathéodory (or Kobayashi) metric to reconstruct the biholomorphic automorphism group of Banach manifolds from those of its certain submanifolds via holomorphic projections. This result seems to be very well suited in calculating explicitly $\text{Aut} B(E)$ in various Banach spaces $E$ admitting a sufficiently large family of contractive linear projections. In Section 3 we illustrate the use of this projection principle by two typical examples where the con-
clusion seems hardly available with other already published methods: After numerous partial solutions, recently T. FRANZONI [4] gave the complete description of \( \text{Aut} B(\mathcal{L}(H_1, H_2)) \) where \( \mathcal{L}(H_1, H_2) = \{ \text{bounded linear operators } H_1 \rightarrow H_2 \} \) and \( H_1, H_2 \) are arbitrary Hilbert spaces. As we shall see, the projection principle makes it possible to obtain the exact description of \( \text{Aut} B(H_1 \otimes \cdots \otimes H_n) \) in an elementary way where \( H_1 \otimes \cdots \otimes H_n = \{ \text{continuous } n \text{-linear functionals } H_1 \times \cdots \times H_n \rightarrow \mathbb{C} \} \). Note that \( \mathcal{L}(H_1, H_2) \simeq H_1 \otimes H_2 \) and for \( n \geq 3, H_1 \otimes \cdots \otimes H_n \) cannot be equipped with a suitable \( J^* \)-structure on which Franzoni’s method is based. The key of the reduction by the projection principle is the fact that in finite dimensions the strong precompactness of \( B(H_1 \otimes \cdots \otimes H_n) \) considerably simplifies the treatment of the space (Section 4). The second application concerns atomic Banach lattices. The unit balls of finite dimensional such spaces are exactly the convex Reinhardt domains. In 1974, T. SUNADA [13] characterized \( \text{Aut}_0 D \) for all the bounded Reinhardt domains \( D \). However, his proofs depend on the Cartan theory of finite dimensional semisimple Lie algebras thus cannot be carried out in infinite dimensions. If the finite dimensional ideals form a dense submanifold, the projection principle reduces even the most general case to some straightforward 2-dimensional considerations. We remark that in this way also Sunada’s proof can be simplified and the method applies in parts to other Banach lattices (cf. [12]).

2. Projection principle

Our main abstract result concerns with holomorphic vector fields on complex Banach manifolds (for basic definitions see [17], [7, § 2]). If \( M \) denotes a complex Banach manifold, a vector field \( v: M \rightarrow TM \) is complete in \( M \) iff for every \( x \in M \), there exists a mapping \( e_x: \mathbb{R} \rightarrow M \) such that \( e_x(0) = x \) and \( \frac{d}{dt} e_x(t) = v(e_x(t)) \) \( \forall t \in \mathbb{R} \). In this case we define \( \exp(tv)(x) = e_x(t) \). A function \( \delta: TM \rightarrow \mathbb{R}_+ \) is called a differential Finsler metric on \( M \) if for any fixed \( x \in M \), the functional \( T_xM \ni w \mapsto \delta(x, w) \) is convex and positive-homogeneous and for each coordinate-map \( (U, \Phi) \), the function \( f^\Phi(\nu_0): \Phi U \ni \nu \mapsto \delta(\Phi^{-1} \nu, \nu(\Phi^{-1} \nu)) \) is locally bounded and lower semicontinuous whenever \( \nu \) is a holomorphic vector field on \( M \). We shall write \( d_\mu \) for the Carathéodory distance [3], [17] on \( M \), i.e. \( d_\mu(x, y) = \sup \{ \text{area } F(y); F \text{ is a holomorphic } M \rightarrow \Delta \text{ function, } F(x) = 0 \} \) where \( \Delta = \{ \xi \in \mathbb{C}: |\xi| < 1 \} \). For a holomorphic mapping \( F: M \rightarrow M \), we denote by \( F' \) its Fréchet derivative (recall that for any fixed \( x \in M \), \( F'(x) \) is a bounded linear \( T_xM \rightarrow T_xM \) operator). For a Banach space \( E \), we shall denote by \( E^*, || \cdot || \), and \( B(E) \) its dual, norm, closure operation and open unit ball, respectively.
2.1. Theorem. Let $M$ be a complex Banach manifold, $M'$ a (complex) submanifold of $M$ and $v$ a complete holomorphic vector field on $M$. Suppose $P$ is a holomorphic mapping of $M$ onto $M'$ such that $P|_{M'} = \text{id}_{M'}$ (the identity mapping on $M'$).

Suppose there exists a differential Finsler metric $\delta$ on $M'$ such that

(i) the vector field $P'v|_{M'}$ is $\delta$-bounded (i.e. $\sup_{x \in M} \delta(x, P'(x)v(x)) < \infty$)

and by writing $d$ for the intrinsic distance generated by $\delta$ on $M'$,

(ii) the topology of the metric $d$ is finer than that of $M'$,

(iii) for any sequence $x_1, x_2, \ldots \in M'$ which is a Cauchy sequence with respect to $d$ but which is not convergent in $M'$ we have $d_{M'}(x_1, x_n) \to \infty$ ($n \to \infty$).

Then the vector field $P'v$ is complete in $M'$.

Proof. For the sake of simplicity, the proof will be divided into three steps.

1) From the definition of Carathéodory distance we see immediately that $d_{M'}(x, y) \equiv d_{M'}(x, y) \forall x, y \in M'$ since $M' \subset M$. It is also well-known [2] that the mapping $P$ is a $d_{M'} - d_{M'}$ contraction. Hence the relation $P|_{M'} = \text{id}_{M'}$ entails $d_{M'}(x, y) \equiv d_{M'}(x, y)$. Thus we obtained $d_{M'} = d_{M'}|_{M'}$.

In the sequel, we set $a_x(t) \equiv \exp\,(tv)(x)\,(x \in M, t \in \mathbb{R})$ and $b_x$ will denote the maximal solution of the initial value problem \( \left\{ \begin{align*}
\frac{d}{dt}y &= P'(y)v(y); \\
y(0) &= x
\end{align*} \right. \)

We show that for arbitrarily fixed $z \in M'$,

(1) \[ d_{M'}(Pa_z(h), b_z(h)) = o(h) \quad (h \to 0). \]

Indeed: Consider any coordinate-map $(U, \Phi)$ from the atlas of $M'$ for which $z \in U$. We may assume without loss of generality that there is a biholomorphism between $U$ and the open unit ball of some Banach space $E$. Then for all $h \in \{ \} : \exists \text{dom } b_z$

\[ b_z(t) \in \Phi^{-1}\left( \frac{1}{2}B(E) \right) \]

we have

\[ d_{M'}(Pa_z(h), b_z(h)) \equiv d(Pa_z(h), b_z(h)) = d_{B(E)}(\Phi Pa_z(h), \Phi b_z(h)) \leq \mu\|\Phi Pa_z(h) - \Phi b_z(h)\| \]

where $\mu \equiv \sup \left\{ \frac{d_{B(E)}(f, g)}{\|f - g\|} : f, g \in \frac{1}{2}B(E) \right\}$. It is easily seen that $\mu \leq 2 \sup \left\{ \|f\|^{-1} \text{ areath } \|f\| : \|f\| \leq \frac{1}{2} \right\} \leq \infty$.

The estimate $\|\Phi Pa_z(h) - \Phi b_z(h)\| = o(h) \quad (j \to 0)$ can be verified as follows:

By definition, $a$ is the solution of the initial value problem \( \left\{ \begin{align*}
\frac{d}{dt}y &= v(y); \\
y(0) &= z
\end{align*} \right. \)
Therefore  \[ \|\Phi a_z(h) - (\Phi z + h\Phi'v(z))\| = o(h). \] Thus  \[ \frac{d}{dh} \bigg|_0 \Phi b_z(h) = \Phi b_z(h) = \Phi P a_z(h) = 0. \]

An application of (1) directly yields that for any \( x, y \in M' \),

\[
\lim_{h \to 0} \frac{1}{|h|} \left[ d_{M'}(b_x(h), b_y(h)) - d_{M'}(x, y) \right] = \lim_{h \to 0} \frac{1}{|h|} \left[ d_{M'}(Pa_x(h), Pa_y(h)) - d_{M'}(x, y) \right] \equiv 0
\]

since \( P \) is a contraction \( d_{M'} \to d_{M'} \) and \( d_{M'} = d_{M'}|M' \).

2) Henceforth we proceed by contradiction. Assume that the vector field \( P'v \) is not complete in \( M' \).

Now we may fix a point \( x \in M' \) such that \( \text{dom } b_x \neq \mathbb{R} \). Let \( t_0 \) be a boundary point of the interval (or ray) \( \text{dom } b_x \). Since \( 0 \in \text{dom } b_x \), we have \( t_0 \neq 0 \). So (by passing to the vector field \( \frac{1}{t_0}v \)) we may assume \( t_0 = 1 \). Then consider the function

\[ q(t) = d_{M'} \left( b_x(t), b_x \left( t + \frac{1}{2} \right) \right), \quad \left( t \in \left[ 0, \frac{1}{2} \right) \right). \]

Since \( b_x(t+h) = b_{b_x(t)}(h) \) and \( b_x \left( t + \frac{1}{2} + h \right) = b_{b_x} \left( t + \frac{1}{2} \right)(h) \) whenever \( t, t+h, t+\frac{1}{2}, t+\frac{1}{2} + h \in [0, 1) \), from step 3) it follows that

\[
\lim_{h \to 0} \frac{q(t+h) - q(t)}{|h|} \equiv 0 \quad \forall t \in \left[ 0, \frac{1}{2} \right).
\]

We show that the function \( q \) is locally Lipschitzian. Since the conclusion of the previous step can be interpreted as \( q'(t) = 0 \) for all such values \( t \) where \( q'(t) \) exists, hence we obtain that \( q \) is constant i.e.

\[ d_{M'} \left( b_x(t), b_x \left( t + \frac{1}{2} \right) \right) = d_{M'} \left( x, b_x \left( \frac{1}{2} \right) \right), \quad \forall t \in \left[ 0, \frac{1}{2} \right). \]

Proof. By triangle inequality, it suffices to see that for any \( z \in M' \), the mapping \( t \mapsto b_x(t) \) is locally Lipschitzian with respect to the metric \( d_{M'} \). Denote by \( \delta_{M'} \) the Carathéodory differential Finsler metric of the manifold \( M' \) (for definition see \([2, 17]\)). Then the function \( \gamma : \tau \mapsto \delta_{M'}(b_x(\tau), P'\delta(b_x(\tau))) \) is locally bounded (cf.
(17)). Hence if \( \mathcal{S} \) is a compact subinterval of \( \text{dom } b_x \) then \( \sup_{t \in \mathcal{S}} \gamma(t) < \infty \) and therefore

\[
\begin{align*}
&d_{M'}(b_x(t'), b_x(t'')) \equiv \left| \int_{t'}^{t''} \delta_{M'}(b_x(t), b_x'(t)) \, dt \right| = \left| \int_{t'}^{t''} \gamma(t) \, dt \right| \leq \\
&\leq \sup_{t \in \mathcal{S}} \gamma(t) \cdot |t'' - t'| \quad \text{whenever } \ t', \ t'' \in \mathcal{S}.
\end{align*}
\]

3) Write \( K \equiv \sup_{x \in \mathcal{M}'} \delta(x, P'v(x)) \) and consider the sequence \( t_n \equiv \frac{1}{2} - \frac{1}{2n} \) \( (n=1, 2, \ldots) \). For \( m \equiv n \) we have

\[
\begin{align*}
d\left(b_x\left(t_n + \frac{1}{2}\right), b_x\left(t_n + \frac{1}{2}\right)\right) &\equiv \int_{t_n}^{t_n+1} \delta(b_x(t), b_x'(t)) \, dt = \\
&= \int_{t_n}^{t_n+1} \delta(b_x(t), P'v(b_x(t))) \, dt \equiv \int_{t_n}^{t_n+1} K \, dt = \frac{K}{2} \left(\frac{1}{m} - \frac{1}{n}\right).
\end{align*}
\]

Thus \( \left\{ b_x\left(t_n + \frac{1}{2}\right) \right\} \) is a Cauchy sequence with respect to the metric \( d \). Suppose

\[
d\left(b_x\left(t_n + \frac{1}{2}\right), z\right) \to 0 \quad (n \to \infty)
\]

for some point \( z \in \mathcal{M}' \). Then we would have \( P'v(b_x(t_n)) \to P'v(z) \) \( (n \to \infty) \), as a consequence of (ii). However, in this case the function \( \tilde{b}(t) = \begin{cases} b_x(t) & \text{if } t \in \text{dom } b_x \\ b_x(t-1) & \text{if } 0 \equiv (t-1) \in \text{dom } b_x \end{cases} \) is a solution of the initial value problem \( \left\{ \frac{d}{dt} y = P'v(y), y(0) = x \right\} \) with \( \text{dom } \tilde{b} \not\subseteq \text{dom } b_x \) which is excluded by the maximality of \( b_x \). Thus \( \left\{ b_x\left(t_n + \frac{1}{2}\right) \right\} \) does not converge in the metric \( d \).

