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On curvature measures

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1. Introduction

It is well-known that Steiner's famous polynomial formula for the volume function of convex parallel sets is based on the following heuristical idea:

If A is a convex open subset of $\mathbf{R}^n$ (the Euclidean n-space) whose boundary ∂A is a C^2 submanifold of $(n-1)$-dimensions of $\mathbf{R}^n$ and $\varrho>0$ then for its parallel set (of radius ϱ) $A_\varrho \equiv \{c\in\mathbf{R}^n\colon \mathrm{dist}\,(x, A)<\varrho\}$ we have that $\partial(A_\varrho)$ is also an $(n-1)$-dimensional C^2-submanifold of $\mathbf{R}^n$, and denoting its infinitesimal surface piece by dF one can find the following relation between the $(n-1)$-dimensional Hausdorff measures of dF and its projection on $\bar{A}$ (the closure of A): [1])

$$\mathrm{vol}_{n-1}\, dF = (1+\varrho\varkappa_1)\dots(1+\varrho\varkappa_{n-1})\,\mathrm{vol}_{n-1}\, dF^0 \quad \text{with} \quad dF^0 \equiv pr_{\bar{A}}\, dF$$

where $\varkappa_1, \dots, \varkappa_{n-1}$ denote the values of the main curvatures of ∂A at the place dF^0.

Hence one easily deduces that for all bounded Borel sets $Q\subset\mathbf{R}^n$ the n-dimensional Hausdorff measure (which, by definition, coincides with Lebesgue measure on $\mathbf{R}^n$) of the figures $T(Q, \varrho)\equiv A\cap\{t\in\mathbf{R}^n\colon pr_{\bar{A}}\,t\in Q\}$ is a polynomial of degree n in the variable ϱ, of the form

$$\mathrm{vol}_n\, T(Q, \varrho) = \sum_{j=0}^{n} a_j(Q)\varrho^j \tag{1}$$

where for the coefficients we have

$$a_0(Q) = \mathrm{vol}_n\, Q\cap A, \quad a_1(Q) = \mathrm{vol}_{n-1}\, Q\cap\partial A,$$

and

$$a_j(Q) = \int_{Q\cap\partial A} \frac{1}{j} \sum_{\substack{I\subset\{1,\dots,n-1\}\\ \mathrm{card}\, I=j}} \prod_{i\in I} \varkappa_i(p)\, d(\mathrm{vol}_{n-1}\, p)$$

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[1]) For any closed subset B of $\mathbf{R}^n$ and for $x\in\mathbf{R}^n$ we define $\mathrm{pr}_B\, x\equiv\{b\in B\colon \mathrm{dist}\,(x, b)=\mathrm{dist}\,(x, B)\}$. For $G\subset\mathbf{R}^n$ we define $\mathrm{pr}_B\, G\equiv \bigcup_{x\in G} \mathrm{pr}_B\, x$.

for $j=2, \ldots, n$ (card=cardinality); $\varkappa_1(p), \ldots, \varkappa_{n-1}(p)$ are the main curvatures of ∂A at the point $p \in \partial A$.

This result was considerably generalized by FEDERER [1]: *If a closed set $A \subset \mathbf{R}^n$ is such that*

$$\text{reach}\, A \equiv \sup \{\delta \geqq 0 \colon \forall x \in A_\delta,\ \text{card}\, pr_A x = 1\} > 0 \quad (\text{with } A_0 = A),$$

then there exist (uniquely determined) *signed Borel measures $a_0, \ldots, a_n$ over $\mathbf{R}^n$ such that* (1) *holds for all bounded Borel subsets Q of $\mathbf{R}^n$ and for all ϱ with* $0 < \varrho < \text{reach}\, A$.

Our purpose in the present article is to prove a result analogous to this theorem which applies to every $A \subset \mathbf{R}^n$ and $\varrho > 0$ and which allows us to extend the concept of curvature measure to the boundary of every $A \subset \mathbf{R}^n$ in a reasonable manner.

2. Summary and alternative formulation of some of Federer's arguments

Theorem A. *Let A be a non-empty closed subset of $\mathbf{R}^n$ and f denote the function $x \mapsto \text{dist}\,(x, A)$ on $\mathbf{R}^n \setminus A$. The function f is totally derivable exactly at those points of $\mathbf{R}^n \setminus A$ which admit a unique projection on A, and for such a point x,* $\text{grad}\, f(x)$ *coincides with the unit vector $(x - pr_A x)/\text{dist}\,(x, A)$. The function f satisfies a Lipschitz condition of order one with (exact) Lipschitz constant* 1, *and the set of the singular points $Z \equiv \{x \in \mathbf{R}^n \setminus A \colon \text{card}\, pr_A x > 1\}$ has* vol_n*-measure* 0. *Removing Z from $\mathbf{R}^n \setminus A$, the remaining set $Q \equiv \mathbf{R}^n \setminus (A \cup Z) = \{x \in \mathbf{R}^n \setminus A \colon \text{card}\, pr_A x = 1\}$ can de uniquely decomposed into a family $\mathbf{Q}$ of pairwise disjoint straight line segments so that for any member L of $\mathbf{Q}$ there exists a (unique) point p in ∂A such that $\{p\} = pr_A L = \bar{L} \cap \partial A$.*

Proof. See [2] p. 93, [3] pp. 271 and 216.

Definition. We shall call the members of the family $\mathbf{Q}$ described in Theorem A the *prenormals* of the set A. I.e. $L (\subset \mathbf{R}^n)$ is a prenormal of A if there exist a point $p \in \partial A$ and a unit vector $k (\in \mathbf{R}^n)$ such that $L = \{x \in \mathbf{R}^n \setminus A \colon pr_A x = \{p\}$ and $(x-p)/\|x-p\| = k\}$.

Definition. A mapping f will be called C^{1+}*-smooth* if it is defined on some open subset Ω of some space $\mathbf{R}^s$ with $f \in C^1(\Omega)$ (i.e. if f has a continuous gradient on Ω) and its gradient locally satisfies a Lipschitz condition (i.e. for all compact subsets K of Ω, $\text{Lip}\,(\text{grad}\, f|_K) < \infty$).

Since the composition of C^{1+}-mappings is also a C^{1+}-mapping, it makes sense to speak of $k (\leqq n)$-dimensional C^{1+}-submanifolds of the space $\mathbf{R}^n$. In particular,

F is an $(n-1)$-dimensional C^{1+}-submanifold of $\mathbf{R}^n$ if, for any $y \in F$, one can find an open neighborhood G of the point y so that for some C^{1+}-smooth function $f: G \to \mathbf{R}$ with nonvanishing gradient and a suitable constant γ we have $G \cap F = \{x: f(x) = \gamma\}$.

Theorem B. *If $A \subset \mathbf{R}^n$ is a closed set with $\partial A \neq \emptyset$ such that $\varrho_0 \equiv \text{reach}\, A > 0$ then the function $f(.) \equiv \text{dist}(., A)$ is C^{1+}-smooth on the domain $G \equiv \equiv \{x \in \mathbf{R}^n: 0 < \text{dist}(x, A) < \varrho_0\}$. The figures $\partial(A_\varrho) = \{x: \text{dist}(x, A) = \varrho\}$ $(0 < \varrho < \varrho_0)$ are $(n-1)$-dimensional C^{1+}-submanifolds of $\mathbf{R}^n$. By setting $B \equiv A_{\varrho_1}$, we have* $\text{reach}\, B \geqq \varrho_1$ *and $\partial(A_\varrho) = \partial(B_{\varrho - \varrho_1})$ whenever $0 < \varrho < \varrho_1 \leqq \varrho_0$, that is, also introducing parallle sets of negative radius*[2]) *we have $\partial(A_\varrho) = \partial[(A_{\varrho_1})_{\varrho - \varrho_1}]$ for all $0 < \varrho < \infty$. The main curvatures $\varkappa_1(p), \ldots, \varkappa_{n-1}(p)$ of the hypersurface ($(n-1)$-dimensional C^{1+}-submanifold) $M \equiv \partial(A_{\varrho_1})$ of $\mathbf{R}^n$ oriented by its normal* $\text{grad} f$ *exist at* vol_{n-1}*-almost every point $p \in M$ and their elementary symmetrical polynomials, i.e. the functions $\varkappa_1(.) + \ldots + \varkappa_{n-1}(.), \ldots, \varkappa_1(.) \ldots \varkappa_{n-1}(.)$, are* vol_{n-1}*-measurable. Further, we have $-1/(\varrho_0 - \varrho_1) \leqq \varkappa_i \leqq 1/\varrho_1$ $(i = 1, \ldots, n-1)$. If T is any subset of $\mathbf{R}^n$ formed by the union of some prenormals of the set A such that $T \cap A_{\varrho_0}$ is* vol_n*-measurable then, for* $0 < \varrho < \text{reach}\, A$,

(2) $$\text{vol}_{n-1} T \cap \partial A_\varrho = \int_{T \cap M} [1 + (\varrho - \varrho_1)\varkappa_1] \ldots [1 + (\varrho - \varrho_1)\varkappa_{n-1}]\, d\, \text{vol}_{n-1}$$

and

(2') $$\text{vol}_n T \cap A_\varrho = \int_0^\varrho \int_{T \cap M} [1 + (\tau - \varrho_1)\varkappa_1] \ldots [1 + (\tau - \varrho_1)\varkappa_{n-1}]\, d\, \text{vol}_{n-1}\, d\tau.$$

Proof. See sections "Sets with positive reach" and "Curvature measures" in [1].

