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Holomorphic invariants for continuous bounded symmetric Reinhardt domains

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Abstract. We study the analytic classification and the continuity properties of the holomorphic invariants of bounded continuous symmetric Reinhardt domains.

0. Introduction

Let Ω be a locally compact Hausdorff topological space, and denote by $E := \mathcal{C}_0(\Omega)$ the algebra of all continuous complex valued functions that vanish at infinity on Ω, endowed with the norm $\|\cdot\|_\infty$ of the supremum. A *continuous Reinhardt domain* is a domain $D \subset E$ such that

$$(f \in D,\ \ g \in E,\ \ |g| \leq |f|\,) \Longrightarrow g \in D.$$

In particular D is circular and $\theta \in D$. Recently one has shown [7] that *symmetric* continuous Reinhardt domains are continuous mixtures of finite-dimensional Euclidean balls admitting richer structure in general than continuous product (described in [10], [11]) in the following sense. Given a bounded symmetric continuous

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Reinhardt domain D in $C_0(\Omega)$, there exists a partition $\{\Omega_i\colon i \in I\}$ of the underlying topological space Ω along with a positive valued function $m\colon \Omega \to \mathbb{R}$ such that

$$D = \Big\{ f \in C_0(\Omega)\colon \sum_{\eta\in\Omega_i} m(\eta)|f(\eta)|^2 < 1 \quad \text{for all } i \in I \Big\} \tag{0.1}$$

and $\sup_{i\in I} \mathbf{card}(\Omega_i) < \infty$, $0 < \inf m \le \sup m < \infty$ (where **card** abbreviates cardinality).

In this paper we are going to deal with the question (remained open in [7]) which couples $(\{\Omega_i\colon i \in I\}, m)$ give rise to symmetric continuous Reinhardt domains in the form (0.1). We answer this question completely by a theorem stating among other less elementary equivalent formulations that $(\{\Omega_i\colon i \in I\}, m)$ corresponds to a symmetric continuous Reinhardt domain if and only if the set valued function $\omega \mapsto S(\omega) \cup \{\infty\}$ with

$$S(\omega) := [\text{the unique } \Omega_i \text{ with } \omega \in \Omega_i] \tag{0.2}$$

is continuous in Hausdorff sense where $\Omega \cup \{\infty\}$ is the one point compactification of Ω and the function m has the following continuity property: given any point $\omega \in \Omega$ and $\varepsilon > 0$, for every closed neighborhood U of ω with $U \cap S(\omega) = \{\omega\}$ there exists a neighborhood V of ω such that $\left| m(\omega) - \sum_{\eta\in S(\theta)\cap U} m(\eta) \right| < \varepsilon$ whenever $\theta \in V$.

On the basis of this result we attack the problem of the linear (and hence even *holomorphic* [4]) equivalence of continuous Reinhardt domains. The isomorphism of the Jordan structures associated with two linearly equivalent domains $D, \widetilde{D}$ corresponding to the couples $(\{\Omega_i\colon i \in I\}, m)$ resp. $(\{\widetilde{\Omega}_i\colon i \in \widetilde{I}\}, \widetilde{m})$ implies easily that any linear mapping $L\colon C_0(\Omega) \leftrightarrow C_0(\widetilde{\Omega})$ with $LD = \widetilde{D}$ has the matrix form

$$Lf(\widetilde{\omega}) = \sum_{\omega\in\Omega} u(\widetilde{\omega}, \omega) f(\omega) \qquad \text{for } f \in C_0(\Omega)$$

where there exists a bijection $\pi\colon I \leftrightarrow \widetilde{I}$ such that $\mathbf{card}(\widetilde{\Omega}_{\pi(i)}) = \mathbf{card}(\Omega_i)$ for all $i \in I$ and $u(\widetilde{\omega}, \omega) = 0$ whenever $\widetilde{\omega} \in \widetilde{\Omega}_{\pi(i)}$ but $\omega \notin \Omega_i$. We give a precise elementary description for the continuity properties of the matrix entry functions $u\colon \widetilde{\Omega} \times \Omega \to \mathbb{C}$ corresponding to linear isomorphisms of symmetric continuous Reinhardt domains. Despite this positive result, for the time being we are far from the description of all complete sets of holomorphic invariants of symmetric continuous Reinhardt domains. E.g. topologically non-equivalent locally compact spaces may admit linearly equivalent symmetric continuous Reinhardt domains in their C_0 function spaces while this is not possible with symmetric continuous products of discs.