By condition (iii), \( d_{M'}\left(b_x\left(t_n + \frac{1}{2}\right), b_x\left(1 - \frac{1}{2n}\right)\right) = d_{M'}\left(b_x\left(t_n + \frac{1}{2}\right), b_x\left(1 + \frac{1}{2}\right)\right) \to \infty \quad (n \to \infty) \). From (2) we see

\[
\begin{align*}
&d_{M'}\left(b_x\left(1 - \frac{1}{2n}\right), b_x\left(1 - \frac{1}{2n}\right)\right) \equiv d_{M'}\left(b_x\left(1\right), b_x\left(1 - \frac{1}{2n}\right)\right) - \\
&-d_{M'}\left(b_x\left(1 - \frac{1}{2n}\right), b_x\left(1 - \frac{1}{2n}\right)\right) = d_{M'}\left(b_x\left(1\right), b_x\left(1 - \frac{1}{2n}\right)\right) - \\
&-d_{M'}\left(x, b_x\left(\frac{1}{2}\right)\right) \to \infty \quad (n \to \infty).
\end{align*}
\]

But this is impossible because the topology of a complex Banach manifold is always finer than that generated by its associated Carathéodory metric (cf. [17]) whence \( d_{M'}\left(b_x\left(1\right), b_x\left(1 - \frac{1}{2n}\right)\right) \to 0 \quad (n \to \infty) \) since the mapping \( t \mapsto b_x(t) \) is differentiable.
The obtained contradiction completes the proof.

2.2. Remark. From step 1) one immediately reads that in general we have

2.2a Lemma. If \( d^*: N \to d^*_N \) is a metric valued functor on the category of complex Banach manifolds such that for all manifolds \( N, N' \),

(iv) \( d^*_N \) is a metric on \( N \),

(v) each holomorphic map \( \phi_N \to N \) is a \( d^*_N \to d^*_N \), contraction,

then \( d^*_N |_{M'} = d^*_M \), whenever \( M' \) is a submanifold of \( M \) and there can be found a holomorphic projection of \( M \) onto \( M' \).

The proof of Theorem 2.1 can be carried out as well for any metric functor \( d^* \) with properties (iv), (v) and

(vi) \( \sup \left\{ d^*_E(f, 0)/\| f \| : \| f \| \leq 1 \right\} < \infty \) for any Banach space \( E \).

The Kobayashi invariant metric (def. see [17], [9]) also satisfies these requirements. Hence Theorem 2.1 holds when replacing Carathéodory distances by those of Kobayashi. Moreover we have the following important special case of Lemma 2.2a.

2.2b Lemma. If \( E \) denotes a Banach space and \( P \) is a contractive linear projection \( E \to E \) then \( d_P(E) \to B(PE) = B(PE) \) and \( d^*_E(B(PE)) = d^*_B(B(PE)) \) where \( d^k \) stands for the Kobayashi distance.

Proof. Since \( \| P \| = 1 \) (otherwise we have the trivial case \( P = 0 \)), \( PE \) is a closed subspace of \( E \) and \( PB(E) = B(PE) \subset B(E) \). Thus Lemma 2.2a can be applied to \( M = B(E) \) and \( M' = B(PE) \).

This latter result can be further specialized as follows: Consider any unit vector \( e \in E \). By the Hahn—Banach theorem, there exists \( \Phi \in E^* \) with \( \| \Phi \| = \langle e, \Phi \rangle = 1 \). Then the mapping \( P: f \mapsto \langle f, \Phi \rangle e \) is a contractive linear projection of \( E \) onto \( Ce \). Thus Lemma 2.2b contains Vesentini's following observation.

2.2c Lemma (Vesentini [16]). Let \( E \) be a Banach space, \( e \in E \) a unit vector and \( \zeta_1, \zeta_2 \in \Delta \). Then we have \( d^*_E(\zeta_1 e, \zeta_2 e) = d_B(\zeta_1 e, \zeta_2 e) \) where

\[
\frac{\zeta_1 - \zeta_2}{1 - \zeta_1 \zeta_2},
\]

i.e. the curve \( \Delta \ni \zeta \to \zeta e \) is a complex geodesic with respect to both the Carathéodory and Kobayashi distances in \( B(E) \).

Later on, we restrict our attention to Banach space unit balls. Recall ([8], [18]) that in a Banach space \( E \), the elements of \( Aut(E) \) (the connected component of \( Aut(B(E)) \) w.r.t. the topology \( T_0 \) defined in [15]) are exactly the exponential images of the second degree polynomial vector fields being complete in \( B(E) \) whose Lie-algebra will be denoted by \( \log Aut(B(E)) \) Moreover, the orbit \( \{ Aut(B(E)) \} \{ 0 \} = \{ F(0) : F \in Aut(B(E)) \} \) is the intersection of \( B(E) \) with a subspace which, in the sequel, we shall denote by \( E_0 \) and we have \( E_0 = [\log Aut(B(E))] \{ 0 \} \).
2.3. Theorem. If $E$ is a Banach space and $P : E \to E$ is a contractive linear projection then $P[\log^* \text{Aut } B(E)]|_{PE} \subset \log^* \text{Aut } B(PE)$.

Proof. Let $u \in \log^* \text{Aut } B(E)$ be arbitrarily fixed. We have to show that the vector field $Pu|_{B(PE)}$ is complete in $B(PE)$. As in the proof of Lemma 2.2b, let us consider the manifolds $M = B(E)$, $M' = B(PE)$, the projection $P|_{B(PE)}$ of $M$ onto $M'$ and the vector field $v = u|_{B(PE)}$ which is by definition complete in $M$. Take the differential Finsler metric $\delta(x, w) = \|w\|_E (x \in B(PE), w \in PE)$ on $M'$ whose generated intrinsic distance is obviously $d(x, y) = \|x - y\|_{E} (x, y \in B(PE))$. To complete the proof, we need only to verify (i), (ii), (iii).

(i): For $x \in B(PE)$ we have $P'(x)v(x) = Pu(x)$ whence by a theorem of KAMP–UPMEIER [8],

$$
\delta(x, P'v(x)) = \|Pu(x)\| = \left\|u(0) + u'(0)x + \frac{1}{2} u''(0)(x, x)\right\| = \\
\geq \left\|u(0)\right\| + \left\|u'(0)\right\|_{E} + \frac{1}{2} \left\|u''(0)\right\|_{E}.
$$

(ii): Trivial.

(iii): Assume $x_1, x_2, \ldots$ is a Cauchy sequence with respect to the metric $d$ without a limit in $M'$. Then for some unit vector $f \in PE$, $\|x_n - f\| \to 0 (n \to \infty)$ i.e. $\|x_n\| \to 1$. Therefore, by Lemma 2.2c, $d_{M'}(x_1, x_n) = d_{B(PE)}(x_1, x_n) = d_{B(PE)}(x_n, 0) - d_{B(PE)}(x_1, 0) = \text{arath} \|x_n\| = \text{arath} \|x_1\| \to \infty$.

2.4. Corollary. If $E$ is a Banach space and $P : E \to E$ is a contractive linear projection then $P(\{0\}) \subset (PE)_0$. In particular, if $B(E)$ is a symmetric manifold then so is $B(PE)$, too.

2.5. Corollary. Let $E$ be a Banach space. If one can find a family $\mathcal{P}$ of contractive linear projections $E \to E$ such that for every $P \in \mathcal{P}$, Aut $B(PE)$ consists only of linear transformations and $\bigcap_{P \in \mathcal{P}} \ker P = \{0\}$ then all the elements of Aut $B(E)$ are also linear.

Proof. If $v \in \log^* \text{Aut } B(E)$ then $Pv(0) = 0 \forall P \in \mathcal{P}$ whence $v(0) = 0$ i.e. the vector field $v$ is linear. On the other hand Aut $B(E) = \text{Aut}^0 B(E) \cup \text{Aut} B(E) = \text{Aut}^0 B(E) \cup \text{exp} \log^* \text{Aut } B(E)$, where Aut$^0 \equiv \{E$-unitarities$\}$.

3. Applications

Let $(X, \mu)$ denote a measure space. In [1], [11] it is proved

3.1. Theorem. The unit ball of $E = L^p(X, \mu)$ admits only linear biholomorphic automorphisms unless $\dim E = 1$ or $p = 2, \infty$.

As the first illustration of the projection principle, we show how can this result be reobtained from Thullen's classical 2 dimensional theorem [14].
Proof. Suppose \( p \in [1, \infty) \setminus \{2\} \) and \( \dim E > 1 \). If \( g_1, g_2 \) are functions in \( E \) with norm 1 having disjoint supports then it is easily seen that the mapping \( P_{g_1, g_2} : E \to \sum_{j=1}^{2} \int g_j |g_j|^{p-2} d\mu \cdot g_j \) is a contractive linear projection of \( E \) onto the subspace \( E_{g_1, g_2} = \sum_{j=1}^{2} C g_j \). Now \( B(E_{g_1, g_2}) = \{ \xi_1 g_1 + \xi_2 g_2 : |\xi_1|^p + |\xi_2|^p < 1 \} \) is a Reinhardt domain whose biholomorphic automorphisms are all linear by Thullen’s theorem. Furthermore we have \( \ker P_{g_1, g_2} = \{ f \in E : \int \int |g_j|^{p-2} d\mu = 0 \ (j=1, 2) \} \). Thus \( \bigcap_{g_1, g_2} \ker P_{g_1, g_2} = \{ f \in E : \forall g \in E \ [\exists h \in E \ [\min(|\xi_1|, |\xi_2|) = 0] \Rightarrow \int \int |g_j|^{p-2} d\mu = 0] \subset \{ f \in E : \forall X_1 \subset X \ [\exists X_2 \subset X \setminus X_1 \ [0 < \mu(X_1), \mu(X_2) < \infty] \Rightarrow \int \int d\mu = 0] \} \}. \) Hence Corollary 2.5 establishes the linearity of \( \text{Aut} B(E) \).

To the next application, let \( H_1, \ldots, H_n \) be arbitrarily fixed Hilbert spaces\(^1\) of at least 2 dimensions and consider the biholomorphic automorphism group of the unit ball \( B \equiv B(E) \) of the space \( E \equiv H_1 \otimes \cdots \otimes H_n \), the Banach space of \( n \)-linear functionals endowed with the usual norm \( \| F \| = \sup \{|F(h_1, \ldots, h_n) : h_j \in H_j, \| h_j \| = 1 \ (j=1, \ldots, n)\} \) for \( F \in E \). For \( n = 1, 2 \), the description of \( \text{Aut} B \) is completely settled \([5],[4]\). It is worth to remark that, in the light of the Kaup Vigué theory, the difficulties in this case can be concentrated to the description of linear \( E \)-unitary operators: If \( n = 1, E \) can be identified with \( H_1 \) and for any fixed \( c \in H_1 \), the quadratic vector field \( q \equiv [H_1 \ni f \mapsto -(f|c)f] \) satisfies \([11, (1)]\) i.e. tangent to the boundary of \( B \).

Similarly, if \( n = 2 \), \( E \) can be identified with \( \mathcal{L}(H_1, H_2) \) and for fixed \( C \in \mathcal{L}(E_1, E_2) \), the vector field \( [\mathcal{L}(H_1, H_2) \ni F \mapsto -FC^*F] \) is quadratic and satisfies \([11, (1)]\). It is easily seen, in both cases that, we have \( \{ \exp (tq)(0) : t \in \mathbb{R} \} = (-1, 1) C, \) thus \( B \) is symmetric and \( \text{Aut} B = (\text{Aut}^{*} B) \exp \{ q_c : c \in E \} \). Here we turn our attention first of all to the case \( n \geq 3 \) which seems heavily treatable with other methods and is not touched by the literature.

3.2. Lemma. Span \( \{ UC : U \text{ linear} \in \text{Aut}_B \} = E \) whenever \( C \in E \setminus \{0\} \) and \( \dim H_j < \infty \ (j=1, \ldots, n) \).

Proof. If \( C \neq 0 \) then we may fix unit vectors \( e_j \in H_j \ (j=1, \ldots, n) \) such that \( \gamma \equiv C(e_1, \ldots, e_n) \neq 0 \). Then let \( P_{\gamma} \) denote the orthogonal projection of \( H_j \) onto \( C e_j \) and set \( U_j \equiv \exp (i \delta_j) P_{\gamma}, \ C(\delta_1, \ldots, \delta_n) \equiv (U_j \otimes \cdots \otimes U_n) C \ (\delta_j \in \mathbb{R} ; j=1, \ldots, n) \). Since the operators \( U_j \) are \( H_j \)-unitary, \( U_1 \otimes \cdots \otimes U_n \in \text{Aut}_B \), therefore \( e_1 \otimes \cdots \otimes e_n = F(e_1, \ldots, e_n) \).

\(^1\) Without danger of confusion, we write simply \( (\cdot, \cdot) \) for the inner product in any of \( H_1, \ldots, H_n \). For \( A_j \in \mathcal{L}(H_j, H_j) \) and \( e_j \in H_j \ (j=1, \ldots, n) \), we define \( A_1 \otimes \cdots \otimes A_n \equiv [H_1 \otimes \cdots \otimes H_n \ni F \mapsto F(A_1 e_1, \ldots, A_n e_n) \ (e_1, \ldots, e_n)] \) and \( \delta_{e_1, \ldots, e_n} \equiv F \mapsto F(e_1, \ldots, e_n) \), respectively.
= \frac{i}{\gamma} \frac{\partial^n}{\partial \theta_1 \cdots \partial \theta_n} \mid C \subseteq S \equiv \text{Span} \{ UC : U \text{ linear} \in \text{Aut}_B \}. \text{ Thus for all } H_j\text{-unitary operators } V_j, (V_j e_1) \otimes \cdots \otimes (V_j e_n) = (V_1 \otimes \cdots \otimes V_n) (e_1 \otimes \cdots \otimes e_n) \in S \text{ i.e. } f_1 \otimes \cdots \otimes f_n \in S \text{ whenever } f_1 \in H_1, \ldots, f_n \in H_n, \text{ whence } S = E \text{ (since } \dim E = \infty). \)

3.3. Proposition. For \( n > 2 \), all the elements of \( \text{Aut} B(H_1 \otimes \cdots \otimes H_n) \) are linear.

Proof. Observe that the family \( \mathcal{P} \equiv \{ P_i \otimes \cdots \otimes P_n \text{ all } P_i \text{-s are orthogonal } H_j\text{-projections with } \dim P_j H_j = [2 \text{ if } j \equiv 3 \text{ and } 1 \text{ if } j \neq 3] \} \) consists of contractive E-projections and \( \bigcap_{P \in \mathcal{P}} \ker P = \{0\} \). Since for arbitrary \( P \in \mathcal{P} \), the subspace \( PE \) is isometrically isomorphic to \( \mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3 \) (\( C \) is endowed with its usual euclidean norm), by Corollary 2.5 it suffices to see only that the elements of the group \( \text{Aut} B(\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3) \) are linear. Thus we may assume \( n = 3 \) and \( H_j = C \) (\( j = 1, 2, 3 \)). Assume now that \( E_0 \neq 0 \). Now Lemma 3.2 establishes \( E_0 = E \) i.e. symmetry of \( B \). We show that this is impossible.