We remark that the connection between (2) and (2') is established by the following more general observation:

Lemma 1. *If $\emptyset \neq A \subset \mathbf{R}^n$ and T is a* vol_n*-measurable subset of $\mathbf{R}^n \setminus \bar{A}$ then*

(3) $$\text{vol}_n T = \int_0^\infty (\text{vol}_{n-1} T \cap \partial(A_\varrho))\, d\varrho.$$

Proof. See e.g. [3] p. 271.

[2]) For $\delta < 0$ and $A \subset \mathbf{R}^n$, $A_\delta \equiv \{x \in \mathbf{R}^n: \text{dist}(x, \mathbf{R}^n \setminus A) > -\delta\}$.

3. A separability argument

Definition. We shall call a subset $S \neq \emptyset$ of the product space $\mathbf{R}^n \times \mathbf{R}^n$ a *generalized oriented surface* (GOS) if for all $(y, k) \in S$ we have $\|k\| = 1$ and one can find an $\varepsilon > 0$ (depending on (y, k)) so that

$$\operatorname{dist}(y, y+\varrho k) = \varrho \geqq \operatorname{dist}(y', y+\varrho k) \quad \text{for any} \quad (y', k') \in S \quad \text{and} \quad 0 \leqq \varrho \leqq \varepsilon.$$

If A is a non-empty proper subset of $\mathbf{R}^n$ then let d^+A denote the figure in $\mathbf{R}^n \times \mathbf{R}^n$ defined by

$$d^+A \equiv \{(y, k)\colon y \in \partial A,\ \|k\| = 1 \text{ and } \exists L \text{ prenormal of } A\ L \supset y + (0, \operatorname{length} L) \cdot k\}.$$

It is clear from Theorem A that all the sets d^+A are GOS-s.

Lemma 2. *Suppose that A is a subset of non-empty compact boundary in $\mathbf{R}^n$ with $\varrho_0 \equiv \operatorname{reach} A > 0$. Then*

a) *the figure d^+A is compact (with respect to the topology of $\mathbf{R}^n \times \mathbf{R}^n$)*

b) *the mapping $\Phi\colon (d^+A) \times (0, \varrho_0) \to \mathbf{R}^n$, $\Phi((y, k), \varrho) \equiv y + \varrho \cdot k$ is a homeomorphism between the sets $(d^+A) \times (0, \varrho_0)$ and $A_{\varrho_0} \setminus \bar{A}$, and $\Phi(d^+A \times \{\varrho\}) = \partial A_\varrho$ whenever $0 < \varrho < \varrho_0$.*

Proof. a) The GOF d^+A is bounded in $\mathbf{R}^n \times \mathbf{R}^n$ because it is contained in the product of the compact figures ∂A and $\partial \mathbf{B}^n = \{k \in \mathbf{R}^n \colon \|k\| = 1\}$. On the other hand, it is also closed since in case of any sequence $\{(y_i, k_i)\colon i \in I\} \subset d^+A$ with $(y_i, k_i) \to (y, k)$ we necessarily have $y \in \partial A$ and $\|k\| = 1$, and for $0 < \varrho < \varrho_0$ the equalities $\varrho = \operatorname{dist}(y_i + \varrho \cdot k_i, y_i) = \operatorname{dist}(y_i + \varrho \cdot k_i, \partial A) = \operatorname{dist}(y_i + \varrho \cdot k_i, A)$ imply (by continuity of the function $\operatorname{dist}(\,.\,, A)$) $\varrho = \operatorname{dist}(y + \varrho \cdot k, y) = \operatorname{dist} y + \varrho \cdot k, A)$ i.e, $y \in pr_A(y + \varrho k)$. This shows that $\{y\} = pr_A(y + \varrho k)$ (since $\varrho < \operatorname{reach} A$). Therefore, by taking $L \equiv \{y + \varrho \cdot k\colon 0 < \varrho < \infty$ and $\{y\} = pr_A(y + \varrho k)\}$, we obtain from Theorem A that L is a prenormal of A and $L = y + (0, \operatorname{length} L) k$ i.e. $(y, k) \in d^+A$.

b) By Theorem A and the definition of d^+A, the condition $\operatorname{reach} A = \varrho_0 > 0$ means that the mapping Φ is one-to-one. By fixing an arbitrary pair ϱ_1, ϱ_2 such that $0 < \varrho_1 < \varrho_2 < \varrho_0$, we see that the figure $D(\varrho_1, \varrho_2) \equiv (d^+A) \times [\varrho_1, \varrho_2]$ is a compact subset of $\operatorname{dom} \Phi$ (since the GOS d^+A is compact). Since Φ is obviously continuous, $\Phi | D(\varrho_1, \varrho_2)$ is a homeomorphism (because the inverse of any continuous function with compact domain between Hausdorff spaces is coontinuous). But then the inverse of Φ is necessarily continuous over the open set $A_{\varrho_2} \setminus \overline{A_{\varrho_1}}$ contained in $\Phi(D(\varrho_1, \varrho_2))$. Thus the relation $\operatorname{range} \Phi = A_{\varrho_0} \setminus \bar{A} = \bigcup_{0 < \varrho_1 < \varrho_2 < \varrho_0} (A_{\varrho_2} \setminus \overline{A_{\varrho_1}})$ immediately implies continuity of Φ^{-1}.

Lemma 3. *Let A, ϱ_0, Φ be defined as in the previous lemma with the same assumptions. Then there exists a Borel measure μ and there are μ-measurable func-*

tions $a_0, \ldots, a_{n-1}$ *over* d^+A *such that for each* $0<\varrho<\varrho_0$ *and* vol_{n-1}*-measurable* $F\subset\partial(A_\varrho)$, *we have*

$$\mathrm{vol}_{n-1} F = \int_{d^+A} 1_F(y+\varrho\cdot k) \sum_{j=0}^{n-1} a_j(y,k)\varrho^j \, d\mu(y,k). \tag{4}$$

(Here $1_F(.)$ *stays for the characteristic function of F.)*

Proof. Fix (arbitrarily) a value $0<\varrho_1<\varrho_0$. Consider the mapping $\Psi(.)\equiv\equiv(.,\varrho_1)$. Observe that $\Psi\colon d^+A\leftrightarrow\partial(A_{\varrho_1})$ and that $\Phi(\Psi^{-1}(.),\varrho)\colon\partial(A_{\varrho_1})\leftrightarrow\partial(A_\varrho)$ for $0<\varrho<\varrho_0$ are homeomorphisms. Therefore the measure

$$d\mu \equiv d\,\mathrm{vol}_{n-1}\circ\psi \; {}^{3)} \tag{5'}$$

is a Borel measure on d^+A. Further, if $\varkappa_1, \ldots, \varkappa_{n-1}$ denote the main curvatures of the hypersurface $M\equiv\partial(A_{\varrho_1})$ oriented by its normal directed outward from A_{ϱ_1} then the functions $a_0, \ldots, a_{n-1}$ defined implicitly by

$$[1+(\tau-\varrho_1)\cdot\varkappa_1(y+\varrho_1 k)]\ldots[1+(\tau-\varrho)\cdot\varkappa_{n-1}(y+\varrho_1 k)] \equiv \sum_{j=0}^{n-1} a_j(y,k)\tau^j \quad (\text{for } 0<\tau<\varrho_0,\ (y,k)\in d^+A) \tag{5''}$$

are μ measurable (cf. Theorem B). Now let $T(F)$ denote the union of those pre-normals of A which intersect F (the surface piece of $\partial(A_\varrho)$ occurring in (4)). Then we have $T(F)\cap A_{\varrho_0}=\Phi(\Psi^{-1}(F)(0,\varrho_0))$. This shows that for any Borel measurable F, the figure $T(F)\cap A_{\varrho_0}$ is also Borel measurable. Then performing the substitutions (5′) and (5″) in the right hand side of (4), we obtain from Theorem B (cf. also (2)) that (4) holds for any Borel subset F of $\partial(A_\varrho)$. Hence we derive (4) for any vol_{n-1} measurable F from the Borel regularity of the measures vol_{n-1} and μ, respectively.

