Dedicated to Prof. L. Hatvani on the occasion of his 70-th birthday

On the holomorphic hull of generalized Reinhardt domains in spaces of continuous functions

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Abstract. We introduce concept of logarithmic convex hull for Reinhardt domains of continuous functions, and show that the holomorphic hull of a complete Reinhardt domain in $\mathcal{C}_0(\Omega)$ over a locally compact topological space contains the logarithmic convex hull in a natural manner.

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1. Introduction.

Throughout the whole paper, let Ω be an arbitrarily fixed locally compact topological space, and let $E := \mathcal{C}_0(\Omega)$ be the space of all complex valued continuous functions vanishing at infinity equipped with the spectral norm $\|f\| := \max|f|$ along with the standard notations $E_+ := \{f \in E : \ f \geq 0\}$, $E_0 := \{f \in E : \ \text{support}(f) \text{ compact}\}$. As a natural extension of the classical concept of Reinhardt domains in $\mathbb{C}^N \simeq \mathcal{C}_0(\{1, \ldots, N\})$, we say that a non-empty open connected set D in E is a *Reinhardt domain* (*R-domain* for short) if

$$f \in D \Longrightarrow \{g \in E : \ |g| = |f|\} \subset D.$$

Extending further the familiar finite dimensional concepts, we say that a R-domain D is *complete* if $f \in D \Longrightarrow \{g \in E : \ |g| \leq |f|\} \subset D$, and a complete R-domain D is *logarithmic convex* (*log-convex* for short), if

$$f_1, f_2 \in D, \ \lambda \in (0,1) \quad \Longrightarrow \quad \left\{g \in E : \ |g| \leq |f_1|^\lambda |f_2|^{1-\lambda}\right\} \subset D.$$

We can regard the *complete*- resp. *log-convex hull* of a R-domain in a natural manner as the intersection of the covering complete resp. log-convex domains. They are open sets as being the unions of the sets $\{g \in E : \ |g| < |f|\}$ $(f \in D)$ resp. $\left\{g \in E : \ |g| < \prod_{k=1}^n |f|^{\lambda_k}\right\}$ with $f_1, \ldots, f_n \in D$, $0 \leq \lambda_1, \ldots, \lambda_n$, $\sum_k \lambda_k = 1$ $(n = 1, 2, \ldots)$. Though it is tempting to expect that the classical arguments apply in the infinite dimensional setting, it is worth to notice that even the natural infinite dimensional version of Sunada's Theorem fails for general complete R-domains [3].

It is well-known [2] that the holomorphic hull of any circular domain $S \subset E$ ($0 \in S = e^{it}S$ for all $t \in \mathbb{R}$) is schlicht. That is there exists a unique circular domain $\widehat{S} \subset E$ such that $S \subset \widehat{S}$, every holomorphic function $f : S \to \mathbb{C}$ extends holomorphically to $\widehat{S}$ but for any boundary point $p \in \partial\widehat{S}$ there is a holomorphic function $\widehat{f}_p : \widehat{S} \to \mathbb{C}$ without holomorphic extension to any domain $U \supset \widehat{S} \cup \{p\}$ in E. It is also well-known [2,1] that, in the above sense, the holomorphic hull of a Reinhardt domain D in $\mathbb{C}^N$ is a log-convex complete Reinhardt domain, namely the log-convex hull of D (the intersection of all log-convex R-domains containing D). During the open discussion of my Thesis [4] for the degree DSc of Hungarian Academy of Sciences, Prof. R. Szőke has raised the question if logarithmic convexity can be generalized to complete R-domain with analogous conclusions to the mentioned ones in finite dimensions. The aim of this short note is to give the following partial answer.

1.1 Theorem. *If D is a complete R-domain in $E := \mathcal{C}_0(\Omega)$ then every holomorphic function $D \to \mathbb{C}$ extends holomorphically to the log-convex hull $\widehat{D}$ of D.*

1.2 Remark. In contrast with the finite dimensional case $E = \mathbb{C}^N$ where $\widehat{D}$ can be represented as the domain of absolute convergence of some suitable classical power series $\sum_{j_1,\dots,j_N=0}^{\infty} \alpha_{j_1,\dots,j_N} z_1^{j_1} \cdots z_N^{j_N}$, entailing $\widehat{D}$ being a complete R-domain, a similar argument is not available in infinite dimensions, even in the case $\Omega = [0,1]$. To establish that the holomorphic hull of any complete R-domain is a log-covex R-domain, on the basis of Theorem 1.1 it only suffices to prove the following conjecture.