Denote by \( e_1, e_2 \) the vectors \( (1, 0) \) and \( (0, 1) \) in \( \mathbb{C}^3 \), respectively, and consider the elements \( C \equiv e_1 \otimes e_1 \otimes e_1 \) and \( F \equiv e_2 \otimes e_2 \otimes e_2 + e_1 \otimes e_2 \otimes e_1 + e_1 \otimes e_1 \otimes e_1 \) of \( E \). Since the space \( E \) is finite dimensional, for every \( A \in E \) we can find \( f_1, f_2, f_3 \in \partial B(\mathbb{C}^3) \) with \( \| A \| = A(f_1, f_2, f_3) \). In particular, for arbitrarily given \( \lambda \in \{0, \frac{1}{3}\} \) we can fix unit vectors \( f_j(\lambda) \) such that \( \| C + \lambda F \| = \langle C + \lambda F, \delta f_1(\lambda), f_2(\lambda), f_3(\lambda) \rangle \). Since \( C, F \equiv 0 \) (i.e. \( C(g_1, g_2, g_3), F(g_1, g_2, g_3) \equiv 0 \; \forall g_1, g_2, g_3 \equiv 0 \) and since \( \langle C + \lambda F, \delta e_2 e_2, e_2 \rangle = \langle C + \lambda F, \delta e_1 e_2, e_1 \rangle = e_j(\lambda) = 1 \), for some \( r_j(\lambda) \equiv 0 \) we can write \( f_j(\lambda) = \frac{e_1 + r_j(\lambda)e_2}{(1 + r_j(\lambda))^{\frac{1}{2}}} \) (\( j = 1, 2, 3 \)). Thus introducing the function \( \Phi_1(e_1, e_2, e_3) \equiv \langle C + \lambda F, \delta e_1 e_2, e_1 \rangle = \frac{1}{(1 + r_2(\lambda))^{\frac{1}{2}}} \), we have \( \frac{\partial}{\partial \Phi_1 |_{r_1(\lambda), r_2(\lambda), r_3(\lambda)}} \Phi_1 = 0 \) (\( j = 1, 2, 3 \)).

So \( \{ \lambda(1 + r_1^2) - (1 + \lambda(1 + r_2 + r_3)) \cdot \sum_{k=1}^3 (1 + r_k^2)^{-\gamma/2} = 0 \) (\( j = 1, 2, 3 \)) and hence \( \lambda = \frac{r_1}{1 - r_1(r_2 + r_3)} = \frac{r_2}{1 - r_2(r_1 + r_3)} = \frac{r_3}{1 - r_3(r_1 + r_2)}. \) Therefore \( r_j \equiv 0 \) (\( j = 1, 2, 3 \)) and \( \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} \left( = \frac{1}{\lambda} + \sum_{j=1}^3 r_j \right) \). Observe that from this and from the assumption \( \lambda \in \{0, \frac{1}{3}\} \) it follows that \( r_1 = r_2 = r_3 \). (Otherwise there would be \( r > 0 \) such that two of the numbers \( r_1, r_2, r_3 \) coincided with \( r \) and the third with \( 1/r \), respectively. But then \( \lambda = \frac{1/r}{1 - (1/r)(r + r)} < 0 \)). Thus the relation \( \lambda = \frac{r}{1 - 2r} \) holds where \( r(\lambda) \equiv r_1(\lambda) = r_2(\lambda) = r_3(\lambda) \). This fact can be so interpreted that for sufficiently small
values of $r > 0$ (namely for $\lambda > \frac{1}{3}$ i.e. $r < \frac{\sqrt{17} - 3}{4}$), $F_r \equiv C + \frac{r}{1 - 2r^2} F$, $\Phi_r \equiv \delta_{e_1 + re_2, e_1 + re_2, e_1 + re_2}$ fulfill $\|F_r\| \cdot \|\Phi_r\| = \langle F_r, \Phi_r \rangle$. Then by [11, Lemma]
\begin{equation}
\|F_r\|^2 \langle C, \Phi_r \rangle + \langle q(F_r, F_r), \Phi_r \rangle = 0 \quad \left(0 < r < \frac{\sqrt{17} - 3}{4}\right),
\end{equation}
for some symmetric bilinear map $q : E \times E \rightarrow E$. Here $\langle C, \Phi_r \rangle = 1, \|F_r\| \equiv \|\Phi_r\|^{-1} \langle F_r, \Phi_r \rangle = (1 + r^2)^{-1/2} \left(1 + 3r^2 - \frac{r}{1 - 2r^2}\right) = (1 + r^2)^{-1/2}(1 - 2r^2)^{-1}$ and $\langle q(F_r, F_r), \Phi_r \rangle = \langle q(C, C), \Phi_r \rangle + 2\frac{r}{1 - 2r^2} \langle q(C, F), \Phi_r \rangle + \left(\frac{r}{1 - 2r^2}\right)^2 \langle q(F, F), \Phi_r \rangle$. Taking into consideration that for fixed $V \in E$, the function $r \mapsto \langle V, \Phi_r \rangle$ is a polynomial of 3rd degree in $r$, from (2) we obtain
\begin{equation}
(1 + r^2)^{-1}(1 - 2r^2)^{-2} + p_1(r) + p_2(r)(1 - 2r^2)^{-1} + p_3(r)(1 - 2r^2)^{-2} = 0
\end{equation}
for some polynomial-triplet $p_1, p_2, p_3$. However, (2') immediately implies the contradiictory fact that the function $r \mapsto (1 + r^2)^{-1}$ is a polynomial.

3.4. Theorem. The linear $H_1 \otimes \ldots \otimes H_n$-unitary operators are exactly those operators $F$ for which there exists a permutation $\pi$ of the index set $\{1, \ldots, n\}$ and there are surjective linear isometries $U_k : H_k \rightarrow H_{\pi(k)}$ $(k = 1, \ldots, n)$ such that
\begin{equation}
F(L) = [(f_1, \ldots, f_n) \mapsto L(U_1^{-1} f_{\pi(1)}, \ldots, U_n^{-1} f_{\pi(n)})].
\end{equation}
A linear vector field $V$ belongs to $\log^* \text{Aut} B$ if and only if it is of the form
\begin{equation}
V = i \cdot \sum_{k=1}^n \text{id}_{H_1} \otimes \ldots \otimes \text{id}_{H_k-1} \otimes A_k \otimes \text{id}_{H_{k+1}} \otimes \ldots \otimes \text{id}_{H_n}
\end{equation}
where the $A_k$-s are arbitrary self-adjoint $H_k$-operators.

Proof. Based on some compactness arguments, in the next section we shall establish independently the validity of (3') if the spaces $H_k$ are all finite dimensional. Our starting point here is (3') for finite dimensional $E$. First we extend it to infinite dimensions.

Let $V \in \log^* \text{Aut} B$ and $e_1^* \in \partial B(H_1), \ldots, e_n^* \in \partial B(H_n)$ be arbitrarily fixed and define the operator $\tilde{V} \equiv V - \langle V(e_1^* \otimes \ldots \otimes e_n^*), \delta_{e_1^*, \ldots, e_n^*} \rangle \text{id}_E$. Since $i \cdot \text{id}_E \in \log^* \text{Aut} B$, we have $\tilde{V} \in \log^* \text{Aut} B$. Remark that $\tilde{V}(e_1^* \otimes \ldots \otimes e_n^*) = 0$. Then consider the family of mappings $\mathcal{P} \equiv \{P_1 \otimes \ldots \otimes P_n : P_k$ is an orthogonal $H_k$-projection, dim $P_k H_k < \infty, e_k \in P_k H_k$ $(k = 1, \ldots, n)\}$. Any element $P \equiv P_1 \otimes \ldots \otimes P_n$ of $\mathcal{P}$ is a contractive linear projection of the space $E$ onto its subspace $(P_1 H_1) \otimes \ldots \otimes (P_n H_n)$. Thus by the projection principle, $P \tilde{V}|_{PE} \in \log^* \text{Aut} B(PE) \forall P \in \mathcal{P}$. Hence (applying (3') to the finite dimensional $(P_1 H_1) \otimes \ldots \otimes (P_n H_n)$ for each $P \in \mathcal{P}$, there exists a
unique choice of $A_k^p \in \{\text{self-adj. } H_1, \text{op. } s\}, \ldots, A_n^p \in \{\text{self-adj. } H_n, \text{op. } s\}$ such that $A_k^p H_k \subseteq P_k H_k$ (i.e. $P_k A_k^p P_k = A_k^p$) and $(A_k^p e_k^p | e_k^p) = 0$ ($k = 1, \ldots, n$),

$$P \tilde{V} P = \sum_{k=1}^{n} i \cdot \text{id}_{H_1} \otimes \cdots \otimes \text{id}_{H_{k-1}} \otimes A_k \otimes \text{id}_{H_{k+1}} \otimes \cdots \otimes \text{id}_{H_n}.$$  

Introduce the following partial ordering $\preceq$ in $\mathcal{P}$: If $P = P_1 \otimes \cdots \otimes P_n$ and $Q = Q_1 \otimes \cdots \otimes Q_n$ then let $P \equiv Q \text{ if } P \preceq Q \iff P_k H_k \subseteq Q_k H_k$ (i.e. $P_k \equiv Q_k$) $k = 1, \ldots, n$. From the relation $P \equiv Q \Rightarrow P \tilde{V} P = PQ \tilde{V} Q$ we immediately see

(4) \quad $A_k^p = P_k A_k^p P_k$ ($k = 1, \ldots, n$) whenever $P \equiv Q$.

Observe that for any fixed $P \in \mathcal{P}$ and index $k$,

$$|(A_k^p e | f)| = |\langle (P \tilde{V}) (e_1^* \otimes \cdots \otimes e_{k-1}^* \otimes e \otimes e_{k+1}^* \otimes \cdots \otimes e_n^*) \rangle| \equiv \|P \tilde{V}\| \cdot \|e_1^* \otimes \cdots \otimes e \otimes e_{k+1}^* \otimes \cdots \otimes e_n^*\| \cdot \|e_1^*, \ldots, e_{k-1}, e_{k+1}, \ldots, e_n^*\| \|P \tilde{V}\| \equiv \|P \tilde{V}\| \forall e, f \in \partial B(H_k),$$

that is

(5) \quad $\|A_k^p\| \equiv \|P \tilde{V}\|$ ($k = 1, \ldots, n$) $\forall P \in \mathcal{P}$.

Since obviously $\forall P, Q \in \mathcal{P} \exists R \in \mathcal{P}$, $P \equiv Q$ and since by (4), (5) the relation $P \equiv Q$ entails $|(A_k^p e | f) - (A_k^q e | f)| = |\langle A_k^q (e-P_k e) | f \rangle + (A_k^q P_k e | f-P_k f) \rangle| \equiv \|P \tilde{V}\| (\|e-P_k e\| + \|f-P_k f\|) \forall e, f \in \partial B(H_k), $ $k = 1, \ldots, n$, the definitions

$$a_k(e, f) \equiv \lim_{P \in \mathcal{P}} (A_k^p e | f)$ (e, f \in H_k, \quad k = 1, \ldots, n)$$

make sense and determine bounded sesquilinear functionals. Therefore there exist self-adjoint operators $A_1: H_1 \to H_1, \ldots, A_n: H_n \to H_n$ such that $a_k(e, f) = (A_k e | f)$ and hence $\langle A_k^p e | f \rangle = \langle A_k^q (P_k e) | P_k f \rangle = \langle A_k P_k e | P_k f \rangle = \langle A_k P_k e | P_k f \rangle = \langle P_k A_k P_k e | f \rangle \forall e, f \in H_k$ i.e. $A_k^p = P_k A_k P_k$ ($P \in \mathcal{P}, k = 1, \ldots, n$). Now for arbitrary $L \in E, e_1 \in H_1, \ldots, e_n \in H_n$ the projections $P \equiv \text{proj}_{\text{Span}(e_1, A_k e_1, e_n)}$ ($k = 1, \ldots, n$) satisfy

$$[\tilde{V} L](e_1, \ldots, e_n) = [\tilde{V} L](P_1 e_1, \ldots, P_n e_n) = [P \tilde{V} L](e_1, \ldots, e_n) =$$

$$= \sum_{k=1}^{n} L(e_1, \ldots, P_k A_k e_k, \ldots, e_n) = \sum_{k=1}^{n} L(e_1, \ldots, A_k e_k, \ldots, e_n).$$

Thus we can write $VL(e_1, \ldots, e_n) = \sum_{k=1}^{n} L(e_1, \ldots, B_k e_k, \ldots, e_n)$ where $B_j \equiv A_j$ for $j = 1, \ldots, n-1$ and $B_n \equiv A_n + \langle V(e_1^*, \ldots, e_n^*) \otimes \delta_{k_1, \ldots, k_n} \rangle \text{id}_E$, proving (3') in general.