Remark. a) It is clear that the system $\mu, a_0, \ldots, a_{n-1}$ is not uniquely determined. However, it is discovered from the proof that the measures $d\nu\equiv a_0 d\mu, \ldots, d\nu_{n-1}\equiv\equiv a_{n-1}d\mu$ depend only on the GOS d^+A (in the sence that if $A^{(1)}$ and $A^{(2)}$ are sets in $\mathbf{R}^n$ of positive reach and $(\mu^{(i)}, a_0^{(i)}, \ldots, a_{n-1}^{(i)})$ are systems satisfying (4) for $A=A^{(i)}$ $(i=1,2)$, respectively, then for the measures $d\nu_j^{(i)}\equiv a_j^{(i)}d\mu^{(i)}$ $(i=1,2,;\ j=0,\ldots,n-1)$ we have

$$d\nu_j^{(1)}|(d^+A^{(1)})\cap(d^+A^{(2)}) = d\nu_j^{(2)}|(d^+A^{(1)})\cap(d^+A^{(2)}) \quad (j=0,\ldots,n-1).$$

b) For any $(y,k)\in d^+A$, the roots of the polynomial $\sum_{j=0}^{n-1} a_j(y,k)\varrho^j$ are real (cf. (5″)) and lie outside of the open interval $(0,\varrho_0)$ (cf. with the relations $-1/(\varrho_0-\varrho_1)\leqq\varkappa_1, \ldots, \varkappa_{n-1}\leqq 1/\varrho_1$ in Theorem B).

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[3]) The measure $\mathrm{vol}_{n-1}\circ\Psi$ is defined on the family of subsets of d^+A $\mathscr{F}\equiv\{\Psi^{-1}(E)\colon E\subset\partial(A_{\varrho_1}),$ E is vol_{n-1}-measurable$\}$ by $(\mathrm{vol}_{n-1}\circ\Psi)(D)\equiv\mathrm{vol}_{n-1}(\Psi(D))$ for any $D\in\mathscr{F}$.

Corollary (with the notations and assumptions of Lemma 2). *Formula* (4) *implies that for all* vol_n*-measurable subsets* T *of* $A_{\varrho_0}\setminus\bar{A}(=\Phi((d^+A)\times(0,\varrho_0)))$ *we have*

$$\mathrm{vol}_n T=\int_{d^+A}\int_0 1_T(y+\varrho\cdot k)\sum_{j=0}^{n-1}a_j(y,k)\varrho^j\,d\varrho\,d\mu(y,k). \tag{4'}$$

Proof. Consider the family of surface pieces $F(\varrho)\equiv T\cap\partial(A_\varrho)$. For $\varrho\geqq\varrho_0$ we have $F(\varrho)=\emptyset$ and for almost every $0<\varrho<\varrho_0$, $F(\varrho)$ is a vol_{n-1}-measurable subset of $\partial(A_\varrho)$. Thus we can apply Lemma 2 for almost every $0<\varrho<\varrho_0$ whence we obtain that

$$\mathrm{vol}_{n-1}T\cap\partial(A_\varrho)=\mathrm{vol}_{n-1}F(\varrho)=\int_{d^+A}1_{T\cap\partial(A_\varrho)}(y+\varrho k)\sum_{j=0}^{n-1}a_j(y,k)\varrho^j\,d(y,k)=$$

$$=\int_{d^+A}1_T(y+\varrho k)\sum_{j=0}^{n-1}a_j(y,k)\varrho^j\,d\mu(y,k).$$

Hence, by Lemma 1,

$$\mathrm{vol}_n T=\int_0^{\varrho_0}\int_{d^+A}1_T(y+\varrho k)\sum_{j=0}^{n-1}a_j(y,k)\varrho^j\,d\mu(y,k)\,d\varrho.$$

Observe that in the above formula, $y+\varrho k=\Phi((y,k),\varrho)$ stays in the argument of the function $1_T(.)$. Since Φ is a homeomorphism between $(d^+A)\times(0,\varrho_0)$ and $A_{\varrho_0}\setminus\bar{A}$ and since the measures $d\mu$, $d\,\mathrm{vol}_n$ and $d\varrho$ are Borel regular measures, respectively, this means that the product measure $d\tau\equiv d\mu\times d\varrho$ (i.e. $=d\mu\times d\,\mathrm{vol}_1$) satisfies

$$\mathrm{vol}_n T=\int_{(d^+A)\times(0,\varrho_0)}1_T(y+\varrho k)\sum_{j=0}^{n-1}a_j(y,k)\varrho^j\,d\tau(y,k,\varrho).$$

This immediately yields (4') by Fubini's theorem.

Notation. If $A\subset\mathbf{R}^n$ is closed with $\partial A\neq\emptyset$, then for any $(y,k)\in d^+A$ let $L^A(y,k)$ denote in the sequel the prenormal of A issued from the point $y(\in\partial A)$ in the direction of the vector k, and let $h^A(y,k)$ denote the length of the line segment $L^A(y,k)$.

Remark. It is easy to see that the value reach A is not other than the greatest lower bound of the function h^A (i.e. reach $A=\inf h^A(=\inf\{h^A(y,k):(y,k)\in d^+A\})$).

Lemma 4. *Let A be closed and $\partial A\neq\emptyset$.*

a) *For any* $\varepsilon>0$, *the GOS* $\{(y,k)\in d^+A: h^A(y,k)\geqq\varepsilon\}$ *is closed (in* $\mathbf{R}^n\times\mathbf{R}^n$*)*

b) d^+A *is Borel measurable (moreover it is an* $\mathcal{F}_\sigma$*).*

c) *For almost every* $\varrho>0$, *the set* $\partial(A_\varrho)$ *is of σ-finite* vol_{n-1}*-measure.*

d) *For the set* $Z^*\equiv\{y+h^A(y,k)k:(y,k)\in d^+A$ with $h^A(y,k)<\infty\}$, *we have* $\mathrm{vol}_{n-1}Z^*\cap\partial(A_\varrho)=0$ *except for countably many values of* $\varrho>0$.

Proof. a) From Theorem A we know that

$$d^+A = \{(y, k): \exists x \in \mathbf{R}^n \setminus A,\ y \in \mathrm{pr}_A x \text{ and } k = (x-y)/\|x-y\|\}. \tag{6}$$

Now if $\{(y_i, k_i): i \in I\} \subset d^+A$ is a convergent sequence such that $h^A(y_i, k_i) \geqq \varepsilon$ $(i \in I)$ and $(y_i, k_i) \to (y, k)$, then for $x \equiv y + \varepsilon k$ we have $x_i \to x$ and $y_i \in pr_A x_i$ with $k_i = (x_i - y_i)/\|x_i - y_i\|$ (for all $i \in I$). Since, in general, the condition $y' \in pr_A x'$ is equivalent to $\mathrm{dist}(x', A) = \mathrm{dist}(x', y')$, we infer from the continuity of the functions $\|.\|$ and $\mathrm{dist}(., A)$ that $\mathrm{dist}(x, y) = \mathrm{dist}(x, A) = \varepsilon$ i.e. $y \in pr_A x$ and $k = (x-y)/\|x-y\|$. This shows by (6) that $(y, k) \in d^+A$.

b) Since $d^+A = \bigcup_{m=1}^{\infty} \{(y, k): h^A(y, k) \geqq 1/m\}$.

c) Applying Lemma 1 we obtain

$$\infty > \mathrm{vol}_n [r\mathbf{B}^n \cap (\mathbf{R}^n \setminus A)] = \int_{-\infty}^{\infty} \mathrm{vol}_{n-1} [r\mathbf{B}^n \cap \partial(A_\varrho)]\, d\varrho$$

for all $r > 0$ [4]. Thus for any $r > 0$, there exists a set $\Delta_r \subset (0, \infty)$ such that $\mathrm{vol}_{n-1} [r\mathbf{B}^n \cap \partial(A_\varrho)] < \infty$ whenever $\varrho \in (0, \infty) \setminus \Delta_r$. Thus if $\varrho \notin \bigcup_{m=1}^{\infty} \Delta_m$ then the vol_{n-1}-measure of $\partial(A_\varrho)$ $(= \bigcup_{m=1}^{\infty} [m\mathbf{B}^n \cap \partial(A_\varrho)])$ is σ-finite.