1. Admissibility of partitions and weight functions

Throughout this section Ω denotes a locally compact topological Hausdorff space with the one point compactification $\Omega \cup \{\infty\}$. As usually, we write $\mathcal{C}_0(\Omega)$ for the algebra of all complex valued continuous functions $\Omega \to \mathbb{C}$ vanishing at ∞ and we regard ∞ as an isolated point if Ω is compact already. We equip the family of all compact subsets $\mathcal{K}(\Omega \cup \{\infty\})$ of $\Omega \cup \{\infty\}$ with the *Hausdorff topology*. For $K \in \mathcal{K}(\Omega \cup \{\infty\})$, a subfamily $\mathcal{U} \subset \mathcal{K}(\Omega \cup \{\infty\})$ is a neighborhood of $K \in \mathcal{K}(\Omega \cup \{\infty\})$ if for some finite open covering $\{U_1, \ldots, U_N\}$ of K in $\Omega \cup \{\infty\}$ we have

$$\mathcal{U} \subset \Big\{ S \subset \Omega \cup \{\infty\} \colon S \subset \bigcup_{k=1}^{N} U_k \,, \quad U_k \cap S \neq \emptyset \quad (k = 1, \ldots, N) \Big\} .$$

It is easy to see that in the case of metrizability of $\Omega \cup \{\infty\}$ by a metric ϱ, the Hausdorff topology of $\mathcal{K}(\Omega \cup \{\infty\})$ is the same as the one generated by the Hausdorff distance $d(S_1, S_2) := \max\{\sup_{\omega_1 \in S_1} \inf_{\omega_2 \in S_2} \varrho(\omega_1, \omega_2), \sup_{\omega_2 \in S_2} \inf_{\omega_1 \in S_1} \varrho(\omega_1, \omega_2)\}$.

Definition 1.1. We say that a mapping $S \colon \Omega \to \{\text{subsets of } \Omega\}$ is a *partition mapping* on Ω if there is a partition $\{\Omega_i \colon i \in I\}$ of Ω such that 0.2 holds. We say that the couple (S, m) of a partition mapping S on Ω and a positive function $m \colon \Omega \to \mathbb{R}$ is *admissible* if the set (0.1) (where $\{\Omega_i \colon i \in I\} = S(\Omega)$) is a bounded symmetric continuous Reinhardt domain in $\mathcal{C}_0(\Omega)$.

Recall [7] that in case (S, m) is admissible and $\{f \in \mathcal{C}_0(\Omega) \colon \max|f| < \varepsilon\} \subset D \subset \{f \in \mathcal{C}_0(\Omega) \colon \max|f| < M\}$ for the domain D in (0.1), we necessarily have $\mathbf{card}(S(\omega)) \leq M^2/\varepsilon^2$ $(\omega \in \Omega)$ and $M^{-2} \leq m \leq \varepsilon^{-2}$.

Theorem 1.2. *Assume $m \colon \Omega \to \mathbb{R}$ is a function such that $0 < \inf m \leq \sup m < \infty$ and S is a partition mapping on Ω such that $\sup_{\omega \in \Omega} \mathbf{card}(S(\omega)) < \infty$. Then the following statements are equivalent:*

(i) *the couple (S, m) is admissible;*

(ii) *for all $f \in \mathcal{C}_0(\Omega)$, the function $\omega \mapsto \sum_{\eta \in S(\omega)} m(\eta)|f(\eta)|^2$ is continuous;*

(iii) *with the measures $\delta_\eta(X) := \mathbf{card}(X \cap \{\eta\})$, the mapping $\omega \mapsto \mu_\omega := \sum_{\eta \in S(\omega)} m(\eta)\delta_\eta$ is weak*-continuous as a function $\Omega \to \mathcal{C}_0(\Omega)'$;*

(iv) *the mapping $\omega \mapsto S(\omega) \cup \{\infty\}$ is Hausdorff-continuous and for every (ω, U, ε) where $\omega \in \Omega$, $\varepsilon > 0$ and U is a closed neighborhood of ω in Ω such that $U \cap S(\omega) = \{\omega\}$ there exists a neighborhood V of ω in Ω such that*

$$\Big| m(\omega) - \sum_{\eta \in U \cap S(\theta)} m(\eta) \Big| < \varepsilon$$

whenever $\theta \in V$.