To prove (3), let $F$ be an arbitrarily given linear $E$-unitary operator and introduce the families $\mathcal{P}_k \equiv \{P_1 \otimes \cdots \otimes P_n: P_k$ is an orthogonal $H_k$-projection, $P_j = \text{id}_{H_j}$ for $j \neq k\}$ ($k = 1, \ldots, n$). From (3') we see $i \mathcal{P} \subseteq \text{log}^* \text{Aut } B$ and hence for every $P \in \mathcal{P}$, the mapping $Q \equiv FP^{-1}$ also has the properties $iQ \subseteq \text{log}^* \text{Aut } B$ and $Q^2 = Q$.
(since $P^2 = P$) which is possible (by (3')) only if $Q \in \mathcal{P}_{\ell_k(P)}$ for some index $\ell_k(P)$ ($k = 1, \ldots, n$).

Let $k \in \{1, \ldots, n\}$ be fixed. We show that $\ell_k(P_1) = \ell_k(P_2) \Rightarrow P_1, P_2 \in \mathcal{P}_k \setminus \{id_k\}$. Indeed, if $\ell_k(R_1) \neq \ell_k(R_2)$ then the operators $Q_j \equiv F R_j F^{-1}$ ($j = 1, 2$) commute (i.e. $[Q_1, Q_2] \equiv Q_1 Q_2 - Q_2 Q_1 = 0$) whence we would have $[R_1, R_2] = 0$. Observe that $\forall P_1, P_2 \in \mathcal{P}_k \setminus \{id_k\}, \exists P_3 \in \mathcal{P}_k$ such that $[P_1, P_2, P_3] \neq 0$, thus (by taking $R_1 \equiv P_j$ and $R_2 \equiv P_3$ $j = 1, 2$) $\ell_k(P_j) = \ell_k(P_3)$ holds for $j = 1, 2$.

Therefore there exists a permutation $\pi$ with

(6) 

$$F \mathcal{P}_k F^{-1} = \mathcal{P}_{\pi(k)} \quad (k = 1, \ldots, n).$$

Since the finite linear combinations of orthogonal projections form a dense submanifold of the algebra of linear operators in any Hilbert space, it directly follows the existence of surjective linear isometries $S_k: \mathcal{L}(H_k, H_k) \rightarrow \mathcal{L}(H_{\pi(k)}, H_{\pi(k)})$ such that

$$F(\text{id}_{H_1} \otimes \ldots \otimes \text{id}_{H_{k-1}} \otimes A_k \otimes \text{id}_{H_{k+1}} \otimes \ldots \otimes \text{id}_{H_n}) F^{-1} =$$

$$= \text{id}_{H_1} \otimes \ldots \otimes \text{id}_{H_{\pi(k)-1}} \otimes S_k(A_k) \otimes \text{id}_{H_{\pi(k)+1}} \otimes \ldots \otimes \text{id}_{H_n}$$

$$(A_k \in \mathcal{L}(H_k, H_k); \quad k = 1, \ldots, n).$$

As a consequence of the relations (6), the mappings $S_k$ send orthogonal projections into orthogonal projections and therefore they constitute *-isomorphisms between the C*-algebras $\mathcal{L}(H_k, H_k)$ and $\mathcal{L}(H_{\pi(k)}, H_{\pi(k)})$. It is well-known that now we can write

$$S_k: A_k \mapsto U_k A_k U_k^{-1} \quad (k = 1, \ldots, n)$$

for some surjective linear isometries $U_k: H_k \rightarrow H_{\pi(k)}$. Thus if we denote by $\sigma$ the inverse of the permutation $\pi$, for any linear E-operator $A$ of the form $A = A_1 \otimes \ldots \otimes A_n$ (where $A_k \in \mathcal{L}(H_k, H_k); \quad k = 1, \ldots, n$) we have

$$(F A F^{-1}) L = [(f_1, \ldots, f_n) \mapsto L(U_{\sigma(1)} A_{\sigma(1)} U_{\sigma(1)}^{-1} f_1, \ldots, U_{\sigma(n)} A_{\sigma(n)} U_{\sigma(n)}^{-1} f_n)] \quad \forall L \in E.$$

This means that $F A F^{-1} = U A U^{-1}$ $\forall A \in \mathcal{L}(E, E)$ holds for the E-unitary operator $U$ defined by

$$U(L) \equiv [(f_1, \ldots, f_n) \mapsto L(U_1^{-1} f_{\sigma(1)}, \ldots, U_n^{-1} f_{\sigma(n)})] \quad (L \in E).$$

It is easily seen that this is possible only if $F = e^{i\delta} U$ for some $\delta \in \mathbb{R}$ which completes the proof.

In the remainder part of this section, by making use of the projection principle, we shall examine the structure of biholomorphic unit ball automorphisms in case of minimal atomic Banach lattices (abbr. by min. B-lattices).

A Banach lattice $E$ is called a min. B-lattice if it is norm-spanned by its 1 dimensional ideals. Henceforth we reserve the symbol $E$ to designate a fixed min. B-lattice.
According to a well-known representation lemma [10, p. 143, Ex. 7 (b)], we may assume that for a fixed set \( X, E \) is a sublattice of \( \{ X \to C \text{ functions} \} \) such that

\[
1_x \in E \quad \text{and} \quad \|1_x\| = 1 \quad \forall x \in X, \tag{7}
\]

(8) Span \( \{1_x; x \in X\} = E \). (\( 1_x \) stand for \( [X \ni y \mapsto 1 \text{ if } y = x \text{ and } 0 \text{ elsewhere}] \)).

Remark that then

\[
(8') \quad \text{Every } f \in E \text{ and } \text{every } f = \lim_{y \text{ finite } \subseteq X} w1_y f \text{ whenever } f \in E, \quad \sup_{x \in X} |w(x)| \leq 1. \tag{8'}
\]

For the sake of simplicity we write \( B \equiv B(E) \) and the functional \( [E \ni f \mapsto f(x)] \) will be denoted by \( \ell_x \).

First we describe the linear part of \( \text{Aut } B \).

3.5 Definition. For \( x, y \in X \), let \( x \sim y \) if \( \langle \ell(y), 1_x \rangle \neq 0 \) for some linear element \( \ell \) of \( \text{log}^* \text{Aut } B \).

3.6. Lemma. (i) \( x \sim y \) if and only if for all \( f, g \in E \), \( f - g \in 1_{x,y} E \) and

\[
\sum_{z = x, y} |f(z)|^2 = \sum_{z = x, y} |g(z)|^2 \text{ entail } \|f\| = \|g\|.
\]

(ii) The relation \( \sim \) is an equivalence. Moreover, in case of \( x_1 \sim \cdots \sim x_n \),

\[
f - g \in 1_{x_1, \ldots, x_n} \quad \text{and} \quad \sum_{j=1}^n |f(x_j)|^2 = \sum_{j=1}^n |g(x_j)|^2 \text{ imply } \|f\| = \|g\|
\]

for all \( f, g \in E \) whenever \( x_1, \ldots, x_n \) are distinct points.

Proof. (i) Let \( Y = \{y_1, \ldots, y_n\} \) be an arbitrary finite subset of \( X \) and \( \ell \) linear \( \in \text{log}^* \text{Aut } B \). Set

\[
\alpha_{jk} = \langle \ell(y_j), 1_{x_k} \rangle \text{ and assume } \alpha_{jk} \neq 0 \text{ (i.e. } y_1 \sim y_2). \quad \text{Since the mapping}
\]

\[
P: f \mapsto 1_Y f \text{ is a band projection of } E \text{ onto } \sum_{j=1}^n \text{Cl}_{y_j}, \text{ the projection principle establishes}
\]

\[
\ell \in \text{log}^* \text{Aut } PB \text{ where } \ell \equiv P \ell |_{P E}. \quad \text{Thus by [11, Lemma]} \tag{9}
\]

\[
\text{Re } \langle \ell(f), \Phi \rangle = 0 \iff \langle f, \Phi \rangle = \|f\| \|\Phi\| \quad \forall f \in PE, \Phi \in (PE)^*.
\]

---

\( ^a \) Proof: Given \( \varepsilon > 0 \), by (8), there are \( Z \) finite \( \subseteq X, g \in 1_Z f \) with \( \|f - g\| < \varepsilon/2 \). Now \( Z \subseteq Y_1, Y_2 \) finite \( \subseteq X \) implies \( \|f - g\| \leq \|f - 1_Z f\| \leq w(1_Z - 1_Z f) \leq w(1_{Y_1 \cup Y_2} f - 1_{Y_1} f) \text{ (} j \neq 1, 2 \text{) i.e. by triangle inequality } \varepsilon \equiv \|w1_{Y_1} f - w1_{Y_2} f\|. \text{ Thus } \{w1_{Y_j} f\}_{Y_j \text{ finite} \subseteq \text{E}} \text{ is a Cauchy net in } E. \text{ Hence for some } h \in E^*, \text{ we have } w1_Y f \to h. \text{ But } h(x) = \langle h, 1_x^* \rangle = \lim_x \langle w1_Y f, 1_x \rangle = w(x)f(x) \forall x. \)

\( ^a \) In the same way as in [11, Lemma], one can see that if a linear vector field \( \ell \) on Banach space \( F \) belongs to \( \text{log}^* \text{Aut } B(F) \) then \( \text{Re } \ell(f), \Phi) = 0 \iff \langle f, \Phi \rangle = \|f\| \|\Phi\| \quad \forall f \in F, \Phi \in F^*. \)

Proof: Since \( \ell \) is tangent to \( \partial B(F) \), we have \( \ell(f)(H - f) \text{ whenever } \|f\| = 1 \) and \( H \) is a real hyperplane in \( F \) supporting \( B(F) \) at \( f. \text{ But the general form of such a supporting hyperplane is } H = \{h \in F; \text{Re } \langle h, \Phi \rangle = 1\} \text{ where } \Phi \in F^* \text{ with } \|\Phi\| = \langle f, \Phi \rangle = 1. \)
Introduce the function \( p(\varrho_1, \ldots, \varrho_n) := \sum_{j=1}^{n} \varrho_j \gamma_j \) on \( \mathbb{R}_+^n \) and set \( C := \{ \varrho \in \mathbb{R}_+^n : \text{grad} \varrho p \text{ does not exist} \} \). Since \( p \) is an increasing positively homogenous convex function, \( C \) is a cone of Lebesgue measure 0. Let us fix arbitrary vectors \( \varrho \in \mathbb{R}_+^n \setminus C \), \( \vartheta \in \mathbb{R}^n \) and set \( \pi \equiv \text{grad} \varrho p, f_0 \equiv \sum_{j=1}^{n} \varrho_j e^{i \theta_j} \gamma_j, \varPhi \equiv \sum_{j=1}^{n} \varrho_j e^{-i \theta_j} \gamma_j \). Since the function \( p \) is increasing, \( \pi, \ldots, \pi_n \equiv 0 \). Since \( \pi \) is positive homogeneous and convex, \( \sum_{j=1}^{n} \pi_j \varrho_j = p(\varrho_1, \ldots, \varrho_n) \) i.e. \( \langle f_0, \varPhi \rangle = \| f_0 \| \). On the other hand, for any \( f \in PE \)

\[
\langle f, \varPhi \rangle = \left| \sum_{j=1}^{n} \pi_j e^{-i \theta_j} f(y_j) \right| \leq \sum_{j=1}^{n} \pi_j |f(y_j)| \leq p(|f(y_1)|, \ldots, |f(y_n)|) = \| f \|
\]

i.e. \( \| \varPhi \| = 1 \). Hence (9) can be applied to \( f_0 \) and \( \varPhi \). Thus

\[(9') \quad \text{Re} \left( \ell \left( \sum_{j=1}^{n} \varrho_j e^{i \theta_j} \gamma_j, \sum_{j=1}^{n} \pi_j e^{-i \theta_j} \gamma_j \right) \right) = 0.
\]

By the arbitrary choice of \( \vartheta \in \mathbb{R}^n \), an equivalent form to (9') is

\[(9'') \quad \text{Re} \left[ \sum_{j} \varrho_j \pi_j \dot{\alpha}_{jj} + \sum_{j \neq k} (\varrho_k \pi_k \dot{\alpha}_{jk} + \varrho_k \pi_j \dot{\alpha}_{kj}) z_j z_k^{-1} \right] = 0
\]

whenever \( |z_1| = \ldots = |z_n| = 1 \).

This is possible only if the rational expression (w.r.t. \( z_1, \ldots, z_n \)) in the argument of the \( \text{Re} \) operation vanishes. Thus in particular \( \varrho_1 \pi_2 \dot{\alpha}_{12} + \varrho_2 \pi_1 \dot{\alpha}_{21} = 0 \). I.e. we obtained the following partial differential equation

\[(10) \quad \varrho_1 \frac{\partial p}{\partial \varrho_2} \varrho_1 \varrho_2 + \varrho_2 \frac{\partial p}{\partial \varrho_1} \varrho_2 \varrho_1 = 0 \quad (\varrho \in \mathbb{R}_+^n \setminus C).
\]

Since \( \varrho_2 = \| \varrho_1 \gamma_2 \| = \| \sum_{j} \varrho_j \gamma_j \| = p(\varrho) \forall \varrho \in \mathbb{R}_+^n \), there exists \( \varrho \in \mathbb{R}_+^n \setminus C \) with \( \frac{\partial p}{\partial \varrho_2} 
rightarrow 0 \).

Therefore \( \varrho_2 \neq 0 \), moreover \( \varrho_2 \varrho_1 \varrho_2 \varrho_1 < 0 \), i.e. \( \varrho_2 \varrho_1 \varrho_2 \varrho_1 = -|\varrho_2|/|\varrho_1| \).