d) Fix (an arbitrary) $\delta > 0$ such that $\partial(A_\delta)$ has σ-finite vol_{n-1}-measure, and for all $\varrho > \delta$ let Λ_ϱ denote the binary relation $\Lambda_\varrho \equiv \{(x, z): z \in \partial(A_\varrho),\ x \in pr_{\bar{A}_\delta} z\} (= \{(y + \delta k, y + \varrho k): (y, k) \in d^+A \text{ and } h^A(y, k) \geqq \varrho\}$. Now we know (see [4] p 254) that for distinct z_1, z_2 $(\in \mathbf{R}^n)$ there cannot be found any $x (\in \mathbf{R}^n)$ with (x, z_1), $(x, z_2) \in \Lambda_\varrho$ and that the mapping λ_ϱ defined by $\lambda_\varrho(x) = z \overset{\mathrm{def}}{\Leftrightarrow} (x, z) \in \Lambda_\varrho$ is Lipschitzian with $\mathrm{dom}\, \lambda_\varrho = pr_{\bar{A}_\varrho} \partial(A_\varrho)$ and $\mathrm{range}\, \lambda_\varrho = \partial(A_\varrho)$ for any $\varrho > \delta$. So for each $\varrho > 0$, we have $\mathrm{vol}_{n-1} Z^* \cap \partial(A_\varrho) = 0$ whenever the set $\lambda_\varrho^{-1}(Z^* \cap \partial(A_\varrho)) = \{y + \delta k : h^A(y, k) = \varrho\}$ has vol_{n-1}-measure 0. But the sets $\{y + \delta k: h^A(y, k) = \varrho\}$ $(\varrho > \delta)$ are all pairwise disjoint subsets of $\partial(A_\delta)$. From a) we infer that they are Borel measurable. Therefore the σ-finiteness of $\mathrm{vol}_{n-1} \partial(A_\delta)$ implies that there exist at most countably many $\varrho > \delta$ such that $\mathrm{vol}_{n-1}\{y + \delta k: h^A(y, k) = \varrho\} > 0$. This suffices for d) since the value of $\delta > 0$ can be chosen arbitrarily small.

Theorem 1. *Let $A \subset \mathbf{R}^n$ be closed and $\partial A \neq \emptyset$. If one can find a sequence $A^1, A^2, \ldots (\subset \mathbf{R}^n)$ of sets with non empty compact boundary such that*

a) $d^+A \subset \bigcup_{i=1}^{\infty} d^+A^i$,

b) $h_i \equiv \mathrm{reach}\, A^i > 0$ *for* $i = 1, 2, \ldots,$

[4]) $\mathbf{B}^n$ is the standard notation for the open unit ball of $\mathbf{R}^n$.

c) *for all* $(y,k)\in d^+A$ *we have* $h^A(y,k)\equiv\sup\{h_i:(y,k)\in d^+A^i\}$, *then there exists a Borel measure* μ *on* d^+A *and there are* μ*-measurable functions* $a_0,\dots,a_{n-1}$ *(over* d^+A*) such that*

$$\text{(7)}\qquad \mathrm{vol}_n T=\int\limits_{d^+A}\int\limits_0^{h^A(y,k)} 1_T(y+\varrho k)\sum_{j=0}^{n-1} a_j(y,k)\varrho^j\,d\varrho\,d\mu(y,k)$$

for all vol_n*-measurable* $T\subset\mathbf{R}^n\setminus A$.

Proof. Set $S_1\equiv(d^+A)\cap(d^+A^1),\dots,S_i\equiv[(d^+A)\cap(d^+A^i)]\setminus\bigcup_{j<i}S_j,\dots$ and for $i=1,2,\dots$ let $(\mu^i,a_0^i,\dots,a_{n-1}^i)$ be a fixed system satisfying (7) (putting A^i in the place of A, μ^i instead of μ etc. in Lemma 3). Now $S_1,S_2,\dots$ is a sequence of Borel-measurable GOS-s forming a partition of d^+A. We also have $S_i\subset d^+A^i$ $(i=1,2,\dots)$. So we can define the system $(\mu,a_0,\dots,a_{n-1})$ in the following way:

$$\text{(8')}\qquad \mu(E)\equiv\sum_{i=1}^{\infty}\mu^i(E\cap S_i)\quad\text{for}\quad E\subset d^+A\quad(\Leftrightarrow d\mu|S_i\equiv d\mu^i|S_i\ \text{for}\ i=1,2,\dots)$$

(in the sense that a set E is μ-measurable if and only if for all indices i, the sets $E\cap S_i$ are μ^i-measurable), and

$$\text{(8'')}\qquad a_j(y,k)\equiv a_j^i(y,k)\quad\text{for}\quad(y,k)\in S_i\quad(j=0,\dots,n-1\ \text{and}\ i=1,2,\dots).$$

Consider now a simple Borel function $f: d^+A\to[0,\infty]$ such that $f<h^A$ and range $f=\{c_1,c_2,\dots\}$, and set $G_f\equiv\{y+\varrho k:(y,k)\in d^+A,\ 0<\varrho<f(y,k)\}$. Then it easily follows from Lemma 3 that

$$\text{(9)}\qquad \mathrm{vol}_n T\cap G_f=\int\limits_{d^+A}\int\limits_0^{f(y,k)} 1_T(y+\varrho k)\sum_{j=0}^{n-1} a_j(y,k)\varrho^j\,d\varrho\,d\mu(y,k)$$

for each vol_n-measurable $T\subset\mathbf{R}^n\setminus A$.

To prove (9), take the following Borel-measurable partition $\{S_{im}:i,m=1,2,\dots\}$ of d^+A defined by

$$S_{im}\equiv\{(y,k)\in f^{-1}(\{c_m\}):\ i\ \text{is the smallest index with}\ (y,k)\in d^+A^i\ \text{and}\ h_i>c_m\}.$$

Then consider the partition $\{B_{im}:i,m=1,2,\dots\}$ of G_f defined by $B_{im}\equiv\{y+\varrho k:(y,k)\in S_{im},\ 0<\varrho<c_m\}$. Then fix an arbitrary pair of indices i,m. Applying Lemma 2b) to A^i, we see that the domain B_{im} is Borel measurable. Since for any $(y,k)\in S_{im}$ and $0<\varrho<h^A(y,k)$ we have $1_{T\cap B_{im}}(y+\varrho k)=1_T(y+\varrho k)\cdot 1_{S_{im}}(y,k)1_{(0,c_m)}(\varrho)$, using Lemma 3 (with A^i instead of A and

with $\varrho_0=h_i$), we have

$$\mathrm{vol}_n T\cap B_{im}=\int_{d^+A}\int_0^{h_i} 1_T(y+\varrho k)\cdot 1_{S_{im}}(y,k)1_{(0,c_m)}(\varrho)\sum_{j=0}^{n-1} a_j^i(y,k)\varrho^j\,d\varrho\,d\mu(y,k)=$$

$$=\int_{S_{im}}\int_0^{c_m} 1_T(y+\varrho k)\sum_{j=0}^{n-1} a_j(y,k)\varrho^j\,d\varrho\,d\mu(y,k)=$$

$$=\int_{d^+A}\int_0^{f(y,k)} 1_T(y+\varrho k)\cdot 1_{S_{im}}(y,k)1_{(0,c_m)}(\varrho)\sum_{j=0}^{n-1} a_j(y,k)\varrho^j\,d\varrho\,d\mu(y,k).$$

Summing this for $i, m=1, 2, \ldots,$ we obtain (9).

In possession of (9) we can conclude as follows: Lemma 4a) shows that the function $h^A\colon d^+A\to(0,\infty]$ is Borel-measurable (moreover that it is upper semi-continuous). Therefore there exists a sequence $0\leqq f_1\leqq f_2\leqq\ldots$ of simple Borel-functions such that $f_i\nearrow h^A$ (pointwise). For any such a sequence $\{f_i\}_1^\infty$, we have $\bigcup_{i=1}^\infty G_{f_i}=\{y+\varrho k\colon (y,k)\in d^+A,\ 0<\varrho<h^A(y,k)\}=\mathbf{R}^n\setminus(A\cup Z^*)$ where $Z^*\equiv\equiv\{y+h^A(y,k)\cdot k\colon h^A(y,k)<\infty\}$. So, for $i\to\infty$, it follows from (9) that

$$\text{(7')}\qquad \mathrm{vol}_n T\setminus Z^*=\int_{d^+A}\int_0^{h^A(y,k)} 1_T(y+\varrho k)\sum_{j=0}^{n-1} a_j(y,k)\varrho^j\,d\varrho\,d\mu(y,k).$$

But now the relation $Z^*=(\mathbf{R}^n\setminus A)\setminus\bigcup_{i=1}^\infty G_{f_i}$ shows that Z^* is a Borel-set. Thus we may apply Lemma 1 to Z^* (in place of T there) which implies (by Lemma 4d)) that $\mathrm{vol}_n Z^*=0$.

4. Some convexity properties of parallel sets

Our aim in this section will be to prove that there always exist sets $A^1, A^2, \ldots$ satisfying the conditions of Theorem 1.