Proof. The equivalence (i) $\Leftrightarrow$ (ii) is shown in [7] Theorem 2. Notice that, in terms of the operations

$$Qf(\omega) := \sum_{\eta \in S(\omega)} m(\eta)|f(\eta)|^2 , \quad Af(\omega) := \int_\Omega f \, d\mu_\omega = \sum_{\eta \in S(\omega)} m(\eta) f(\eta) ,$$

(ii) means $Q(\mathcal{C}_0(\Omega)) \subset \mathcal{C}(\Omega)$ while (iii) means $A(\mathcal{C}_0(\Omega)) \subset \mathcal{C}(\Omega)$. Hence the equivalence (ii) $\Leftrightarrow$ (iii) is immediate since $Af = \sum_{k=1}^4 i^k Q(\sqrt{[Re(i^{-k}f)]_+})$ and $Qf = A(|f|^2)$ for all $f \in \mathcal{C}_0(\Omega)$.

Proof of (iii) $\Rightarrow$ (iv). Assume (iii) and consider any point $\omega \in \Omega$. Then we can write $S(\omega) \cap \{\infty\} = \{\omega_1, \dots, \omega_N\}$ with $N :=$ **card** $S(\omega) + 1$ where $\omega_1 = \omega$ and $\omega_N = \infty$. To establish (iv), it suffices to show

(iv$_\omega$) *for any $\varepsilon > 0$ and for any disjoint open sets $U_1, \dots, U_N \subset \Omega \cup \{\infty\}$ with $\omega_k \in U_k$ $(k = 1, \dots, N)$ there exists a neighborhood $V \subset \Omega$ of the point ω such that $S(\theta) \subset \bigcup_{k=1}^N U_k$ and $\left| m(\omega_k) - \sum_{\eta \in S(\theta) \cap U_k} m(\eta) \right| < \varepsilon$ $(k = 1, \dots, N-1)$ for the points $\theta \in V$.*

Let $U_1, \dots, U_N$ be given disjoint open neighborhoods of the points ω_k. Unless we have the trivial case $\Omega = S(\omega)$, we may assume that the compact set $K := \Omega \setminus \bigcup_{k=1}^N U_k$ is not empty and there exists a function $g \in \mathcal{C}_0(\Omega)$ such that $1 = g(K) \geq g \geq 0 = g(S(\omega))$. By the assumption (iii), the function Ag is continuous. Since $Ag(\omega) = 0$, there is a neighborhood W of ω with $0 \leq Ag(W) < \inf m$. On the other hand, $Ag(\theta) = \sum_{\eta \in S(\theta)} m(\eta) g(\eta) \geq \inf m$ if $S(\theta) \cap K \neq \emptyset$. Therefore $S(\theta) \cap K = \emptyset$ that is $S(\theta) \subset \bigcup_{k=1}^N U_k$ for the points $\theta \in W$. For $k = 1, \dots, N-1$ let $\widetilde{U}_k$ be a compact neighborhood of the point ω_k such that $\widetilde{U}_k \subset U_k$. Since the space Ω is locally compact, such a system $\widetilde{U}_1, \dots, \widetilde{U}_{N-1}$ exists. Moreover, we can find functions $f_k \in \mathcal{C}_0(\Omega)$ $(k = 1, \dots, N)$ such that $1 = f(\widetilde{U}_k) \geq f \geq 0 = f_K(\Omega \setminus U_k)$. By the previous argument, there exists a neighborhood $\widetilde{W} \subset W$ of ω such that $S(\theta) \subset U_N \cup \bigcap_{k=1}^{N-1} \widetilde{U}_k$ whenever $\theta \in \widetilde{W}$. Observe that simply

$$Af_k(\theta) = \sum_{\eta \in S(\theta) \cap \widetilde{U}_k} m(\eta) = \sum_{\eta \in S(\theta) \cap U_k} m(\eta) \qquad \text{for } \theta \in \widetilde{W}.$$