For \( (\varrho_3, \ldots, \varrho_n) \in \mathbb{R}_+^{n-2} \), define \( \varphi : \mathbb{R} \to \mathbb{R} \) by \( \varphi_{\varrho_3, \ldots, \varrho_n}(t) \equiv p(|\varrho_2| \cos t, |\varrho_1| \sin t, \varrho_3, \ldots, \varrho_n) \). Since \( C \) is a cone of measure 0 in \( \mathbb{R}_+^n \), (10) implies

\[(11) \quad \varphi_{\varrho_3, \ldots, \varrho_n}(t) = 0 \quad \text{for almost every} \quad t \in (0, \pi/2) \quad \text{and} \quad (\varrho_3, \ldots, \varrho_n) \in \mathbb{R}_+^{n-2}.
\]

From the convexity of \( p \) it follows that it is locally Lipschitzian in the interior of \( \mathbb{R}_+^n \). Hence, by (11),

\[(11') \quad \varphi_{\varrho_3, \ldots, \varrho_n}(t) = \varphi_{\varrho_3, \ldots, \varrho_n}(0) \quad \forall t \in [0, \pi/2], (\varrho_3, \ldots, \varrho_n) \in \mathbb{R}_+^{n-2}.
\]
But then $|\alpha_{12}| = \varphi_0, \ldots, \varphi_{n/(2)} = |\alpha_{21}|$ whence

$$p|\alpha_{12}|^{-1}(q_1^2 + q_2^2)^{1/2} \cdot \varphi_{q_3}, \ldots, \varphi_{n} \left( \arccos \frac{q_1}{(q_1 + q_2)^{1/2}} \right) = |\alpha_{21}|^{-1}(q_1^2 + q_2^2)^{1/2} \varphi_{q_3}, \ldots, \varphi_{n}(0) =$$

$$= p(\sqrt{q_1^2 + q_2^2}, 0, q_3, \ldots, q_n).$$

Let now $f, g \in E$ be functions such that $f-g \in 1_{(y_1, y_2)} E$ and $\sum f(y_j)^2 = \sum g(y_j)^2$. Then $\|1_Y f\| = p \left( \left( \sum f(y_j) \right)^{1/2}, 0, |f(y_3)|, \ldots, |f(y_n)| \right) = \|1_Y g\|$. Taking into consideration the fact that $Y$ may be any finite subset of $X$, from (8') we obtain $\|f\| = \|g\|$. 

Conversely: Assume that $f-g \in 1_{(y_1, y_2)} E$ and $\sum f(y_j)^2 = \sum g(y_j)^2$ imply $\|f\| = \|g\|$ for all $f, g \in E$. Then the mappings $U^t = [f \mapsto 1_Y f(y_3) f + (\cos t) \cdot f(y_3) + + (\sin t) \cdot f(y_2), ((-\sin t) \cdot f(y_3) + c + (\cos t) \cdot f(y_2))]_{\mathbb{R}}$ ($t \in \mathbb{R}$) form a one-parameter $E$-unitary operator group. Hence the linear field $\frac{d}{dt} \bigg|_{0} U^t = [f \mapsto f(y_3) f - f(y_2)]_{\mathbb{R}}$ belongs to $\log^* A B$.

Proof of (ii): Say that $f \sim^Y g$ if $f \in E, g \in E, f-g \in 1_Y E$ and $\sum f(y_j)^2 = \sum g(y_j)^2$. Obviously, the relations $\sim^Y$ are all equivalences. Consider the set $\forall \{m: \exists x_1 \sim \ldots \sim x_m \ni f, g \in E f \sim^Y (x_1, \ldots, x_m) g, \|f\| \neq \|g\| \}$. Suppose $N \neq 0$ and set $n = 1 \text{ min } N$. From (i) it follows $n \geq 2$. Fix a set $Y = \{y_1, \ldots, y_n\}$ and functions $f_1, f_2 \in E$ such that $f_i \sim^Y f_2, y_1 \sim \ldots \sim y_n$, but $\|f_i\| \neq \|f_2\|$. Consider the functions $g_j = 1_{(y_1, \ldots, y_n)} f_j + + \left( \sum_{x=3}^n f_j(y_x) \right)^{1/2} f_{y_2}$ ($j = 1, 2$). Observe that $f_j \sim^{(y_1, \ldots, y_n)} g_j$ whence $\|f_j\| = \|g_j\|$ ($j = 1, 2$). However, $g_1 \sim^{(y_1, y_2)} g$ and therefore by (i) we have $\|g_1\| = \|g\|$ contradicting the assumption $\|f_1\| \neq \|f_2\|$. Thus $N = 0$. Hence if $y_1 \sim y_2 \sim y_3$ then $f, g \in E f \sim^{(y_1, y_2, y_3)} g \mapsto f \sim^{(y_1, y_2, y_3)}$ i.e. by (i), $y_1 \sim y_3$ holds.

3.7. Corollary. The proof of (i) shows that $\langle f, 1_{y_2}, \rangle_{\ell^2}^* = -\langle f, 1_{y_2}, \rangle_{\ell^2}$ whenever $y_1, y_2 \in X$ and $\ell$ linear $\log^* A B$.

3.8. Definition. From now on we reserve the notation $\{S_i: i \in \mathcal{F}\}$ to denote the partition of $X$ formed by the equivalence classes of the relation $\sim$. For each $i \in \mathcal{F}$, we shall denote the projection band $1_{S_i} E$ of $E$ by $H_i$.

3.9. Proposition. (i) If $f, g \in E$ are functions with finite support and $\|f_i\|_{\ell^2} = \|g_j\|_{\ell^2} = \left( \sum_{x \in S_i} |g(x)|^2 \right)^{1/2}$ $\forall i \in \mathcal{F}$ then $\|f\| = \|g\|$. 

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(ii) For any \( i \in \mathcal{S}, \mathcal{H}_i \) is a Hilbert space (i.e., the norm \( \| \cdot \| \) restricted to \( \mathcal{H}_i \) satisfies parallelogram identity). Namely, a function \( h: X \to \mathbb{C} \) belongs to \( \mathcal{H}_i \) iff \( \text{supp}(h) \subset \mathcal{S}_i, \sum_{x \in \mathcal{S}_i} |h(x)|^2 < \infty \), furthermore we have \( \| f \| = \| f \|_{\mathcal{H}_i} \) \( \forall f \in \mathcal{H}_i \).

(iii) If \( f, g \in \mathcal{E} \) and \( \| f \|_{\mathcal{E}} = \| g \|_{\mathcal{E}} \) \( \forall i \in \mathcal{S} \) then \( \| f \| = \| g \| \).

(iv) If \( g: X \to \mathbb{C}, f \in \mathcal{E} \) and \( \| f \|_{\mathcal{E}} = \| g \|_{\mathcal{E}} \) \( \forall i \in \mathcal{S} \) then \( g \in \mathcal{E} \).

(v) Assume \( \varrho \in \mathcal{L}(\mathcal{E}, \mathcal{E}) \). Then \( \varrho \in \log^* \text{Aut} \mathcal{B} \) if and only if there exists a family of linear mappings \( \{ \varrho_j : j \in \mathcal{S} \} \) such that \( \varrho \cdot \varrho_j \) is a self-adjoint \( \mathcal{H}_j \)-operator for each \( j \in \mathcal{S}, \sup_{j \in \mathcal{S}} \| \varrho_j \| < \infty \) and \( \varrho = \bigotimes_{j \in \mathcal{S}} \varrho_j \).

Proof: (i) is a direct consequence of Lemma 3.6 (i).

(ii): Let \( f \in \mathcal{H} \) and \( x_0 \in \mathcal{E} \) be arbitrarily fixed. By (i), \( \| 1_{\mathcal{F}} f \| = \| \left( \sum_{y \in \mathcal{Y}} |f(y)|^2 \right)^{1/2} 1_{x_0} \| = \left( \sum_{y \in \mathcal{Y}} |f(y)|^2 \right)^{1/2} \) for all \( Y \) finite \( \subset X \). Hence by (8'), \( \| f \|_{\mathcal{E}} = \| f \|_{\mathcal{H}} \). Furthermore, if \( g \) is a function \( X \to \mathbb{C} \) having support in \( \mathcal{S}_i \) and \( \| g \|_{\mathcal{H}} < \infty \) then (i) ensures \( Y_1, Y_2 \) finite \( \subset X \), \( 1_{Y_1} f - 1_{Y_2} f \| = \| 1_{Y_1} f - 1_{Y_2} f \|_{\mathcal{E}} = \| 1_{Y_1 \Delta Y_2} f \| \) i.e. the net \( \{ 1_{Y_i} f \}_Y \) is a Cauchy net when \( f \in \mathcal{E} \).

(iii): Let \( \varepsilon > 0 \) be fixed. According to (8'), one can find \( Y \) finite \( \subset X \) with \( \| f - 1_{Z_1} f \|, \| g - 1_{Z_2} g \| < \varepsilon \) \( \forall Z \subset Y \). Since the index set \( J \equiv \{ i \in \mathcal{S} : Y \cap \mathcal{S}_i = \emptyset \} \) is finite, there exists a family of sets \( \{ Z_i : i \in J \} \) such that \( Y \cap \mathcal{S}_i \subset Z_i \) finite \( \subset \mathcal{S}_i \) (\( i \in J \)) and \( \sum_{i \in J} \| 1_{Z_i} f - 1_{Z_i} f \|_{\mathcal{E}} < \varepsilon \). Consider now the functions \( f_i = \sum_{i \in J} 1_{Z_i} f \|_{\mathcal{E}} 1_{x_i} \) and \( g_i \equiv \sum_{i \in J} 1_{Z_i} g \|_{\mathcal{E}} 1_{x_i} \) where \( x_i \) denotes an arbitrarily fixed point of \( \mathcal{S}_i \) (\( i \in J \)). By writing \( Z = \bigcup_{i \in J} Z_i \), we can see \( \| f_i \| = \| 1_{Z_1} f \|_{\mathcal{H}} = \| 1_{Z_1} g \|_{\mathcal{H}} \) and \( \| f - 1_{Z_1} f \|, \| g - 1_{Z_2} g \| < \varepsilon \). Using the triangle inequality, \( \| f_i - g_i \| = \sum_{i \in J} \| 1_{Z_i} f - 1_{Z_i} g \|_{\mathcal{E}} = \sum_{i \in J} \| 1_{Z_i} f - 1_{Z_i} f \|_{\mathcal{E}} + \| 1_{Z_i} g - 1_{Z_i} g \|_{\mathcal{E}} \equiv (\sum_{i \in J} \| 1_{Z_i} f - 1_{Z_i} f \|_{\mathcal{E}} + \| 1_{Z_i} g - 1_{Z_i} g \|_{\mathcal{E}}) < 2\varepsilon \).

(iv): By (8'), to every number \( n \in \mathbb{N} \), we can choose \( Z_n \) finite \( \subset X \) such that \( \| f - 1_{Z_n} f \| < \frac{1}{n} \). We may assume without loss of generality \( Z_1 \subset Z_2 \subset \ldots \). Then set \( \mathcal{S}_n \equiv \{ i \in \mathcal{S} : Z_n \cap \mathcal{S}_i = \emptyset \} \), \( g_n \equiv \sum_{i \in \mathcal{S}_n} 1_{S_i} g \). By (ii) and the finiteness of the sets \( \mathcal{S}_n, g_n \in \mathcal{E} \) \( \forall n \in \mathbb{N} \). If \( n = m \) then \( \| g_n - g_m \| = \| \sum_{i \in \mathcal{S}_n} 1_{S_i} g \| = (\text{by (iii)}) = \| \sum_{i \in \mathcal{S}_n} 1_{S_i} f \| \leq (\text{since} \| \sum_{i \in \mathcal{S}_n} 1_{S_i} f - f \| = \| f - 1_{Z_n} f \| < \frac{1}{m} \). Thus \( \{ g_n \}_n \) is a Cauchy sequence in \( \mathcal{E} \). For all \( x \in X \), \( \lim_{n \to \infty} g_n(x) = g(x) \) whence \( g = \lim_{n \to \infty} g_n \).

(v) First let \( \varrho \in \log^* \text{Aut} \mathcal{B} \). If \( j, k \in \mathcal{S}, j \neq k, x \in \mathcal{S}_j, y \in \mathcal{S}_k \) then by the definition of the classes \( \mathcal{S}_i \) and by Lemma 3.6 (i), \( \langle \varrho(x), 1_j \rangle = 0 \). This fact shows \( \varrho(\mathcal{H}_j) \subset \mathcal{H}_j \).
\[ \forall \xi \in \mathcal{F}. \text{ Thus by setting } \ell_j = \ell |_{H_j} \text{ we obviously have } \| \ell_j \| \equiv \| \ell \| \text{ and } \ell = \bigoplus_j \ell_j. \text{ Furthermore, } [11, \text{ Lemma}] \text{ establishes } \ell_j \in i \cdot \{ \text{self-adj. } H_j\text{-op.-s} \} \forall j \in \mathcal{F}. \]

The converse statement is immediate from (ii) since then we have \( \exp(\ell) = \bigoplus_j \exp(\ell_j) \) and, by assumption, all the operators \( \exp(\ell_j) \) are \( H_j \)-unitary here.

3.10. Corollary. For some subset \( \mathcal{F}_0 \subset \mathcal{F} \), by writing \( X_0 = \bigcup_{i \in \mathcal{F}_0} S_i \), we have \( E_0 = \{ x \in E \text{ where } E_0 = C \cdot [\text{Aut } B] \{ 0 \} \} \) cf. Introduction.