Lemma 5. *Let $x_0\in\mathbf{R}^n$ and $\varrho_0>0$. Then the function* $g(.)\equiv\mathrm{dist}\,(.\,,x_0)-\frac{1}{2\varrho_0}\|.\|^2$ *is concave on the domain* $G\equiv\{x\colon \mathrm{dist}\,(x,x_0)>\varrho_0\}$. (A function f is said to be *concave on a domain H* if it is concave in the usual sence when restricted to any convex subset of H.)

Proof. Evaluate the eigenvalues of the second derivative tensor[5]) of the function f at a point $x_1\in G$. It is convenient to use a Cartesian coordinate system

[5]) The second derivative tensor of a function $f\colon H(\subset\mathbf{R}^n)\to\mathbf{R}$ at a point $x\in H$ is considered here as the bilinear form $D_2 f(x)\colon \mathbf{R}^n\times\mathbf{R}^n\to\mathbf{R}$, $(v_1,v_2)\mapsto\partial_{v_1}\partial_{v_2}f(x)$ where the symbol ∂_v means the directional derivation in the direction $v(\in\mathbf{R}^n)$ i.e. $\partial_v f(y)\equiv\lim_{\lambda\searrow 0}\lambda^{-1}[f(y+\lambda v)-f(y)]$.

with origin x_0 and first unit vector $e_1=\frac{x_1-x}{\|x_1-x\|}$. Then, independently of the choice of the further basic vectors $e_2, \dots, e_n$, the function $f(.)\equiv\text{dist}(., x)$ is represented by the form $\varphi(\xi_1, \dots, \xi_n)=f(x_0+\xi_1 e_1+\dots+\xi_n e_n)=\sqrt{\xi_1^2+\dots+\xi_n^2}$ in this coordinate system. Since $x_1=x_0+\|x_1-x_0\| e_1$, the eigenvalues of $D_2 f(x_1)$ coincide with those of the matrix $M\equiv\left(\frac{\partial^2\varphi}{\partial\xi_i\,\partial\xi_j}\bigg|_{(\|x_1-x_0\|, 0,\dots,0)}\right)_{i,j=1}^n$. But it is easy to see that M is of diagonal form with $0, \|x_1-x_0\|^{-1}, \dots, \|x_1-x_0\|^{-1}$ in its main diagonal. On the other hand, $D_2\|.\|^2$ is represented in any Cartesian system by the matrix $I\equiv(2\cdot\delta_{ij})_{i,j=1}^n$ (δ_{ij} denotes the "Kronecker δ"). Therefore the eigenvalues of $D_2 f(x_1)$ are $-\frac{1}{\varrho_0}$ and $\|x_1-x_0\|^{-1}-\frac{1}{\varrho_0}$ (with multiplicity $n-1$), all negative numbers. This completes the proof by recalling that any function of negative definite second derivative tensor is concave on any open convex subset of its domain.

Theorem 2. *Let $A\subset\mathbf{R}^n$ be such that $\partial A\neq\emptyset$ and fix $\varrho_0>0$. Then the function $g(.)\equiv\text{dist}(., A)-\frac{1}{2\varrho_0}\|.\|^2$ is concave on the domain $G\equiv\{x\in\mathbf{R}^n\colon \text{dist}(x, A)>\varrho_0\}$.*

Proof. f is the infimum of the function family $F\equiv\left\{\text{dist}(., A)-\frac{1}{2\varrho_0}\|.\|^2 : x\in A\right\}$. By Lemma 5, all members of F are concave functions on G. But the infimum of any family of concave functions in concave.

Corollary. *All directional derivatives of the function $f(.)\equiv\text{dist}(., A)$ exist in $\mathbf{R}^n\setminus A$. For a fixed $x_0\in\mathbf{R}^n\setminus A$, the function $t\mapsto\partial_t f(x_0)$ is continuous and superlinear (i.e. positive homogeneous and concave).*

Proof. Apply Theorem 2 with $\varrho_0\equiv\frac{1}{2}\text{dist}(x_0, A)$. This shows that the function $g(.)=f(.)-\frac{1}{2\varrho_0}\|.\|^2$ is concave on some neighborhood of the point x_0. Therefore $\partial_t f(x_0)$ exists for all $t\in\mathbf{R}^n$ and satisfies $\partial_t f(x_0)=\partial_t g(x_0)+\frac{1}{\varrho_0}\langle t, x_0\rangle$. Thus $t\mapsto\partial_t f(x_0)$ is the sum of a continuous superlinear and a linear form of t (since the directional derivatives at a fixed point of any concave $\mathbf{R}^n\to\mathbf{R}$ function are continuous and superlinear.)

Theorem 3. *Let $A\subset\mathbf{R}^n$ be closed and $f(.)\equiv\text{dist}(., A)$. Then for any $x_0\notin A$ and for any $t\in\mathbf{R}^n$ we have*

$$\partial_t f(x_0)=\min\left\{\left\langle t, \frac{y-x_0}{\|y-x_0\|}\right\rangle\colon y\in pr_A x_0\right\}.$$

Proof. Consider an arbitrary $y_0 \in pr_A x_0$. Now we have $f(x_0+\lambda t)-f(x_0)= = \operatorname{dist}(x_0+\lambda t, A)-\operatorname{dist}(x_0, A) = \operatorname{dist}(x_0+\lambda t, A)-\operatorname{dist}(x_0, y_0) \leqq \operatorname{dist}(x_0+\lambda t, y_0)- -\operatorname{dist}(x_0, y_0)$. Thus, by writing $h(.)\equiv\operatorname{dist}(., y_0)$, we obtain $\partial_t f(x_0)\leqq\partial_t h(x_0)= =\langle t, \operatorname{grad} h(x)_0\rangle = \left\langle t, \frac{y_0-x_0}{\|y_0-x_0\|}\right\rangle \leqq \min\left\{\left\langle t, \frac{y-x_0}{\|y-x_0\|}\right\rangle : y\in pr_A x_0\right\}$.

The proof of the inequality in the converse direction: Let us associate with any $x\in\mathbf{R}^n\setminus A$ a point $y(x)$ from the set $pr_A x$ and then let φ_x denote the function $\varphi_x(.)\equiv\operatorname{dist}(., y(x))$. Now we have $f=\inf_{x\in\mathbf{R}^n\setminus A}\varphi_x$ and for all $x\notin A$, $f(x)=\varphi_x(x)$. Thus, by writing $\psi(.)\equiv\varphi_{x_0+\lambda t}(.)$, we obtain

$$\frac{1}{\lambda}[f(x_0+\lambda t)-f(x_0)] \geqq \frac{1}{\lambda}[f(x_0+\lambda t)-\psi(x_0)] \geqq \frac{1}{\lambda}[\psi(x_0+\lambda t)-\psi(x_0)] \geqq \partial_t\psi(x_0)$$

for any arbitrarily fixed $t\in\mathbf{R}^n$ and $\lambda>0$. (The last inequality is a consequence of the convexity of ψ.) Hence from the relation $\operatorname{grad}\psi(x_0)=\frac{x_0-y(x_0+\lambda t)}{\|x_0-y(x_0+\lambda t)\|}$, we deduce that

$$\frac{1}{\lambda}[f(x_0+\lambda t)-f(x_0)] \geqq \left\langle t, \frac{x_0-y(x_0+\lambda t)}{\|x_0-y(x_0+\lambda t)\|}\right\rangle \quad \text{whenever} \quad \lambda>0. \tag{10}$$

Since for any bounded $G\subset\mathbf{R}^n\setminus A$ the set $\{y(x): x\in G\}$ is also bounded, there can be found a sequence $\lambda_i\searrow 0$ such that the sequence $\{y(x_0+\lambda_i t)\}_1^\infty$ be convergent. Fix such a sequence $\{\lambda_i\}_1^\infty$ and set $y^*\equiv\lim_i y(x_0+\lambda_i t)$. Now by (10) we have

$$\partial_t f(x_0) \geqq \left\langle t, \frac{x_0-y^*}{\|x_0-y^*\|}\right\rangle. \tag{10'}$$

On the other hand from the equivalence of the relations $\operatorname{dist}(x_0+\lambda_i t, A)= =\operatorname{dist}(x_0+\lambda_i t, y(x_0+\lambda_i t))$ and $y(x_0+\lambda_i t)\in pr_A(x_0+\lambda_i t)$ we infer for $i\to\infty$ that $y^*\in pr_A x_0$. Thus for some $y^*\in pr_A x_0$, (10′) holds.