1.3 Conjecture. *If* $\dim(E) > 1$ *then the holomorphic hull of any R-domain $D \subset E$ is a complete R-domain.*

2. Auxiliary results

Henceforth Ω denotes an arbitrarily fixed locally compact topological space and D is a complete R-domain in $E := \mathcal{C}_0(\Omega)$. We assume without loss of generality that $\dim(E) = \infty$.

2.1. Lemma. *Let $u_1, \dots, u_N$ be linearly independent functions in E. Suppose $H + U \subset D$ where $H := \{ \sum_k \zeta_k u_k : (\zeta_1, \dots, \zeta_N) \in \Delta \}$ with some classical complete Reinhardt domain $\Delta \subset \mathbb{C}^N$ and U is a convex neighborhood of the origin in E. Then the holomorphic hull $\widehat{D}$ regarded as a circular domain in E contains the figure $\widetilde{H} + U$ where $\widetilde{H} := \{ \sum_k \zeta_k u_k : (\zeta_1, \dots, \zeta_N) \in \widetilde{\Delta} \}$ with the log-convex hull $\widetilde{\Delta}$ of Δ.*

Proof. It suffices to see that any fuction $\phi \in \mathrm{Hol}(D, \mathbb{C})$ admits a holomorphic extension to $D \cup [\widetilde{H} + U]$. It is well-known [1] that $\widetilde{\Delta}$ coincides with the Hartogs convex hull of Δ. Recall that the holomorphic extension of a function from a Hartogs figure can be constructed by the aid of integral means over compact circles lying in the figure. Hence, for every point $\xi \in \widetilde{\Delta}$, we can find a complex Baire measure on H with bounded total variation and compact support such that $\widetilde{\varphi}(\xi) = \int \varphi \, d\mu_\xi$ whenever we have $\varphi \in \mathrm{Hol}(\Delta, \mathbb{C})$ with the holomorphic extension $\widetilde{\varphi} \in \mathrm{Hol}(\widetilde{\Delta}, \mathbb{C})$. As a consequence, given any point $x \in \widetilde{H}$, there exists a compactly supported complex Baire measure μ_x of bounded total variation on Δ such that

$$\widetilde{\psi}(x) = \int \psi \; d\mu_x \quad \text{if} \;\; x \in \widetilde{H}, \;\; \psi \in \mathrm{Hol}(H, \mathbb{C}), \;\; \widetilde{\psi} \in \mathrm{Hol}(\widetilde{H}, \mathbb{C}), \psi = \widetilde{\psi}\big|_H$$

holds for the holomorphic extensions on the finite dimensional complex linear subspace $L := \big\{ \sum_k \zeta_k u_k : \;\zeta_1, \ldots, \zeta_N \in \mathbb{C} \big\}$ of E. Given any $\phi \in \mathrm{Hol}(D, \mathbb{C})$ with a complementary subspace F such that $E = L \oplus F$, the function

$$\widetilde{\phi}_F(x+u) := \int_{y \in H} \phi(y+u) \; d\mu_x(y) \qquad (x \in \widetilde{H}, \;\; u \in F \cap U)$$

is a well-defined extension of the restriction $\phi|_{H+[F \cap U]}$ to the set $\widetilde{H} + [F \cap U]$. Notice that both $H + [F \cap U]$ and $\widetilde{H} + [F \cap U]$ are open subsets in E. Observe that $\widetilde{\phi}_F$ is holomorphic along the slices $\widetilde{H}{+}u$ $(u{\in}F{\cap}U)$ and $x{+}[F{\cap}U]$ $(x{\in}\widetilde{H})$, respectively. Indeed, $\widetilde{\phi}_F(x{+}u) = \widetilde{\psi}$ with the holomorphic function $\psi(x) := \phi(x+u)$ $(x \in H)$, that is $\widetilde{\phi}_F|_{\widetilde{H}+u}$ is a holomorphic extension of $\phi|_{H+u}$. On the other hand, we have $\zeta^{-1}[\widetilde{\phi}_F(x+u+\zeta v) - \widetilde{\phi}_F(x+u)] = \int_{y \in H} \zeta^{-1}[\phi(y+u+\zeta v) - \phi(y+u)] d\mu_x(y) \to \int_{y \in H} \phi'(y+u)v \; d\mu_x(y)$ entailing the Fréchet differentiability of $\widetilde{\phi}_F$ in the directions $v \in F$. Since continuous partially holomorphic function are holomorphic, we conclude $\widetilde{\phi} \in \mathrm{Hol}(\widetilde{H} + [F \cap U], \mathbb{C})$. The arbitrariness of the choice of ϕ from $\mathrm{Hol}(H + [F \cap U], \mathbb{C})$ along with the arbitrariness of the complementary spaces F implies that