Proof. Set \( Z = \{ x \in X : \exists c \in E_0 \text{ such that } c(x) \neq 0 \} \). Clearly \( E_0 \subset 1_z E \). On the other hand, if \( x \in Z, c \in E_0 \) and \( c(x) \neq 0 \) then, by (v), the linear field \( \ell = [f \mapsto i \cdot f(x) 1_x] \) satisfies \( 1_x \cdot c(x) \exp(\ell t) \in E_0 \forall t \in \mathbb{R} \) whence \( E_0 \supset \text{Span } \{ 1_x : x \in Z \} = 1_z E \). i.e. \( E_0 = 1_z E \). Suppose now \( x \in Z, c \in E_0 \), \( c(x) \neq 0 \) and \( x \in S_i \). Let \( y \in S_i \setminus \{ x \} \) and \( \ell_1 = [f \mapsto if(x) 1_y + if(y) 1_x] \). As in the previous case, \( c_1 = \ell_1(c) = \frac{d}{dt} \exp(\ell t)c \in E_0 \) since by (v), \( \ell_1 \in \text{Log}^* \text{Aut } B \). However, \( c_1(y) = ic(x) \neq 0 \) i.e. \( y \in S_i \). Thus \( S_i \subset Z \).

Next we turn our attention to the quadratic part of \( \text{Log}^* \text{Aut } B \). In the sequel we shall use the notations \( \mathcal{F}_0, X_0 \) introduced in Corollary 3.10. Recall that for any \( c \in E_0 \), there is a unique symmetric bilinear form \( q_c : E \times E \to E \) with \( [f \mapsto c + q_c(f, f)] \in \text{Log}^* \text{Aut } B \) and that the mapping \( c \mapsto q_c \) is conjugate-linear and continuous. Since the finitely supported functions are dense in \( E \), to get the complete description of \( \text{Log}^* \text{Aut } B \) it is enough to determine only the values \( \langle q_{\zeta_1, \ldots, \zeta_n}, 1 \rangle \) \( (\zeta_1 \in X_0, x_2, \ldots, x_n) \subset X \). To this task, the projection principle provides an essential reduction.

3.11. Lemma. Let \( x_1, \ldots, x_n \in X, x_1 \in X_0 \) and \( \beta_{jk}^{*} = \langle q_{1_{\zeta_1}, \ldots, \beta_{jk}^{*}} 1_x \rangle \). Then

(i) \( \beta_{jk}^{*} = 0 \) if \( \{ 1, \ell \} \neq \{ j, k \} \),

(ii) \( \beta_{jk}^{*} = -1 \),

(iii) \( \beta_{jk}^{*} \in [-1, 0] \) and \( \beta_{jk}^{*} B = \{ \zeta_1, \ldots, \zeta_n \} \) \( \beta_{jk}^{*} > -1 \) if \( \beta_{jk}^{*} \geq 0 \) or \( \beta_{jk}^{*} = 0 \)

(iv) \( \beta_{jk}^{*} = -1/2 \) if \( x_1 \sim x_2 \in X_0 \) and \( \beta_{jk}^{*} = 0 \) if \( x_1 \sim x_2 \in X_0 \).

Proof. (i) Consider the band projection \( P : f \mapsto 1_{\{ x_1, \ldots, x_n \}} f \). By the projection principle, \( [f \mapsto 1 + P q_{1_{\zeta_1}}(f, f)] \in \text{Log}^* \text{Aut } PB \). Applying [11, Lemma] to \( PB \), we obtain

\[ 0 = \| f \|^2 \langle \phi, \phi \rangle + \langle P q_{1_{\zeta_1}}(f, f), \phi \rangle \leq \| f \| \cdot \| \phi \| = \langle f, \phi \rangle \quad \forall f \in PE, \phi \in (PE)^*. \]
Introducing the same function \( p: \mathbb{R}^n_+ \rightarrow \mathbb{R}_+ \) and set \( C \subset \mathbb{R}^n_+ \) as in the proof of Lemma 3.6,

\[
0 = p(q_1, \ldots, q_n) \sum_{j=1}^{n} \frac{\partial p}{\partial q_j} e^{-i\beta_j q_j^*} + \left< q_1 \left( \sum_{j=1}^{n} q_j e^{i\beta_j q_j^*}, \sum_{k=1}^{n} q_k e^{i\gamma_k q_k^*}, \sum_{\ell=1}^{n} \frac{\partial p}{\partial q_\ell} e^{-i\beta_\ell q_\ell^*} \right) \right> 
\]

for all \( q \in \mathbb{R}^n_+ \setminus C \) and \( \varnothing \in \mathbb{R}^n \). Thus

\[
(12') \quad p^2 \frac{\partial p}{\partial q_1} e^{i\beta_1} + \left( \sum_{j,k,\ell=1}^{n} \beta_{jk}^* q_j q_k \frac{\partial p}{\partial q_\ell} e^{i(\beta_j + \gamma_k - \beta_\ell)} \right) = 0 \quad (q \in C, \varnothing \in \mathbb{R}^n).
\]

Therefore (for fixed \( q \in \mathbb{R}^n_+ \setminus C \)) the rational expression \( p^2 \frac{\partial p}{\partial q_1} z_1 + \left( \sum_{j,k,\ell=1}^{n} \beta_{jk}^* q_j q_k \frac{\partial p}{\partial q_\ell} e^{i(\beta_j + \gamma_k - \beta_\ell)} \right) \) vanishes on \( \partial_0 A^n \) i.e. its homogeneous parts are 0-s. Hence only the coefficients of the form \( \beta_{1k}^* = \beta_{kk}^* \) may differ from 0.

(ii) is immediate from (12') if we take \( n=1 \) because then \( p(q_1) = q_1 \).

For the proof of (iii) and (iv), consider the case \( n=2 \). From (12') and (ii) we then see

\[
(12'') \quad (p^2 - q_1^2) \frac{\partial p}{\partial q_1} + 2q_1 q_2 \frac{\partial p}{\partial q_2} \beta_{12}^2 = 0 \quad (q \in \mathbb{R}^n_+ \setminus C).
\]

Since \( p(0, q) = p(q, 0) \) and since the function \( p \) is increasing and convex, \( \forall q \in \mathbb{R}_+ \exists ! t \equiv 0 \) \( p(q, t) = 1 \). Thus the function \( t: [0, 1) \rightarrow \mathbb{R}_+ \) is welldefined by \( p(q, t(q)) = 1 \). Observe that now \( t \) is a decreasing concave function and \( t(0) = 0 \). By the implicit function theorem, \( t'(q_1) = -\frac{\partial p/\partial q_1}{\partial p/\partial q_2} \) whenever \( (q_1, t(q_1)) \in C \). Thus, since \( C \) is a cone with measure 0 in \( \mathbb{R}^n_+ \), (12'') implies

\[
(12''') \quad t'(q)(1-q^2) = 2q t(q) \beta_{12}^2 \quad \text{for almost every } q \in (0, 1).
\]

Since \( t' \leq 0 \), we have \( \beta_{12}^2 \equiv 0 \). If \( \beta_{12}^2 = 0 \) then \( t(q) = t(0) = 1 \) \( \forall q \in (0, 1) \). In this case, \( p(q_1, q_2) \equiv 1 \) if \( q_1 < 1 \) and \( q_2 \equiv t(q_1) = 1 \) or \( q_1 = 1 \) and \( q_2 \equiv 1 \), i.e. \( p(q_1, q_2) = \max(q_1, q_2) \). If \( \beta_{12}^2 < 0 \) then the solution of (12''') with initial value \( t(0) = 1 \) is \( t(q) = (1-q^2)^{-\beta_{12}^2} \). Thus by setting \( K = \{(q_1, q_2) : p(q_1, q_2) \equiv 1\} \),

\[
K = \{(q_1, q_2) : q_1^2 + q_2^{-2} - \beta_{12}^2 = 1\}.
\]

The convexity of the function \( p \) entails that \( K \) is convex whence \( \beta_{12}^2 \equiv -1 \) yielding (iii).
(iv): If \( x_1 \sim x_2 \neq x_1 \) then \( p(\varrho_1, \varrho_2) = (\varrho_1^2 + \varrho_2^2)^{1/2} \) (cf. Proposition 3.9 (ii)), that is, by (13), we have \( \beta_{12}^2 = -\frac{1}{2} \).

On the other hand, suppose \( x_1 \not\sim x_2 \in X_0 \) and \( \beta_{12}^2 \neq 0 \). Since \( x_4 \in X_0 \), all the previous considerations can be carried out by interchanging \( x_1 \) and \( x_2 \). Thus by (iii),

\[
1_{(x_1, x_2)} B = \{ \varsigma_1 \lambda_{x_1} + \varsigma_2 \lambda_{x_2} : |\varsigma_1|^2 + |\varsigma_2|^2 - 3\langle q_{x_1}, (l_{x_1}, l_{x_2})^* l_{x_2} \rangle \equiv 1 \} = \{ \varsigma_1 \lambda_{x_1} + \varsigma_2 \lambda_{x_2} : |\varsigma_1|^2 + |\varsigma_2|^2 - 3\langle q_{x_1}, (l_{x_2}, l_{x_1})^* l_{x_1} \rangle \equiv 1 \}.
\]

This is possible only if \( \beta_{12}^2 = -\frac{1}{2} \langle \xi_{x_1}, (l_{x_1}, 1_{x_2})^* 1_{x_2} \rangle \) thus \( p(\varrho_1, \varrho_2) = (\varrho_1^2 + \varrho_2^2)^{-1/2} \).

If \( S \) denotes the equivalence class (w.r.t. \( \sim \)) of \( x_1 \) then by Proposition 3.9 (iii),

\[
\| f + 1_{x_1} \| = \| f \|_{\mathbb{R}^*} 1_{x_1} + \varrho_1 1_{x_1} = p(\| f \|_{\mathbb{R}^*}, \varrho) = \| f + \varrho 1_{x_1} \|_{\mathbb{R}^*}
\]

for arbitrary \( f \in H_1 \) whence it follows \( x_2 \in S \). The obtained contradiction proves (iv).

(v): Let \( y_1, \ldots, y_n \in X_0 \) be pairwise non-\( \sim \)-equivalent. Now for arbitrarily fixed \( f, c \in 1_{(y_1, \ldots, y_n)^* E} \),

\[
q_c(f, f) = \sum_{m=1}^{n} c(y_m) q_{y_m} (f, f) = \sum_{m=1}^{n} c(y_m) \sum_{j, j', i=1}^{n} f(y_j) f(y_k) \langle q_{y_m}, (l_{y_j}, l_{y_k})^* l_{y_k} \rangle.
\]

Applying (i) and (iii) to \( x_1 \equiv y_m, x_k \equiv y_k \) and \( x_j \equiv y_j \), hence we obtain

\[
q_c(f, f) = -\sum_{m=1}^{n} c(y_m) f(y_m)^* l_{y_m} = -c \cdot f^2.
\]

Therefore the solution of the initial value problem \( \frac{d}{dt} f_t = c - q_c(f, f), f_0 = 0 \) is \( f_t = \tanh (tc) \). Hence \( \{ \sum_{m=1}^{n} q_m 1_{y_m} : q_1, \ldots, q_n \in [0, 1] \} \subset \operatorname{Aut} [F \rightarrow c + q_c(f, f)](0) \):

\[
c \in 1_{(y_1, \ldots, y_n)^* E} \subset \{ A \} \subset B.
\]

Then \( \max_{m=1}^{n} q_m \equiv \left\| \sum_{m=1}^{n} q_m 1_{y_m} \right\| \equiv 1 \) whenever \( q_1, \ldots, q_n \in [0, 1] \). Consequently \( \left\| \sum_{m=1}^{n} q_m 1_{y_m} \right\| = 1 \) whenever \( \max_{m=1}^{n} |q_m| = 1 \) whence

\[
\left\| \sum_{j=1}^{n} q_j 1_{y_j} \right\| = \max_{m=1}^{n} |q_m|.
\]

The proof is complete.

FromLemma 3.11 (i) and the symmetry of the bilinear mappings \( q_c \) follows directly that introducing the functions

\[
w_{x_1}(x_2) = \begin{cases} -1/2 & \text{if } x_1 = x_2, \\ \langle q_{x_1}(l_{x_1}, l_{x_2})^* l_{x_2} \rangle & \text{if } x_1 \neq x_2. \end{cases}
\]

\( (x_1 \in X_0, x_2 \in X) \),
we have
\[ q_{1_x}(l_x, l_y) = 2w_x(x)l_x \text{ for all } x \in X_0, \]
\[ q_{1_x}(l_x, l_y) = w_x(y)l_y \text{ if } x \in X_0, y \in X \setminus \{x\}, \]
\[ q_{1_x}(l_y, l_x) = 0 \text{ if } x \notin \{y, z\}, x \in X_0. \]

Hence
\[ (14) \quad q_{1_x}(f, g) = f(x)w_xg + g(x)w_xf \quad (x \in X_0) \]
whenever the function \( f \in E \) is finitely supported. Moreover by \((8')\) and Lemma 3.11 (iii), \((14)\) holds for every \( f \in E \).

For sake of brevity, in what follows we shall write \( f^{(0)} \) instead of the function \( 1_{S_t}f \).

3.12. Lemma. (i) \( w_x^{(0)} = -\frac{1}{2} 1_{S_t} \) whenever \( x \in S_t \) \((i \in \mathcal{S}_t)\),

(ii) \( w_x^{(0)} = 0 \) whenever \( x \notin S_t \) \((i \in \mathcal{S}_t)\),

(iii) There exists a unique matrix \( (\gamma_{ij})_{i \in \mathcal{S}_0, j \in \mathcal{S} \setminus \mathcal{S}_0} \) consisting of numbers belonging to [0, 1] such that \( w_x^{(0)} = -\gamma_{ij} 1_{S_j} \) whenever \( x \in S_i \subset X_0 \) and \( j \in \mathcal{S} \setminus \mathcal{S}_0 \).