From now on, throughout the remaining part of this section, let A denote a fixed closed subset of $\mathbf{R}^n$, let $x_0\in\mathbf{R}^n\setminus A$ (also fixed), $r\equiv\operatorname{rad} pr_A x_0$ [6]), $\varrho\equiv\operatorname{dist}(x_0, A)$ and $f(.)\equiv\operatorname{dist}(., A)$.

Lemma 6. *$\max_{t\neq 0}(\partial_t f(x_0)/\|t\|)=\sqrt{1-(r/\varrho)^2}$ if $r<\varrho$ and $\max_{t\neq 0}(\partial_t f(x_0)/\|t\|)\ 0$ if and only if $r=\varrho$. (Since $pr_A x_0\subset\{y: \|y-x_0\|=\varrho\}$, the possibility $r>0$ is excluded).*

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[6]) For any set $H\subset\mathbf{R}^n$, $\operatorname{rad} H\equiv\inf\{\delta\geqq 0: \exists p\in\mathbf{R}^n\ H\subset p+\delta\mathbf{B}^n\}$.

Proof. Since the function $t \mapsto \partial_t f(x_0)$ is superlinear and continuous, a simple compactness argument shows that $\max_{t \neq 0} \partial_t f(x_0)/\|t\|$ is always attained for some $t_0 \in \mathbf{R}^n$ with $\|t_0\| = 1$. Now if $\partial_{t_0} f(x_0) > 0$, then the set $pr_A x_0$ is contained in the spherical cap

$$K \equiv \{y \in \mathbf{R}^n \colon \|y-x\| = \varrho,\ \langle t_0, y-x_0 \rangle \geqq \varrho \cdot \partial_{t_0} f(x_0)\}.$$

But then, by writing $p \equiv x_0 - (\varrho \cdot \partial_{t_0} f(x_0)) t_0$, we have $K \subset \{y \colon \|y-p\| \leqq \sqrt{\varrho^2 - (\varrho \cdot \partial_{t_0} f(x_0))^2}\}$. Thus $\partial_{t_0} f(x_0) > 0$ implies that $r \leqq \sqrt{1-(\partial_{t_0} f(x_0))^2}$ and therefore $\partial_{t_0} f(x_0) \geqq \sqrt{1-(r/\varrho)^2}$.

On the other hand, if $r < \varrho$ then, because of the compactness of the set $pr_A x_0$, there exists a unique closed ball $B (\subset \mathbf{R}^n)$ of radius r such that $pr_A x_0 \subset B$. Consider the spherical cap $K' \equiv \{y \in B \colon \|y-x_0\| = \varrho\}$. It is not hard to prove that the closed ball $B' (\subset \mathbf{R}^n)$ of minimal radius containing the set K' is that whose center and radius coincide with those of the $(n-1)$-dimensional sphere $S' \equiv \{y \in \partial B \colon \|y-x_0\| = \varrho\}$, respectively. Since $pr_A x_0 \subset K' \subset B'$, we necessarily have $B' = B$. Let q denote the center of B and set $t_1 \equiv x_0 - q$. Since the point q is the center of S', we have $\operatorname{angle}(t_1, y-q) = \pi/2$ for all $y \in S'$. Hence we deduce $\|t_1\|^2 = \sqrt{\|x_0-y\|^2 - \|y-q\|^2} = \sqrt{\varrho^2 - r^2}$ (with arbitrary $y \in S'$). Observe now that

$$K' = \{y \colon \|y-x_0\| = \varrho \text{ and } \operatorname{angle}(t_1, y-q) \geqq \pi/2\} = \{y \colon \|y-x_0\| = \varrho,\ \langle t_1, y-q \rangle \leqq 0\}.$$

Therefore, by Theorem 5 we obtain

$$\partial_{t_1} f(x_0) \geqq \min\left\{\left\langle t_1, \frac{x_0-y}{\varrho}\right\rangle \colon \|x_0-y\| = \varrho,\ \langle t, y-q \rangle \leqq 0\right\} \geqq \left\langle t_1, \frac{x_0-q}{\varrho}\right\rangle = \|t_1\|^2/\varrho.$$

So $r < \varrho$ implies that $\max_{t \neq 0} \partial_t f(x_0)/\|t\| \geqq \|t_1\|/\varrho = \sqrt{1-(r/\varrho)^2}$.

Definition. We call a vector $t (\in \mathbf{R}^n)$ a *tangent vector* of a set $S (\subset \mathbf{R}^n)$ at the point $x \in S$ if $t = 0$ if there is a sequence $x \neq x_1, x_2, \ldots \in S$ such that $x_i \to x$ and $\operatorname{angle}(t, x_i - x) \to 0$ (for $i \to \infty$). $\left(\text{For } t_1, t_2 \in \mathbf{R}^n,\ \operatorname{angle}(t_1, t_2) \equiv \arccos\left\langle \frac{t_1}{\|t_1\|}, \frac{t_2}{\|t_2\|}\right\rangle.\right)$ The set of the tangent vectors of S of x will be denoted by $\operatorname{Tan}(x, S)$.

Lemma 7. *If $r < \varrho$ then for any $t \in \mathbf{R}^n$ we have*

a) $t \in \operatorname{Tan}(x_0, \partial(A_\varrho))$ *if and only if* $\partial_t f(x_0) = 0$,

b) $t \in \operatorname{Tan}(x_0, \mathbf{R}^n \setminus A_\varrho)$ *if and only if* $\partial_t f(x_0) \geqq 0$.

(I.e. $\operatorname{Tan}(x_0, \mathbf{R}^n \setminus A_\varrho)$ *is a closed convex cone with non-empty interior and boundary and its boundary coincides with* $\operatorname{Tan}(x_0, \partial(A_\varrho))$.*)*

Proof. Since $\mathbf{R}^n \setminus A_\varrho = \{x : f(x) \geqq \varrho\}$ and $f(x_0) = \varrho$, we can immediately establish that $\partial_t f(x_0) > 0$ implies $t \in \operatorname{Tan}(x_0, \mathbf{R}^n \setminus A_\varrho)$ and that in case of $t \in \operatorname{Tan}(x_0, \mathbf{R}^n \setminus A_\varrho)$ we have $\partial_t f(x_0) \geqq 0$. Therefore it suffices to prove just the statement a).

Since $\partial(A_\varrho) = \{x : f(x) \doteq \varrho\}$, it is clear that $\partial_t f(x_0) = 0$ for all $t \in \operatorname{Tan}(x, \partial(A_\varrho))$. To prove $\partial_t f(x_0) = 0 \Rightarrow t \in \operatorname{Tan}(x_0, \partial(A_\varrho))$ we can proceed as follows. Let $C \equiv \{t : \partial_t f(x_0) = 0\}$ and $F(t) \equiv \partial_t f(x_0)$. From the continuity and superlinearity of the functional F it follows that C is a closed convex cone. Lemma 6 ensures that, for some $t_0 \in C$, we have $F(t_0) > 0$. Since there also exists a vector t_1 such that $F(t_1) < 0$ (e.g. the vector $t_1 \equiv y - x_0$ with an arbitrary $y \in pr_A x_0$), from the superlinearity and continuity of F we easily deduce that

$$F(t) > 0 \Leftrightarrow t \in \mathring{C} \text{ (the interior of } C), \quad F(t) = 0 \Leftrightarrow t \in \partial C, \text{ and } F(t) < 0 \Leftrightarrow t \notin C (\forall t \in \mathbf{R}^n).$$

Therefore we have to show that for any $0 \neq t \in \partial C$ and $\varepsilon > 0$ there exists a point $x \in \partial(A_\varrho)$ such that $0 < \|x - x_0\| < \varepsilon$ and $\operatorname{angle}(t, x - x_0) < \varepsilon$. But it is a directe corollary from continuity of F.

Lemma 8. *If S is any subset of $\mathbf{R}^n$, $x \in S$ and L denotes the smallest cone containing the unit vectors $k (\in \mathbf{R}^n)$ satisfying $(x, k) \in d^+ S$ then* $\operatorname{Tan}(x, S) \subset \operatorname{dual} L$[7] *(or which is the same* $L \subset \operatorname{dual} \operatorname{Tan}(s, S)$*).*

Proof. We must prove that in case of $(x, k) \in d^+ S$, for any $t \in \operatorname{Tan}(x, S)$ we have $\langle t, k \rangle < 0$. Proceed by contradiction. Suppose that $(x, k) \in d^+ S$ and $t \in \operatorname{Tan}(x, S)$ are such that $\langle t, k \rangle > 0$. Since the figure $\operatorname{Tan}(x, S)$ is a cone, we may assume without loss of generality that $\|t\| = 1$. Consider a sequence $x \neq x_1, x_2, \ldots \to x$ in S such that $\operatorname{angle}(t, x_i - x) \to 0$ $(i \to \infty)$ and set $h_i \equiv \|x_i - x\|$ and $t_i \equiv \frac{1}{h_i}(x_i - x)$ $(i = 1, 2, \ldots)$. Observe now that $t_i \to t$ and that for any arbitrarily fixed $\varrho' > 0$, the function $\psi(.) \equiv \operatorname{dist}(., x + \varrho' k)$ satisfies

$$\lim_i \frac{1}{h_i} [\operatorname{dist}(x_i, x + \varrho' k) - \varrho'] = \lim_i \frac{1}{h_i} [\psi(x + h_i t_i - \psi(x)] =$$

$$= \lim_{h \searrow 0} \frac{1}{h} [\psi(x + ht) - \psi(x)] = \partial_t \psi(x) = \langle t, k \rangle > 0.$$

This shows that $\operatorname{dist}(x_i, x + \varrho' k) < \varrho'$ holds for some index i. Thus we necessarily have $(y, k) \notin d^+ S$ by the arbitrariness of $\varrho' > 0$ and the definition of the GOS $d^+ S$.

