A short proof of the fact that biholomorphic automorphisms of the unit ball in certain $L^p$ spaces are linear

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1. As a consequence of his investigations on the Carathéodory and Kobayashi distances on domains in locally convex vector spaces, E. Vesentini [1] proved that biholomorphic automorphisms of the unit ball* of $L^1(\Omega, \mu)$ are all linear, whenever the underlying measure space $(\Omega, \mu)$ is not a unique atom. In this paper we shall provide a quite different approach to the problem which applies to $L^p(\Omega, \mu)$ as well, for every $p \in [1, \infty)$.

Theorem. Let $(\Omega, \mu)$ be a measure space having two disjoint subsets $\Omega', \Omega''$ such that $0 < \mu(\Omega'), \mu(\Omega'') < \infty$. Then for any $p \in [1, \infty) \setminus \{2\}$, all biholomorphic automorphisms of the unit ball of $L^p(\Omega, \mu)$ are linear.

Our method is based on a result of W. Kaup and H. Upmeier [2] concerning $\text{Aut} B(E)$ for general Banach spaces $E$. Here we present a direct proof of the theorem, which may have interest because of its extreme brevity. However, we remark that one can also determine the general algebraic form of an element from $\text{Aut} B(L^1(\Omega, \mu))$ in a similar way.

2. First we prove a lemma. To this end, let $E$ denote an arbitrarily fixed Banach space with norm $\|\cdot\|$, $E^*$ the dual of $E$ endowed with the norm $\|\cdot\|_*$.

Lemma. $\text{Aut} B(E)$ contains only linear mappings if and only if the relation

\[ \langle q(x, x), \varphi \rangle = -\langle c, \varphi \rangle \quad \text{for all} \quad x \in E, \varphi \in E^* \quad \text{with} \quad \|x\| = \|\varphi\|_* = 1 = \langle x, \varphi \rangle \]

entails $c = 0$ whenever $c \in E$ and $q$ is a bilinear form from $E \times E$ into $E$.

Received December 20, 1978.

*) In general, if $B(E)$ denotes the open unit ball of a Banach space $E$ then the biholomorphic automorphisms of $B(E)$ are defined as those one-to-one mappings of $B(E)$ onto itself whose Fréchet derivative exists at every point $x \in B(E)$ as an invertible operator. We shall denote the group formed by the biholomorphic automorphisms of $B(E)$ by $\text{Aut} B(E)$. 
Proof. According to [2, p. 131], there can be found a subspace \( V \) in \( E \) and a conjugate-linear mapping \( \varphi \mapsto g_\varphi \) from \( V \) into the space of the (continuous) \( E \)-bilinear forms such that \( \text{Aut}(D) \) is generated by the group \( G_0 \) of the surjective linear isometries of \( E \) onto itself any by the images under the exponential map of the vector fields \( \left( v + g_\varphi(z, z) \right) \frac{\partial}{\partial z} \) \( (v \in V) \). Thus, for \( \text{Aut} B(E) = G_0 \) it is necessary and sufficient that there exist a \( c \in E \setminus \{0\} \) and a bilinear form \( q : E \times E \to E \) such that the vector field \( (c + q(z, z)) \frac{\partial}{\partial z} \) be tangent to \( \partial B(E) \) (the boundary of \( B(E) \)), i.e.

\[
\text{Re} \langle c + q(z, z), \psi \rangle = 0 \quad \text{whenever} \quad \|z\| = \|\psi\|_* = 1 = \langle z, \psi \rangle.
\]

Suppose now that the vectors \( c, x \in E, \varphi \in E^* \) and the \( E \)-bilinear form \( q \) satisfy \( \|x\| = \|\varphi\|_* = 1 = \langle x, \varphi \rangle \) and (2). Then for all \( \lambda \in \mathbb{C} \) with \( |\lambda| = 1 \) we have \( \|\lambda x\| = \|\lambda \varphi\|_* = 1 = \langle \lambda x, \lambda \varphi \rangle \) whence \( 0 = \text{Re} \langle c + q(\lambda x, \lambda x), \lambda \varphi \rangle = \text{Re} \left[ \langle c, \varphi \rangle + \langle q(x, x), \varphi \rangle \right] \). Therefore \( \langle c, \varphi \rangle + \langle q(x, x), \varphi \rangle = 0 \) which completes the proof of the Lemma.

3. Now we shall proceed to the proof of the Theorem. Henceforth let \( p \in [1, \infty) \) be arbitrarily fixed and set \( E = L^p(\Omega, \mu) \). As usual we shall identify \( E^* \) with \( L^{p/(p-1)}(\Omega, \mu) \) and the pairing operation with \( \langle x, \varphi \rangle = \int \bar{x}(\xi) \cdot \varphi(\xi) \, d\mu(\xi) \) (for all \( x \in E \) and \( \varphi \in E^* \)), respectively.