Proof. (i) and (ii) are contained in Lemma 3.11 (iv).

(iii): Let \( x, x' \in S_t \) and \( y, y' \in S_j \) where \( i \in \mathcal{S}_0, j \notin \mathcal{S}_0 \). From Proposition 3.9 (v) it follows the existence of an \( E \)-unitary operator \( U \) such that \( 1_{x'} = U1_x \) and \( 1_{y'} = U1_y \). From the elementary theory of Lie-groups it is well-known that \( UvU^{-1} \in \log^* \text{Aut } B \) for every \( v \in \log^* \text{Aut } B \). In particular, \( [f \mapsto U(1_x + q_{1_x}(U^{-1}f, U^{-1}f))] \in \log^* \text{Aut } B \) whence \( q_{1_x}(f, f) = q_{U1_x}(f, f) = q_{1_x}(U^{-1}f, U^{-1}f) \). Therefore \( \langle q_{1_x}(1_{x'}, 1_{y'}), 1_{y'}^{(0)} \rangle = \langle Uq_{1_x}(U^{-1}1_{x'}, U^{-1}1_{y'}), 1_{y'}^{(0)} \rangle = \langle Uq_{1_x}(1_{x'}, 1_{y'}), 1_{y'}^{(0)} \rangle = \langle q_{1_x}(1_{x'}, 1_{y'}), 1_{y'}^{(0)} \rangle \) since if \( U = \bigoplus_{i \in \mathcal{S}_i} U_i \) is the directe decomposition of \( U \) provided by Proposition 3.9 (v) and \( f \in E \) then \( \langle Uf, 1_{x'}^{(0)} \rangle = (Uf|U^{-1}1_{x'}) = (f|U^{-1}1_{x'}) = (f|1_{x'}) \).

Henceforth we reserve the notation \( (\gamma_{ij})_{i \in \mathcal{S}_0, j \in \mathcal{S} \setminus \mathcal{S}_0} \) for the matrix introduced in Lemma 3.12 (iii).

3.13. Corollary. For arbitrary finitely supported \( c \in E_0 \) and \( f \in E \),
\[ (15) \quad q_c(f, f) = -\sum_{i \in \mathcal{S}_0} (f^{(0)}|c^{(0)})f^{(0)} - 2\sum_{j \in \mathcal{S} \setminus \mathcal{S}_0} [\sum_{i \in \mathcal{S}_0} \gamma_{ij}(f^{(0)}|c^{(0)})] f^{(0)}. \]

Proof. Applying Lemma 3.12. and \((14)\), we can see that if \( c \in E_0 \) and \( f \in E \) have finite supports then \( q_c(f, f) = -\sum_{x \in X_0} c(x)q_{1_x}(f, f) \sum_{i \in \mathcal{S}_0} \sum_{x \in \mathcal{S}_i} 2c(x)f(x) \cdot \]
\[ \left[ -\frac{1}{2} f^{(0)} - \sum_{j \in \mathcal{S}_0} \gamma_{ij} f^{(0)} \right]. \]

In order to extend \((15)\) to every \( c \in E_0 \) and \( f \in E \), we need the following observations.
3.14. Lemma. (i) $E_0 = \bigoplus_{i \in \mathcal{J}_0} H_i$ i.e. a function $c : X \to C$ belongs to $E_0$ if and only if $\forall i \in \mathcal{J}_0 \, \|c^0\|_{\ell_2} < \infty$ and $\forall \varepsilon > 0 \{ i \in \mathcal{J}_0 : \|c^0\|_{\ell_2} \geq \varepsilon \}$ finite $\subset \mathcal{J}_0$ (in the latter case $\|c\| = \sup_{i \in \mathcal{J}_0} \|c^0\|_{\ell_2}$).

(ii) $\sup_{i \in \mathcal{J}_0} \sum_{j \in \mathcal{J}_0} \gamma_{ij} \equiv 4 \|q\| (\equiv 4 \sup_{c \in B \cap \mathcal{F}_0} \|q\| = 4 \sup_{c \in B \cap \mathcal{F}_0, f \in B} \|q_c(f, g)\|)$.

Proof. (i): Trivial from Proposition 3.9 (v), Lemma 3.11 (v) and the fact that the finitely supported functions are dense in $E$.

(ii): Let $j \in \mathcal{J}_0 \setminus \mathcal{J}_0$, $h_1, \ldots, h_n \in \mathcal{J}_0, y \in \mathcal{J}_j$ and $x_1 \in \mathcal{S}_1, \ldots, x_n \in \mathcal{S}_n$. Consider the functions $c = \sum_{m=1}^{n} 1_{x_m}$ and $f = 1_y + \sum_{m=1}^{n} 1_{x_m}$. By (i) we have $\|c\| = 1$ and $\|f\| \equiv 2$. By (15),

$\langle q_c(f, f), 1_y^* \rangle = \sum_{m=1}^{n} \gamma_{m,j}$. At the same time, $\|\langle q_c(f, f), 1_y^* \rangle\| \equiv \|q\| \cdot \|c\| \cdot \|f\|^\circ \cdot \|1_y^*\| \equiv \|q\| \cdot 2 \cdot 2 \cdot \|1_y^*\| \equiv 4 \|q\|$.

3.15. Corollary. (15) holds for each $c \in E_0$ and $f \in E$.

Proof. The previous lemma shows that the right hand side of (15) makes always sense. Observe that the mapping $Q : E_0 \times E \triangleright (c, f) \mapsto \{\text{right hand side of (15)}\}$ is real-linear in $c$ and real-quadratic in $f$. For $\|c\|, \|f\| \leq 1$ we have $\|Q(c, f)\| \equiv \|\sum_{i \in \mathcal{J}_0} (f^0)|c^0|f^0\| + 2\|\sum_{j \in \mathcal{J}_0} \sup_{k \in \mathcal{J}_0} \sum_{i \in \mathcal{J}_0} \gamma_{ik} \|f^0\|_{\ell_2} \cdot \|c^0\|_{\ell_2} \cdot f^0\| \equiv \|f\|^2 \cdot \|c\| + \|q\| \cdot \|c\| \cdot \|f\|^2$. Thus $Q$ is a continuous map. On the other hand, the relation $Q(c, f) = +q_c(f, f)$ is already established for a dense submanifold of $E_0 \times E$ by Corollary 3.13.

In this way we completely know $\log^* \text{Aut} \, B$. The mappings $\exp [B \ni f \mapsto \rightarrow c + q_c(f, f)]$ are easy to describe: By (15), the equation $\frac{d}{dt} f_t = c + q_c(f_t, f_t)$ is equivalent with

(16') $\frac{d}{dt} f_t^{(0)} = c^{(0)} - (f_t^{(0)}|c^{(0)}) f_t^{(0)} (i \in \mathcal{J}_0)$

(16'') $\frac{d}{dt} f_t^{(0)} = -2 \sum_{i \in \mathcal{J}_0} \gamma_{ij} (f_t^{(0)}|c^{(0)}) f_t^{(0)} (i \in \mathcal{J}_0 \setminus \mathcal{J}_0)$.

If we represent $c^{(0)}$ in the form $c^{(0)} = \xi_f c_0^{(0)}$ where $\xi_f \equiv 0, \|c_0^{(0)}\| = 1$ and if $f_0^{(0)} = \xi_{f_0} c_0^{(0)} + f_0^{(0)}$ where $f_0^{(0)}$ lying orthogonally to $c_0^{(0)}$, one then checks immediately that for arbitrarily given $f_0 \in B$, the solution of (16') is

(17') $f_t^{(0)} = M_{\xi_f}(\xi_f)^{0} c_0^{(0)} + M_{\xi_f}(\xi_f)^{0} f_0^{(0)} (i \in \mathcal{J}_0)$.
where $M_\tau$ and $M_\tau^\perp$ are the Moebius- and co-Moebius transformations

$$
M_\tau(\zeta) = \frac{\zeta + \tanh(\tau)}{1 + \zeta \tanh(\tau)}, \quad M_\tau^\perp(\zeta) = \frac{1 - (\tanh(\tau))^2}{1 + \zeta \tanh(\tau)} \quad (\tau \in \mathbb{R}, |\zeta| < 1).
$$

Substituting (17') into (16'), we obtain

$$
\frac{d}{dt} f^{(j)}_t = \left[ -2 \sum_{i \in \mathcal{F}_0} g_{ij} q_i M_{q_i t}(\zeta_i) \right] f^{(j)}_t \quad (j \in \mathcal{F} \setminus \mathcal{F}_0)
$$

whose solution is given by

$$
f^{(j)}_t = \exp \left[ -2 \sum_{i \in \mathcal{F}_0} g_{ij} q_i \int_0^1 M_{q_i t}(\zeta_i) \, d\tau \right] f^{(j)}_0 = \left[ \prod_{i \in \mathcal{F}_0} M_{q_i t}^\perp(\zeta_i) \right] f^{(j)}_0 \quad (j \in \mathcal{F} \setminus \mathcal{F}_0).
$$

The fact that the right hand side in (17') makes sense, is guaranteed by Lemma 3.14 (ii). Fortunately, by Lemma 3.14 (i) and (17'),

$$
[\text{Aut } B][0] = B \cap E_0 = \left\{ \sum_{i \in \mathcal{F}_0} \lambda_i c_i : 0 \leq \lambda_i \leq 1, c_i \in \partial B(H_i) \quad i \in \mathcal{F}_0 \right\}
$$

$$
[i \mapsto \lambda_i] \in c_0(\mathcal{F}_0) = \left\{ \sum_{i \in \mathcal{F}_0} M_{q_i}(0) c_i : q_i \in \mathbb{R}_+, c_i \in \partial B(H_i) \quad \forall i \in \mathcal{F}_0 \right\}
$$

$$
[i \mapsto \lambda_i] \in c_0(\mathcal{F}_0) = \{ \exp [f \mapsto c + q_c(f,f)](0) : c \in E_0 \}
$$

where $c_0(\mathcal{F}_0) = \{ \mathcal{F}_0 \to \mathbb{C} \text{ functions vanishing at infinity} \}$. A classical theorem of Cartan asserts that the relations $U \in \text{Aut } B$ and $U(0) = 0$ entail the linearity of $U$. Thus given $F \in \text{Aut } B$, if we choose the vector $c \in E_0$ so that the automorphism $G = \exp [B \mapsto -c + q_{c(-)}(f,f)]$ satisfies $G(0) = F^{-1}(0)$ then the automorphism $U = F \circ G$ is necessarily linear, i.e. we have $F \in \text{U \cdot exp}[f \mapsto c + q_c(f,f)]$ for suitable $c \in E_0$ and linear $E$-unitary $U$. Hence we arrive at the following characterization of $\text{Aut } B$.

3.16. Theorem. Let $E$ denote a minimal atomic Banach lattice. The space $E$ is spanned by a family $\{H_i : i \in \mathcal{F}\}$ of its pairwise lattice-orthogonal Hilbertian projection bands such that

(i) the linear members of $\text{Aut}_0 B(E)$ map $B(H_i)$ onto themselves ($\forall i \in \mathcal{F}$),

(ii) conversely, if for any index $i \in \mathcal{F}$, $U_i$ is an $H_i$-unitary operator then $\bigoplus_{i \in \mathcal{F}} U_i |_{B(E)} \in \text{Aut}_0 B(E)$.

Furthermore there exists a matrix $(\gamma_{ij})_{i,j \in \mathcal{F}}$ and an index subfamily $\mathcal{F}_0 \subset \mathcal{F}$ such that

(iii) $E (= C[\text{Aut } B(E)][0]) = \bigoplus_{i \in \mathcal{F}_0} \gamma_i H_i$,
(iv) $0 \leq \gamma_{ij} \leq 1$ for all $i, j \in \mathfrak{S}$; $\gamma_{ii} = \frac{1}{2}$ for all $i \in \mathfrak{S}$; $\gamma_{ij} = 0$ whenever $i, j \notin \mathfrak{S}$ or $i$ and $j$ are distinct elements of $\mathfrak{S}$.

(v) A mapping $F : B(E) \to E$ belongs to $\text{Aut}_\theta B(E)$ if and only if, by denoting the band projection onto $H_1$ by $P_1$, we have

\begin{align*}
P_i F(f) &= U_i \{ M_{\theta_i}( (P_i f | e_i^0) e_i^0 + M_{\overline{\theta}_i}( (P_i f | e_i^0) (P_i f \overline{\theta}_i | e_i^0) ) e_i^0 ) \} \quad (i \in \mathfrak{S}), \\
P_j F(f) &= \left\{ \exp \int_0^1 \sum_{i \in \mathfrak{S}_j} \gamma_{ij} q_i M_{\theta_i}( (P_i f | e_i^0) ) \, d\tau \right\} U_j P_j f \quad (j \in \mathfrak{S} \setminus \mathfrak{S}_0)
\end{align*}

for suitable $H_j$-unitary operators $U_j (j \in \mathfrak{S})$, unit vectors $e_i^0 \in H_i (i \in \mathfrak{S}_0)$ and a function $[\mathfrak{S}_0 \theta_0 = q_0]$ assuming values in $\mathbb{R}^+$ and vanishing at infinity, respectively (the transformations $M_{\theta_i}, M_{\overline{\theta}_i}$ are those defined in (18)).

---

### 4. Appendix

**Linear finite dimensional tensor unit ball automorphisms**

Throughout this section $H_1, ..., H_n$ are fixed finite dimensional Hilbert spaces. We are aimed to describe the structure of the linear unitary operators in the space $E = H_1 \otimes \ldots \otimes H_n$.