$$\widehat{D} \supset \bigcup_F \big(\widetilde{H} + [F \cap U]\big) = \widetilde{H} + \bigcup_F [F \cap U] = \widetilde{H} + \big(\{0\} \cup [U \setminus L]\big) \; .$$

We complete the proof by recalling that, by Riemann's Second Extension Theorem [2, 7.14, p.19], any holomorphic function from $U \setminus L$ admits a holomorphixc extension to U.

2.2 Lemma. *Given any function $f \in D$, there exists $\varepsilon > 0$ such that we have $h \in D$ whenever $h \in E$ and $|h| \le |f| + \varepsilon$.*

Proof. By the completeness of D, we also have $|f| \in D$. Since D is open in E, we can find a radius $\varepsilon > 0$ with $D \supset \{g \in E : \; \|(g - |f|)\| \le \varepsilon\}$. Thus we have $|h| \in D$ and hence also $h \in D$ for all functions $h \in E$ with $|h| \le |f| + \varepsilon$.

2.3 Lemma. *Let $f \in E_0$, $\varepsilon > 0$ and let $\{u_1, \ldots, u_N\} \subset E_+$ be a partition of unity over* $\mathrm{support}(f)$ *such that* $\mathrm{diam} f\big(\mathrm{support}(u_k)\big) \le \varepsilon$ $(k = 1, \ldots, N)$ *and* $1 \ge \sum_k u_k \ge 1_{\mathrm{support}(f)}$. *Assume* $\omega_k \in \mathrm{support}(u_k)$ $(k = 1, \ldots, N)$. *Then* $\big\|f - \sum_k f(\omega_k) u_k\big\| \le \varepsilon$.

Proof. For each point $\omega \in \Omega$ define $I_\omega := \{k : \;\; u_k(\omega) > 0\}$. Observe that $k \in I_\omega \Rightarrow \big|f(\omega) - f(\omega_k)\big| \le \varepsilon$. Consider any point $\omega \in \mathrm{support}(f)$. By assumption $1 = \sum_{k \in I_\omega} u_k(\omega)$, implying

$$\Big|f(\omega) - \sum_k f(\omega_k) u_k(\omega)\Big| = \big| \sum_{k \in I_\omega} \big[f(\omega) - f(\omega_k)\big] u_k(\omega)\big| \le \sum_{k \in I_\omega} \big|f(\omega) - f(\omega_k)\big| u_k(\omega) \le \varepsilon.$$

Considering any point $\omega \in \Omega \setminus \text{support}(f)$, we have $f(\omega) = 0$, $1 \geq \sum_{k \in I_\omega} u_k(\omega) \geq 0$ and $\big|f(\omega_k)\big| \leq \varepsilon$, implying

$$\Big|f(\omega) - \sum_k f(\omega_k) u_k(\omega)\Big| = \big|\sum_{k \in I_\omega} f(\omega_k) u_k(\omega)\big| \leq \varepsilon \sum_{k \in I_\omega} u_k(\omega) \leq \varepsilon.$$

2.4 Lemma. *Assume* $f_1, \ldots, f_m \in E_+$, $g \in E$, $\lambda_1, \ldots, \lambda_m > 0, \sum_j \lambda_j = 1$, *and* $|g| \leq \prod_j f_j^{\lambda_j}$. *Let* $\varepsilon > 0$ *be arbitrarily given. Then we can find* $\widetilde{f}_1, \ldots, \widetilde{f}_m, \widetilde{g} \in E_0$ *such that, for suitable functions* $\varphi_1, \ldots, \varphi_m, \gamma : \Omega \to \mathbb{R}_+$ *we have* $\widetilde{f}_j = \varphi_j g$, $\widetilde{g} = \gamma g$ *and*

$$|\widetilde{f}_1| \leq f_1, \ldots, |\widetilde{f}_m| \leq f_m, \quad \|\widetilde{g} - g\| \leq \varepsilon, \quad |\widetilde{g}| \leq \prod_{j=1}^m |\widetilde{f}_j|^{\lambda_j}.$$