[7]) For any set $H \subset \mathbf{R}^n$ we define its dual by $\operatorname{dual} H \equiv \{t \in \mathbf{R}^n : \forall u \in H \langle t, u \rangle \leqq 0\}$.

Remark. The converse inclusion $L \supset$ dual Tan (x, S) fails in general. Example: in $n=2$ dimensions for $S \equiv \{(\xi, \eta) \in \mathbf{R}^2: \eta \leqq |\xi|^{3/2}\}$, $x \equiv (0, 0)$ and $k \equiv (0, 1)$ we have Tan $(x, S) = \{(\tau_1, \tau_2): \tau_2 \leqq 0\} = \{t \in \mathbf{R}^2: \langle k, t \rangle \leqq 0\}$ while $(y, k) \notin d^+S$. However, one can conjecture that if $S \equiv \mathbf{R}^n \setminus A_\varrho$ and $x \equiv x_0$ then $L =$ dual Tan $(x_0, \mathbf{R}^n \setminus A_\varrho)$ always holds. It will suit our requirements the following simpler special case:

Theorem 4. *Suppose $r < \varrho$. Then*

a) *the figure $D \equiv \{y: x_0 \in pr_{\mathbf{R}^n \setminus A_\varrho} y\}$ is convex and closed (this holds even for $r = \varrho$),*

b) *one can represent the set $D^0 \equiv \operatorname{conv}(\{x_0\} \cup pr_A x_0)$* [8]) *as the union of straight line segments issued from the point x_0 and of length $\sqrt{\varrho^2 - r^2}$.*

c) *If $L \equiv [0, \infty)\{k: (x_0, k) \in d^+(\mathbf{R}^n \setminus A_\varrho)\}$ then we have*

$$L = [0, \infty)(D - x_0) = [0, \infty)(D^0 - x) = \text{dual Tan}\,(x_0, \mathbf{R}^n \setminus A_\varrho)$$

d) $h^{\mathbf{R}^n \setminus A_\varrho}(x_0, k) \geqq \sqrt{\varrho^2 - r^2}$ *whenever* $(x_0, k) \in d^+(\mathbf{R}^n \setminus A_\varrho)$.

Proof. a) From the definition of $pr_{\mathbf{R}^n \setminus A_\varrho} y$ we infer that

$$D = \{y: \forall x \in \mathbf{R}^n \setminus A_\varrho,\ \text{dist}(y, x_0) \leqq \text{dist}(y, x)\} = \bigcap_{x \in \mathbf{R}^n \setminus A_\varrho} \{y: \|y - x_0\| \leqq \|y - x\|\}.$$

Thus D is the intersection of some family of closed half spaces (or $D = \mathbf{R}^n$ if $\{x_0\} = \mathbf{R}^n \setminus A_\varrho$).

b) For the sake of simplicity, we can assume (without loss of generality) that $x_0 = 0$.

It is well-known that, in general, the closed convex hull of any compact subset of $\mathbf{R}^n$ coincides with its algebraic convex hull. Hence

$$\operatorname{conv}(\{x_0\} \cup pr_A x_0) =$$

$$= \Big\{\alpha \sum_1^m \lambda_i y_i: 0 \leqq \alpha \leqq 1, \lambda_1, \ldots, \lambda_m \geqq 0, \sum_1^m \lambda_i = 1 \text{ and } y_1, \ldots, y_m \in pr_A x_0\Big\}.$$

Thus we can write $D^0 = [0, 1] \cdot \operatorname{conv}(pr_A x_0) = \bigcup \{[0, 1] \cdot c: c \in \operatorname{conv}(pr_A x_0)\}$. Therefore it suffices to see that for any $c \in \operatorname{conv}(pr_A x_0)$ we have $\|c\| \geqq \sqrt{\varrho^2 - r^2}$. Let t_0 be a unit vector such that $\partial_{t_0} f(x_0) = \sqrt{1 - (r/\varrho)^2}$ (its existence is established by Lemma 6).

[8]) For $H \subset \mathbf{R}^n$, conf H denotes the closed convex hull of H (i.e. the smallest closed convex subset of $\mathbf{R}^n$ containing H).

From Theorem 5 we infer that for any finite convex linear combination $c=\lambda_1 y_1+\ldots+\lambda_m y_m$ of some points of $pr_A x_0$ we have

$$\langle t_0, c\rangle = \sum_1^m \lambda_i \langle t_0, y_i\rangle = -\sum_1^m \lambda_i \langle t_0, x_0-y_i\rangle = -\varrho \sum_1^m \lambda_i \left\langle t_0, \frac{x_0-y_i}{\|x_0-y_i\|}\right\rangle \geqq$$

$$\geqq -\varrho \sum_1^m \lambda_i \partial_{t_0} f(x_0) = -\varrho\, \partial_{t_0} f(x_0) = -\sqrt{\varrho^2-r^2},$$

whence $\|c\|=\|t_0\|\cdot\|c\|\geqq|\langle t_0, c\rangle|=\sqrt{\varrho^2-r^2}$.

c) The relation $L=[0,\infty)(D-x_0)$ directly follows from the definitions. From Lemma 7b) and Theorem 5 we also have that $t\in \mathrm{Tan}\,(x_0, \mathbf{R}^n\setminus A_\varrho)\Leftrightarrow \partial_t f(x_0)\geqq 0 \Leftrightarrow \Leftrightarrow \forall y\in pr_A x_0 \langle t, x_0-y\rangle \geqq 0, \Leftrightarrow t\in \mathrm{dual}\,[(pr_A x_0)-x_0] \Leftrightarrow t\in \mathrm{dual}\,(D^0-x_0) \Leftrightarrow t\in \mathrm{dual}\,[0,\infty)\cdot \cdot (D^0-x_0)$. Thus $\mathrm{Tan}\,(x_0, \mathbf{R}^n\setminus A_\varrho)=\mathrm{dual}\,[0,\infty)(D^0-x_0)$. Since both $\mathrm{Tan}\,(x_0, \mathbf{R}^n\setminus A_\varrho)$ and $[0,\infty)(D^0-x_0)$ are closed convex cones in $\mathbf{R}^n$, respectively, from Farkas's well-known theorem we infer $[0,\infty)(D^0-x_0)=\mathrm{dual}\,\mathrm{Tan}\,(x_0, \mathbf{R}^n\setminus A_\varrho)$. Then observe that from the definition of the set D it follows $x_0\in D$ and $pr_A x_0\subset D$. This implies by a) that $D^0\subset D$ and therefore $[0,\infty)(D^0-x_0)\subset[0,\infty)(D-x_0)$. At this point the proof of c) is completed by Lemma 8 which shows (for $S\equiv \mathbf{R}^n\setminus A_\varrho$ and $x\equiv x_0$) that $L\subset \mathrm{dual}\,\mathrm{Tan}\,(x, \mathbf{R}^n\setminus A_\varrho)$, since we have proved here $L=[0,\infty)(D-x_0)\supset \supset[0,\infty)(D^0-x_0)=\mathrm{dual}\ \mathrm{Tan}\,(x_0, \mathbf{R}^n\setminus A_\varrho)$.

d) is immediate from b) and c).

Corollary. *If* $\varrho>0$, $A\subset \mathbf{R}^n$ *is closed and* $\mathrm{rad}\, A<\varrho$ *then* $h^{\mathbf{R}^n\setminus A_\varrho}\geqq \geqq \sqrt{\varrho^2-(\mathrm{rad}\, A)^2}$.

Proof. Let $(x_0, k)\in d^+(\mathbf{R}^n\setminus A_\varrho)$. Now we have $x_0\in\partial(\mathbf{R}^n\setminus A_\varrho)=\partial(A_\varrho)$ and $r=\mathrm{rad}\, pr_A x_0\leqq \mathrm{rad}\, A<\varrho$. Thus Theorem 4d) can be applied.