By the continuity of Af_k at the point ω guaranteed by assumption (iii), we can find a neighborhood $V_k \subset \widetilde{W}$ of ω with

$$\varepsilon > |Af_k(\theta) - Af_k(\omega)| = \Big| \sum_{\eta \in S(\theta) \cap U_k} m(\eta) - m(\omega) \Big|$$

for the points $\theta \in V_k$. Therefore the choice $V := \bigcup_{k=1}^{N-1} V_k$ suits in the proof of (iii) $\Rightarrow$ (iv).

Proof of (iv) $\Rightarrow$ (iii). Assume (iv) and fix any point $\omega \in \Omega$ along with a function $f \in \mathcal{C}_0(\Omega)$. We have to establish the continuity of Af at ω. Let us write again $N := \mathbf{card}\ S(\omega) + 1$ and $S(\omega) \cup \{\infty\} = \{\omega_1, \dots, \omega_N\}$ with $\omega = \omega_1$ and $\infty = \omega_N$. Then given $\varepsilon \in (0, \inf m)$, there exist neighborhoods $U_{k,\varepsilon}$ of the respective points ω_k such that

$$|f(\eta) - f(\omega_k)| < \varepsilon^* := \frac{\varepsilon}{2N^*(\sup m + 1)(\max |f| + 1)} \quad \text{for } \eta \in U_{k,\varepsilon}\ (k = 1, \dots, N)$$

where $N^* := \max_{\theta \in \Omega} \mathbf{card}(S(\theta))$. By the assumption (iv), for each index $k < N$ there exists a neighborhood $V_{k,\varepsilon} \subset \Omega$ of ω with $S(\theta) \subset \bigcup_{\ell=1}^N U_{\ell,\varepsilon}$ and

$$\Big| m(\omega_k) - \sum_{\eta \in S(\theta) \cap U_{k,\varepsilon}} m(\eta) \Big| < \varepsilon^*$$

if $\theta \in V_{k,\varepsilon}$. Consider the difference $Af(\theta) - Af(\omega)$ for the points $\theta \in V_\varepsilon := \bigcap_{k=1}^{N-1} V_{k,\varepsilon}$. We have then

$$\begin{aligned}
&|Af(\theta) - Af(\omega)| = \\
&= \Big| \sum_{k=1}^{N-1} \sum_{\eta \in S(\theta) \cap U_{k,\varepsilon}} m(\eta) f(\eta) + \sum_{\eta \in S(\theta) \cap U_{N,\varepsilon}} m(\eta) f(\eta) - \sum_{k=1}^{N-1} m(\omega_k) f(\omega_k) \Big| \\
&\le \sum_{k=1}^{N-1} \Big| \sum_{\eta \in S(\theta) \cap U_{k,\varepsilon}} m(\eta) f(\eta) - m(\omega_k) f(\omega_k) \Big| + \sum_{\eta \in S(\theta) \cap U_{N,\varepsilon}} m(\eta) |f(\eta)| \\
&\le \sum_{k=1}^{N-1} \Big[\sum_{\eta \in S(\theta) \cap U_{k,\varepsilon}} m(\eta) |f(\eta) - f(\omega_k)| + \Big| \sum_{\eta \in S(\theta) \cap U_{k,\varepsilon}} m(\eta) - m(\omega_k) \Big| |f(\omega_k)| \Big] + \\
&\qquad + \sum_{\eta \in S(\theta) \cap U_{N,\varepsilon}} \sup m \varepsilon^* \\
&\le \sum_{k=1}^{N-1} \Big[\sum_{\eta \in S(\theta) \cap U_{k,\varepsilon}} \sup m \varepsilon^* + \varepsilon^* \max |f| \Big] + \sum_{\eta \in S(\theta) \cap U_{N,\varepsilon}} \sup m \varepsilon^* \\
&\le \mathbf{card}(S(\theta)) \varepsilon^* (\sup m + \max |f|) < \varepsilon \,.
\end{aligned}$$

■

Corollary 1.3. *Let (S, m) be an admissible couple and define*

$$\mathcal{S} := \{\Omega_i \cup \{\infty\} : i \in I\} \cup \{\{\infty\}\}.$$

Then the following statements hold.