Proof. By using the continuous transformation $\Pi : \mathbb{C} \to \mathbb{C}$, $\Pi\big(\rho e^{i\theta}\big) := [\rho - \varepsilon]_+ e^{i\theta}$ $(\rho \geq 0 \leq \theta < 2\pi)$ the choice

$$\widetilde{g} := \Pi \circ g, \quad \widetilde{f}_j := f_j \widetilde{g} / \max\{|g|, \varepsilon\} \quad (j = 1, \ldots, m)$$

suits our requirements. Indeed, then all the functions $\widetilde{f}_j, \widetilde{g}$ are continuous as being compositions and products of continuous maps. We also have $\text{support}(\widetilde{f}_j) \subset \text{support}(\widetilde{g}) \subset \{\omega : |g(\omega)| \geq \varepsilon\}$ compact$\subset \Omega$, whence $\widetilde{f}_1, \ldots, \widetilde{f}_m, \widetilde{g} \in E_0$. Finally

$$\prod_{j=1}^m |\widetilde{f}_j|^{\lambda_j} = \frac{|\widetilde{g}|}{\max\{|g|, \varepsilon\}} \prod_{j=1}^m f_j^{\lambda_j} \geq \frac{|g|\,|\widetilde{g}|}{\max\{|g|, \varepsilon\}} \geq |\widetilde{g}|.$$

3. Proof of Theorem 1.1.

According to Lemma 2.1, it suffices to see only that in case of

$$0 \leq f_1, \ldots, f_m \in D, \quad \lambda_1, \ldots, \lambda_m > 0, \quad \sum_j \lambda_j = 1, \quad g \in E \quad \text{with} \quad |g| \leq \prod_j f_j^{\lambda_j},$$

there is a family $u_1, \ldots, u_N \in E_+$ of functions, along with a classical complete Reinhardt domain $\Delta \subset \mathbb{C}^N$ such that for some constant $\delta > 0$ we have

$$\text{(3.1}a) \qquad \Big\{v + \sum_k \zeta_k u_k : \ (\zeta_1, \ldots, \zeta_N) \in \Delta, \ v \in E, \ \|v\| < \delta\Big\} \subset D \ ,$$

$$\text{(3.1}b) \qquad g \in \Big\{v + \sum_k \zeta_k u_k : \ (\zeta_1, \ldots, \zeta_N) \in \widetilde{\Delta}, \ v \in E, \ \|v\| < \delta\Big\}.$$

In view of Lemma 2.2, we can fix a constant $\varepsilon > 0$ such that

$$(3.2)\qquad \bigcup_{j=1}^{m} \big\{f \in E:\ |f| \le f_j + 4\varepsilon\big\} \subset D.$$

According to Lemma 2.4 we can also fix the functions $\widetilde{f}_1, \dots, \widetilde{f}_m, \widetilde{g} \in E_0$ satisfying the conditions

$$|\widetilde{f}_j| \le f_j \quad (j=1\dots,m), \quad \|\widetilde{g} - g\| < \varepsilon, \quad |\widetilde{g}| \le \prod_{j=1}^{m} |\widetilde{f}_j|^{\lambda_j},$$

moreover $\operatorname{support}(\widetilde{f}_j) = \operatorname{support}(\widetilde{g})$ $(j=1,\dots,m)$ [1] and $\widetilde{f}_j(\omega)/\widetilde{g}(\omega) > 0$ whenever $\omega \in \{\xi:\ |g(\xi)| > \varepsilon_1\} = \{\xi:\ \widetilde{g}(\xi) \ne 0\}$. We can find a finite open covering $\mathbf{G} := \{G_1, \dots, G_M\}$ for the compact set $\operatorname{support}(\widetilde{g})$ such that the functions $|\widetilde{f}_1|, \dots, |\widetilde{f}_m|, \widetilde{g}$ and $\widetilde{g}$ map each covering term G_k into a range of diameter $< \varepsilon$. The covering $\mathbf{G}$ admits a finite subordinated family of functions[2] $0 \le u_1, \dots, u_N \in E_0$ forming a partition of unity over $\operatorname{support}(\widetilde{g})$. In particular $\sum_k u_k(\omega) = 1$ at any point $\omega \in \operatorname{support}(\widetilde{g})\big)$ and $\operatorname{diam}\big(f(\operatorname{support}(u_k))\big) < \varepsilon$ for $f = |\widetilde{f}_1|, \dots, |\widetilde{f}_m|, \widetilde{g}$. Let us fix an arbitrary family $\omega_1 \in \operatorname{support}(u_1), \dots, \omega_N \in \operatorname{support}(u_N)$ of points and define