5. Main Theorem

On the basis of the previous section we can construct the sets $A^1, A^2, \ldots$ required by Theorem 1.

Lemma 9. *For any closed subset* A *of the space* $\mathbf{R}^n$ *with* $\partial A\neq\emptyset$ *there exists a countable family* $\mathcal{A}\equiv\{A^\alpha\colon \alpha\in I\}$ *of subsets of* $\mathbf{R}^n$ *with positive reach and compact boundary such that* $\bigcup_{\alpha\in I} d^+A\supset d^+A^\alpha$ *and* $h^A(y,k)\leqq \sup\{\mathrm{reach}\, A^\alpha\colon (y,k)\in d^+A^\alpha\}$ *hold for any* $(y,k)\in d^+A$.

Proof. Let $\varrho_1, \varrho_2, \ldots$ be an enumeration of the positive rational numbers and for $i=1,2,\ldots$ let the set B^i defined by $B^i\equiv\partial(A_{\varrho_i})$. Now we obtain from the definition of the function $h^A(:d^+A\to(0,\infty))$ that

$$B^i=\partial(A_{\varrho_i})=\{y+\varrho_i k\colon (y,k)\in d^+A \text{ and } h^A(y,k)\geqq\varrho_i\}\quad (i=1,2,\ldots). \tag{11}$$

Then let each set B^i be covered by a countable family $K^{i,1}, K^{i,2}, \ldots$ of closed balls of radius $\varrho_i/(2i)$ and define the sets $A^{i,s}$ $(i,s=1,2,\ldots)$ as follows: set $G^{i,s}\equiv\equiv B^i\cap K^{i,s}$ and let $A^{i,s}\equiv\mathbf{R}^n\setminus(G^{i,s})_{\varrho_i}$ $(=\{y\colon \operatorname{dist}(y,G^{i,s})\geqq\varrho_i\})$.

Observe that if $(y,k)\in d^+A$ is such that $h^A(y,k)\geqq\varrho_i$ and $y+\varrho_i k\in G^{i,s}$ then (for the same pair of indices i,s) we have $\operatorname{dist}(y+\varrho_i k, A^{i,s})=\varrho_i$ and hence $(y,k)\in d^+A^{i,s}$ $(i,s=1,2,\ldots)$. Since $\bigcup_{s=1}^{\infty} G^{i,s}=B^i$, this means by (11) that

$$\{(y,k)\in d^+A\colon h^A(y,k)\geqq\varrho_i\}\subset\bigcup_{s=1}^{\infty} d^+A^{i,s}\quad (i=1,2,\ldots). \tag{12}$$

It follows from (12) that $d^+A\subset\bigcup_{i,s=1}^{\infty} d^+A^{i,s}$.

Since the figure $G^{i,s}$ is contained in the ball $K^{i,s}$ whose radius equals to $\varrho_i/(2i)$, we have from the Corollary of Theorem 4 that $\operatorname{reach} A^{i,s}=\inf h^{A^{i,s}}=\inf h^{\mathbf{R}^n\setminus(G^{i,s})_{\varrho_i}}\geqq\geqq\varrho_i\sqrt{1-1/(4i^2)}>0$ $(i,s=1,2,\ldots)$. So from (12) we also infer that

$$\sup\{\operatorname{reach} A^{i,s}\colon (y,k)\in d^+A^{i,s}\}\geqq h^A(y,k)$$

for each $(y,k)\in d^+A$. Finally, the inclusions $\partial A^{i,s}=\partial[\mathbf{R}^n\setminus(G^{i,s})_{\varrho_i}]=\partial((G^{i,s})_{\varrho_i}]\subset\subset\overline{(G^{i,s})}_{\varrho_i}\subset\overline{(K^{i,s})}_{\varrho_i}$ immediately imply compactness of $\partial A^{i,s}$ $(i,s=1,2,\ldots)$. Thus the choice $\mathcal{A}\equiv\{A^{i,s}\colon i,s=1,2,\ldots\}$ suits our requirements.

Theorem 5. *For every closed $A\subset\mathbf{R}^n$ of non-empty boundary there exists a Borel measure μ over the generalized oriented surface d^+A and there can be found μ-measurable functions $a_0(.),\ldots,a_{n-1}(.)$ such that for any Lebesgue integrable function $\varphi\colon\mathbf{R}^n\setminus A\to\mathbf{R}^n$ we have*

$$\int_{\mathbf{R}^n\setminus A}\varphi\,d\mathrm{vol}_n=\int_{d^+A}\int_0^{h^A(y,k)}\varphi(y+\varrho k)\sum_{j=0}^{n-1}a_j(y,k)\varrho^j\,d\varrho\,d\mu(y,k)= \tag{13}$$

$$=\int_D\varphi(y+\varrho k)\sum_{j=0}^{n-1}a_j(y,k)\varrho^j\,d\tau(y,k,\varrho)$$

where $D\equiv\{(y,k,\varrho)\colon (y,k)\in d^+A$ and $0<\varrho<h^A(y,k)\}$ and $d\tau$ denotes the product measure $d\mu\times d$length over (d^+A).

Proof. From Lemma 9 and Theorem 1 we immediately obtain (13) for characteristic functions of vol_n-measurable subsets of $\mathbf{R}^n \setminus A$. By taking linear combinations we can pass to simple $\mathbf{R}^n \setminus A \to \mathbf{R}$ functions and then a standard density argument establishes (13) for arbitrary Lebesgue integrable $\mathbf{R}^n \setminus A \to \mathbf{R}$ functions.

Corollary. *For μ-almost every $(y, k) \in d^+A$, the zeros of the polynomial $\sum_{j=0}^{n-1} a_j(y, k)\varrho^j$ are real and lie outside $(0, h^A(y, k))$.*

Proof. Recall the construction of the measure μ and the functions a_j in Theorem 1 (8′) and (8″). Applying the same notations (and definitions) as in Theorem 1, we can proceed as follows: From Remark a) after Lemma 3 we infer that for any fixed pair of indices i_1, i_2 one can write $a_j^{i_1} d\mu^{i_1} = a_j^{i_1} d\mu^{i_2}$ $(j=0, \dots, n-1)$ when restricted to the set $(d^+A^{i_1}) \cap (d^+A^{i_2})$. This shows now that there exists a subset R^{i_1, i_2} of $(d^+A^{i_1}) \cap (d^+A^{i_2})$ such that $\mu^{i_1}(R^{i_1, i_2}) = \mu^{i_2}(R^{i_1, i_2}) = 0$ and there is a function $c_{i_1, i_2}\colon [(d^+A^{i_1})(d^+A^{i_2})] \setminus R^{i_1, i_2} \to (0, \infty)$ such that $a_j^{i_1}(y, k) = c_{i_1, i_2}(y, k) a_j^{i_2}(y, k)$ $(j=0, \dots, n-1)$ for any $(y, k) \in \mathrm{dom}\, c_{i_1, i_2}$. This is equivalent to the condition that the roots of the polynomials $\sum_{j=0}^{n-1} a_j^{i_1}(y, k)\varrho^j$ and $\sum_{j=0}^{n-1} a_j^{i_2}(y, k)\varrho^j$ are the same with the same multiplicity (for all $(y, k) \in \mathrm{dom}\, c_{i_1, i_2}$). Let then $(y, k) \in (d^+A) \setminus \bigcup_{i_1, i_2 = 1}^{\infty} R^{i_1, i_2}$ be arbitrarily fixed. Now Remark b) after Lemma 3 implies that the zeros of the polynomial $\sum_{j=0}^{n-1} a_j(y, k)\varrho^j$ are real and lie outside the interval $(0, \mathrm{reach}\, A^i)$ for any i, such that $(y, k) \in d^+A^i$. Therefore $p(.)$ cannot have any zero inside $\bigcup \{(0, \mathrm{reach}\, A^i)\colon (y, k) \in d^+A^i\} = (0, \sup\{\mathrm{reach}\, A^i\colon (y, k) \in d^+A^i\}) \supset$ $\supset (0, h^A(y, k))$. Since by (8′) we have $\mu((d^+A) \cap \bigcup_{i_1, i_2 = 1}^{\infty} R^{i_1, i_2} = 0$, the previous statement holds for μ-almost every $(y, k) \in d^+A$.

References

[1] H. Federer, Curvature measures, *Trans. American Math. Soc.*, **93** (1959), 418—491.

[2] G. Bouligand, *Introduction à la géométrie infinitésimale directe*, Vuibert (Paris, 1932).

[3] H. Federer, *Geometric measure theory*, Springer (New York, 1969).

[4] M. Kneser, Über den Rand von Parallelkörpern, *Math. Nachr.*, **5** (1951), 251—258.

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