For any \( x \in E \), let \( x \) denote the function \( \xi \mapsto x(\xi) \cdot |x(\xi)|^{p-2} \) (with the convention \( 0 \cdot 0^{p-2} = 0 \)). Observe that here

\[
\langle x^{*} \rangle \in E^*, \quad \|x^{*}\|_* = \|x\|^{p-1}, \quad \langle x, x^{*} \rangle = \|x\|^p \quad \text{for all} \quad x \in E.
\]

Then assume that the function \( x \in E \) and the \( E \)-bilinear form \( q \) satisfy (1). Applying (3) we see that

\[
\langle q(x, x), x^{*} \rangle = -\|x\|^2 \langle c, x^{*} \rangle \quad \text{for all} \quad x \in E.
\]

In particular, if \( F \) and \( G \) are any two disjoint subsets of \( \Omega \) such that \( 0 < \mu(F), \mu(G) < \infty \) then

\[
\int \Omega q(1_F + \lambda \cdot 1_G, 1_F + \lambda \cdot 1_G)(1_F + \lambda |\lambda|^{p-2} 1_G) \, d\mu =
\]

\[
= -\left( \mu(F) + |\lambda|^p \cdot \mu(G) \right) \int \Omega \bar{c}(1_F + \lambda \cdot |\lambda|^{p-2} 1_G) \, d\mu
\]

for all \( \lambda \in \mathbb{C} \). (For any \( \mu \)-measurable subset \( H \subset \Omega \) of finite \( \mu \)-measure, \( 1_H \) denotes the characteristic function of \( H \), considered as an element in \( E \).)
Thus, by setting
\[ \alpha_0 = \int_F \varrho (1_F, 1_F) \, d\mu, \quad \alpha_1 = \int_F \varrho (1_F, 1_G) + \varrho (1_G, 1_F) \, d\mu, \quad \alpha_2 = \int_F \varrho (1_G, 1_G) \, d\mu, \]
\[ \beta_0 = \int_G \varrho (1_F, 1_F) \, d\mu, \quad \beta_1 = \int_G \varrho (1_F, 1_G) + \varrho (1_G, 1_F) \, d\mu, \quad \beta_2 = \int_G \varrho (1_G, 1_G) \, d\mu, \]
\[ \mu_1 = \mu (F), \quad \mu_2 = \mu (G), \quad \gamma_1 = \int_F \tilde{c} \, d\mu, \quad \gamma_2 = \int_G \tilde{c} \, d\mu \]
we obtain
\[ \sum_{k=0}^3 \alpha_k \lambda^k + \bar{x} |\lambda|^{-2} \cdot \sum_{k=0}^3 \beta_k \lambda^k = -(\mu_1 + |\lambda|^p \mu_2)^{2/p} (\gamma_1 + \lambda \cdot |\lambda|^{-2} \gamma_2) \]
for all \( \lambda \in \mathbb{C} \). Therefore for any \( \varrho > 0 \) and \( \delta \in \mathbb{C} \) with \( |\delta| = 1 \),
\[ (\beta_0 \cdot \varrho^{p-1})^{\delta^{-1}} + (\alpha_0 + \beta_1 \cdot \varrho^p) + (\alpha_1 \cdot \varrho + \beta_2 \cdot \varrho^{p+1}) \delta + (\alpha_2 \cdot \varrho^2) \delta^2 = - (\mu_1 + \mu_2 \cdot \varrho^{p})^{2/p} (\gamma_1 + (\gamma_2 \cdot \varrho^{p-1}) \delta). \]
In particular, we have
\[ \alpha_0 + \beta_1 \cdot \varrho^p = -(\mu_1 + \mu_2 \cdot \varrho^{p})^{2/p} \gamma_1 \]
for all \( \varrho > 0 \).

Hence \( -\mu_2^{2/p} \cdot \gamma_1 \lim_{\varrho \to \infty} [-(\mu_1 + \mu_2 \cdot \varrho^{p})^{2/p} \varrho^{-2} - \gamma_2] = \lim_{\varrho \to \infty} (\alpha_0 + \beta_1 \cdot \varrho^p \cdot \varrho^{-2} \gamma_1). \) This is possible only if \( p = 2 \) or \( \gamma_1 = 0 \). Thus if \( p \neq 2 \) then by definition of \( \gamma_1 \) we have
\[ (4) \quad \int_F \tilde{c} \, d\mu = 0 \] whenever \( 0 < \mu (G) < \infty \) for some \( G \subset \Omega \setminus F \).

But (4) immediately implies \( c = 0 \) because of our assumption on the measure space \((\Omega, \mu)\). Thus, by the Lemma, \( B (E) \) admits in case \( p \neq 2 \) only linear biholomorphic automorphisms. Q.E.D.

References


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