$$Pf := \sum_{k=1}^{N} f(\omega_k) u_k \quad (f \in E), \quad \Delta := \bigcup_{j=1}^{m} \bigcap_{k=1}^{N} \Big\{(\zeta_1, \dots, \zeta_N) \in \mathbb{C}^N:\ |\zeta_k| < \big|\widetilde{f}_j(\omega_k)\big| + \varepsilon\Big\}.$$

From Lemma 2.3 we deduce that

$$(3.3)\qquad \|Pf - f\| < \varepsilon \quad \text{for} \quad f = |\widetilde{f}_1|, \dots, |\widetilde{f}_N|, \widetilde{g}.$$

We show that this choice for Δ with $\delta := 2\varepsilon$ suits the requirements (3.1a,b).

Proof of (3.1b): Since $\|g - \widetilde{g}\|, \|P\widetilde{g} - \widetilde{g}\| < \varepsilon$, by the triangle inequality it follows $\|g - P\widetilde{g}\| < 2\varepsilon = \delta$. Thus $g = \sum_{k=1}^{N} \zeta_k u_k + v$ for the tuple $\boldsymbol{\zeta} := \big(\widetilde{g}(\omega_1), \dots, \widetilde{g}(\omega_N)\big)$ and a function $v \in E$ with $\|v\| < \delta$. Given any index $j \in \{1, \dots, N\}$, we have $\big(\widetilde{f}_j(\omega_1), \dots, \widetilde{f}_j(\omega_N)\big) \in \Delta$. Since the holomorphic hull $\widetilde{\Delta}$ of the classical Reinhardt domain Δ coincides with its log-convex hull, the relation $|\widetilde{g}| \le \prod_{j=1}^{m} |\widetilde{f}_j|^{\lambda_j}$ implies $\boldsymbol{\zeta} \in \widetilde{\Delta}$.

Proof of (3.1a): Consider any function h belonging to the set on the left hand side in (3.1a). Thus for a suitable tuple $(\zeta_1, \dots, \zeta_N) \in \Delta$ we have $\big\|h - \sum_{k=1}^{N} \zeta_k u_k\big\| < \delta = 2\varepsilon$. By the definition of the set Δ, for some index $j \in \{1, \dots, m\}$ we have $|\zeta_k| < |\widetilde{f}_j(\omega_k)| + \varepsilon \le f_j(\omega_k) + \varepsilon$ $(k = 1, \dots, N)$. According to (3.3), $\big\|P|\widetilde{f}_j| - |\widetilde{f}_j|\big\| < \varepsilon$ and hence $P|\widetilde{f}_j| < |\widetilde{f}_j| + \varepsilon \le f_j + \varepsilon$. It follows

$$\big|h\big| < \Big|\sum_{k=1}^{M} \zeta_k u_k\Big| + 2\varepsilon \le \sum_{k=1}^{N} |\zeta_k| u_k + 2\varepsilon \le \sum_{k=1}^{N} |\widetilde{f}_j(\omega_k)| + 3\varepsilon = P|\widetilde{f}_j| + 3\varepsilon < f_j + 4\varepsilon.$$

In view of (3.2), this entails (3.1a).

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[1] By construction $\widetilde{f}_j = \widetilde{g}\big[f_j/\max\{|g|, \varepsilon\}\big]$, implying $\operatorname{supp}(\widetilde{f}_j) \subset \operatorname{supp}(\widetilde{g}$. The relations $\lambda_j > 0$ and $|\widetilde{g}| \le |\widetilde{f}_j|^{\lambda_j} \prod_{k:k\ne j} |\widetilde{f}_k|^{\lambda_k}$ entail $\operatorname{supp}(\widetilde{g}) \subset \operatorname{supp}(\widetilde{f}_j)$

[2] For any $j \in \{1, \dots, N\}$, there exists $G \in \mathbf{G}$ with $\operatorname{support}(u_j) \subset G$.

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