We shall use the notations $B \equiv B(E), B^* \equiv B(E^*)$,

\begin{align*}
K &\equiv \{ F \in \partial B : \exists ! \Phi \in \partial B^* \quad \langle F, \Phi \rangle = 1 \}, \\
K^* &\equiv \{ \Phi \in \partial B^* : \exists F \in K \quad \langle F, \Phi \rangle = 1 \}.
\end{align*}

4.1. **Lemma.** $K^* = \{ \delta_{e_1, \ldots, e_n} : e_1 \in \partial B(H_1), \ldots, e_n \in \partial B(H_n) \}$.

**Proof.** Since $\dim E = \infty$, $B$ is compact, thus for any $n$-linear functional $F \in \partial B$, one can find $e_1 \in \partial B(H_1), \ldots, e_n \in \partial B(H_n)$ with $F(e_1, \ldots, e_n) = 1$. Hence $K^* \subset \{ \delta_{e_1, \ldots, e_n} : e_1 \in \partial B(H_1) \}$. On the other hand, every $E$-unitary operator maps $K$ onto itself and therefore also

\begin{equation}
U^* K^* = K^* 
\end{equation}

for all $E$-unitary operators.

From the compactness of $B$ it follows $K \neq \emptyset$ (indeed: for any smooth norm $\| \cdot \|_1$ on $E$, $\emptyset \neq \{ F \in \partial B : \| F \|_1 \equiv \| G \|_1 \| G \in \partial B \} \subset K \}$ whence $K^* \neq \emptyset$. That is, for some unit vectors $e_1^0 \in H_1, \ldots, e_n^0 \in H_n$ we have $\delta_{e_1^0, \ldots, e_n^0} \in K^*$. Now from (19) we obtain $\delta_{U_1 \otimes \ldots \otimes U_n} = (U_1 \otimes \ldots \otimes U_n)^* \delta_{e_1^0, \ldots, e_n^0} \in K^*$ whenever the $U_j$'s are $H_j$-unitary operators. Thus $\{ \delta_{e_1^0, \ldots, e_n^0} : e_j \in \partial B(H_j) \} \supset K^*$.

4.2. **Lemma.** Let $\Phi \equiv \delta_{f_1, \ldots, f_n}, \Psi \equiv \delta_{g_1, \ldots, g_n}$ and $\Theta \equiv \delta_{h_1, \ldots, h_n}$ where $0 \neq f_j, g_j, h_j \in H_j (j = 1, \ldots, n)$ and assume $\Phi + \Psi = \Theta$. Then there exists $k$ such that for each $j \neq k$ we have $f_j \parallel g_j$ (i.e. $f_j$ and $g_j$ are linearly dependent).
Proof. The statement holds obviously if for some index \( m, f_j \| g_j \) for all \( j \neq m \) or \( f_j \| g_j \) for all \( j \neq m \). In the contrary case \( f_k \| g_k \) and \( f_m \| g_m \) for some pair of indices \( k \neq m \). We may then suppose \( k = 1 \) and \( m = 2 \). First we show that in this case we have \( h_1 \| f_1 \). Indeed: from \( h_1 \| f_1 \) it follows that introducing the tensor \( E = g_1 \otimes g_2 \otimes \ldots \otimes g_n \) where \( g_1 = g_1 - \| f_1 \|^2 (g_1 | f_1) f_1 \) the relations \( \langle E, \Phi \rangle = \langle E, \Theta \rangle = 0 \neq \langle E, \Psi \rangle \) hold. One can see in the same manner that \( h_2 \| g_2 \). Since \( h_1 \| f_1 \), there exists \( u_1 \in H_1 \) with \( f_1 \perp u_1 \perp h_1 \) and since \( h_2 \| g_2 \) one can find \( u_2 \in H_2 \) with \( g_2 \perp u_2 \perp h_2 \). But then the tensor \( T = u_1 \otimes u_2 \otimes h_3 \otimes \ldots \otimes h_n \) satisfies \( \langle T, \Phi \rangle = \langle T, \Psi \rangle = 0 \neq \langle T, \Theta \rangle \) which is impossible.

4.3. Proposition. Let \( r_j = \dim H_j(j = 1, \ldots, n) \) and let \( U \in \mathcal{L}(E, E) \) be fixed so that \( U_{1B} \in \text{Aut}_B \). Then one can choose \( H_j \)-unitary operators \( U_j \) such that \( U = U_1 \otimes \ldots \otimes U_n \).

Proof. It is enough to prove the statement only for \( E \)-unitary operators lying in a suitable neighbourhood of \( id_E \) as it is well-known (see e.g. [6]).

To do this, fix \( \varepsilon > 0 \) such that the functionals \( \Phi = \delta_{e_1} \ldots e_n \), \( \Theta = \delta_{\tilde{e}_1} \ldots \tilde{e}_n \), \( \Psi = \delta_{f_1} \ldots f_n \), \( \tilde{\Phi} = \delta_{\tilde{f}_1} \ldots \tilde{f}_n \) fulfill

\[
\exists k \quad e_k \perp \tilde{e}_k, f_k \perp \tilde{f}_k \quad \text{and} \quad \forall j \neq k \quad e_j \| \tilde{e}_j, f_j \| \tilde{f}_j
\]

whenever we have

\[
\Phi - \tilde{\Phi}, \Psi - \tilde{\Psi} \in K^*, \| \Phi - \tilde{\Phi} \| = \| \Psi - \tilde{\Psi} \| = \sqrt{2} \quad \text{and} \quad \| \Phi - \Psi \|, \| \tilde{\Phi} - \tilde{\Psi} \| < \varepsilon,
\]

\[
\| e_j \| = \| \tilde{e}_j \| = \| f_j \| = \| \tilde{f}_j \| = 1 \quad (j = 1, \ldots, n).
\]

A value \( \varepsilon > 0 \) with the above properties in fact exists: Otherwise there would be a sequence \( \Phi_m = \delta_{e_1^m} \ldots e_n^m \), \( \tilde{\Phi}_m = \delta_{\tilde{e}_1^m} \ldots \tilde{e}_n^m \), \( \Psi_m = \delta_{f_1^m} \ldots f_n^m \), \( \tilde{\Psi}_m = \delta_{\tilde{f}_1^m} \ldots \tilde{f}_n^m \) \((m = 1, 2, \ldots)\) satisfying (21), (22) for \( \varepsilon = - \) but without property (20). For a suitable index subsequence \( \{m_k\}_k \) and for some unit vectors \( e_j, \tilde{e}_j, f_j, \tilde{f}_j \) we have \( e_s^m \rightarrow e_j, \tilde{e}_s^m \rightarrow \tilde{e}_j, f_s^m \rightarrow f_j, \tilde{f}_s^m \rightarrow \tilde{f}_j \) \((s \rightarrow \infty, j = 1, \ldots, n)\). Then the limits \( \Phi, \tilde{\Phi}, \Psi, \tilde{\Psi} \) satisfy \( \Phi = \Psi, \tilde{\Phi} = \tilde{\Psi} \), \( \| \Phi - \tilde{\Phi} \| = \| \Psi - \tilde{\Psi} \| = \sqrt{2} \) and the contrary of (20). At the same time we also have \( \Phi - \tilde{\Phi}, \Psi - \tilde{\Psi} \in K^* \) because of the closedness of \( K^* \). Thus by Lemma 4.2, \( \exists k_0 \forall j \neq k_0 \quad e_j \| \tilde{e}_j \). Since \( \| \Phi - \tilde{\Phi} \| = \sqrt{2} \), hence \( \| e_{k_0} - \tilde{e}_{k_0} \| = \sqrt{2} \) i.e. \( e_{k_0} \perp \tilde{e}_{k_0} \). Similarly \( \exists \ell_0 \quad f_{\ell_0} \perp \tilde{f}_{\ell_0} \) and \( \forall j \neq \ell_0 \quad f_j \| \tilde{f}_j \). Since (20) does not hold, necessarily \( k_0 \neq \ell_0 \). However the relations \( \Phi = \Psi, \tilde{\Phi} = \tilde{\Psi} \) entail \( k_0 = \ell_0 \).

Now assume \( \| U - id_E \| < \varepsilon \). Fix an orthonormal basis \( \{ e_j^k : j = 1, \ldots, r_k \} \) in \( H_k(k = 1, \ldots, n) \), respectively and let us write the functional \( U^* \delta_{e_1^k} \ldots e_k^2 \) in the form \( U^* \delta_{e_1} \ldots e_k^2 \) (cf. Lemma 4.1) where \( f_1^k \) is a fixed unit vector in \( H_k(k = 1, \ldots, n) \). It follows from the choice of \( \varepsilon \) that for arbitrary index \( k \), the singleton \( \{ f_1^k \} \) can be continued to an orthonormal basis \( \{ f_j^k : j = 1, \ldots, r_k \} \) of \( H_k \) in a unique
way so that we have

\[ U^*\delta_{e_1^1, \ldots, e_{n-1}^j, e_n^j} = \delta_{f_1^j, \ldots, f_{n-1}^j, f_n^j} \quad (j = 1, \ldots, r_k). \]

Set \( I_0 = \{ (1, \ldots, 1, j_1, 1, \ldots, 1) : k = 1, \ldots, n; j = 1, \ldots, r_k \}, \)

\( I_1 = \bigotimes_{k=1}^{n} \{ 1, \ldots, r_k \} \)

and let a family \( I \subset I_1 \) of multiindices be called thick if \( \forall i \in I, \forall i' \in I_1 \quad i' \equiv i \Rightarrow i' \in I. \)

Observe that for any multiindex \( i = (i_1, \ldots, i_n) \in I_1 \) there exists a unique complex number which we shall denote by \( \kappa_i \) such that \( |\kappa_i| = 1 \) and

\[ U^*\delta_{e_1^i, \ldots, e_n^i} = \kappa_i \delta_{f_1^i, \ldots, f_n^i}. \]

Indeed: If not, we can find a minimal (w.r.t. \( \equiv \)) \( i \in I_1 \) not satisfying (23). Now \( U^*\delta_{e_1^i, \ldots, e_n^i} = \delta_{h_1, \ldots, h_n} \) for some vectors \( h_k \in \partial B(H_k) (k = 1, \ldots, n) \). Since obviously \( i \not\equiv I_0 \), for arbitrarily fixed \( k \), there is \( k \not\equiv k \) with \( i_k \not\equiv 1 \). Consider the multiindex \( j \)

defined by \( j_k = [i_k] \) if \( \ell \not\equiv k \), 1 if \( \ell = k \) \((\ell = 1, \ldots, n)\). By the minimality of \( i \), \( U^*\delta_{e_1^i, \ldots, e_n^i} = \kappa_j \delta_{f_1^i, \ldots, f_n^i} \). Since \( U^* \left( \frac{1}{\sqrt{2}} \delta_{e_1^i, \ldots, e_n^i} + \frac{1}{\sqrt{2}} \delta_{e_1^i, \ldots, e_n^i} \right) \in K^*, \) using Lemma 4.2

we can see \( h_k \parallel f_k^i \) i.e. \( h_k = \kappa_k f_k^i \) for suitable \( \kappa_k \in \partial B (k = 1, \ldots, n) \).

Then let \( I \) be a maximal thick subset of \( I_1 \) such that \( I_1 \supset I_0 \) and \( \kappa_i = 1 \forall i \in I \).

(Remark: \( \kappa_i = 1 \forall i \in I_0 \).) We shall show that necessarily \( I = I_1 \). Hence and from the linearity of the mapping \( U \), (23) immediately yields the statement of the lemma.

Assume \( I_1 \setminus I \neq \emptyset \). Let \( I \) be a minimal element of \( I_1 \setminus I \). Observation: \( \forall i \in I \)

\( j \neq i \equiv j \Rightarrow i \in I \). I.e. the family \( I' \equiv I \cup \{ j \} \) is thick. Therefore it suffices to prove \( \kappa_j = 1 \)

(which contradicts our assumption). By writing \( J = \{ 1, k_1 \} \times \cdots \times \{ 1, i_n \} \),

\[ U^*\delta_{e_1^i+k_1, \ldots, e_n^i+k_n} = \sum_{j \in J} U^*\delta_{e_1^i, \ldots, e_n^i} = \sum_{j \in J} \kappa_j \delta_{f_1^i, \ldots, f_n^i} = \]

\[ = \kappa_j \delta_{f_1^i, \ldots, f_n^i} + \sum_{j \in J \setminus \{ j \}} \delta_{f_1^i, \ldots, f_n^i} = (\kappa_j - 1) \delta_{f_1^i, \ldots, f_n^i} + \delta_{f_{k_1}^i, \ldots, f_{k_n}^i} + \delta_{f_1^j, \ldots, f_n^j}. \]

However, the function \( U^*\delta_{e_1^i+k_1, \ldots, e_n^i+k_n} \) has the form \( \delta_{h_{k_1}, \ldots, h_n} \) whence directly \( \kappa_j = 1 \).

4.4. Corollary. The vector fields \( V \) being tangent to \( \partial B (E) \) are exactly those of the form

\[ V = i \cdot \sum_{j=1}^{n} \text{id}_{H_1} \otimes \ldots \otimes \text{id}_{H_{j-1}} \otimes A_j \otimes \text{id}_{H_{j+1}} \otimes \ldots \otimes \text{id}_{H_n} \]

where each \( A_j \) is a self-adjoint \( H_j \)-operator.

Proof. For every \( H_j \)-operator \( A_j \) there is a self-adjoint \( A_j \) with \( U_j = \exp (i \cdot A_j) \).

Thus by Proposition 4.3, \( V \) has the form \( V = \frac{d}{dt} \bigg|_0 \exp (it \cdot A_j) \otimes \ldots \otimes \exp (it \cdot A_n) \).
References