(i) *Given any convergent net $S_j \to S_0 \neq \{\infty\}$ in $\mathcal{S}$ wrt. Hausdorff topology, there exists a convergent net $\omega_j \to \omega_0$ in Ω such that $S_j = S(\omega_j)$ for all indices and $S_0 = S(\omega_0)$.*

(ii) *The set $\mathcal{S}$ is compact in the Hausdorff topology.*

(iii) *The functions $A^*f(S) := \sum_{\omega\in S} m(\omega)|f(\omega)|^2$ $(S \in \mathcal{S})$ are continuous for all $f \in \mathcal{C}_0(\Omega)$.*

Proof. (i) Consider a convergent net $S_j \to S_0 \neq \{\infty\}$ in $\mathcal{S}$. Fix any element $\omega_0 \in S_0$. Then $S_0 = S(\omega_0) \cup \{\infty\}$ and we may assume also $S_j = S(\omega_j) \cup \{\infty\}$ with suitable elements $\omega_j \in \Omega$. Choose a compact neighborhood U of ω_0 such that $U \cap S_0 = \{\omega_0\}$. Since $S_j \to S_0$ in Hausdorff topology, there exists an index j_0 such that $S_j \cap U \neq \emptyset$ for $j > j_0$. For every $j > j_0$ choose any $\omega_j \in S_j \cap U$. Since $S_j = S(\omega_j)$ $(j > j_0)$ and since the map $\omega \mapsto S(\omega)$ is Hausdorff continuous, the limit of any convergent subnet of $(\omega_j)_{j>j_0}$ can only be ω_0 the unique point of $S_0 = S(\omega_0)$ in the compact U. Thus necessarily $\omega_j \to \omega_0$.

(ii) Recall a topological space is compact iff any net in it admits a convergent subnet. Let $(S_j)_{j\in J}$ be a net in $\mathcal{S}$. If $S_j \not\to \{\infty\}$ in Hausdorff topology then there is an open neighborhood U of ∞ in $\Omega \cup \{\infty\}$ such that $J_0 := \{j \in J\colon\ S_j \not\subset U\}$ is unlimited in J. In this case, for every index $j \in J_0$, we can choose an element $\omega_j \in S_j$ such that $\omega_j \notin U$. By the compactness of $\Omega \setminus U$, there exists a convergent subnet $(\omega_{j_\alpha})_{\alpha\in A}$ with $\omega_{j_\alpha} \to \omega \in \Omega \setminus U$. By the theorem, this subnet converges, since $S_{j_\alpha} = S(\omega_{j_\alpha}) \to S(\omega)$ in Hausdorff topology.

(iii) Consider a function $f \in \mathcal{C}_0(\Omega)$ and a convergent net $S_j \to S_0$ in $\mathcal{S}$. If $S_0 = \{\infty\}$ then $\max_{\omega\in S_j} |f(\omega)| \to 0$. Hence $A^*f(S_j) \to 0 = A^*f(\{\infty\}) = A^*(S_0)$ because $A^*f(S_j) \leq M \max_{\omega\in S_j} |f(\omega)|$ where $M := \max_{i\in I} \mathbf{card}(\Omega_i) \sup_{\omega\in\Omega} m(\omega) < \infty$. If $S_0 \neq \{\infty\}$ then, by (i), we may assume that $S_j = S(\omega_j)$ $(j \in J)$ and $S_0 = S(\omega_0)$ with some convergent net $\omega_j \to \omega_0 \in \Omega$. Observe that $A^*f(S_j) = Af(\omega_j)$ $(j \in J)$. By the theorem, the function Af is continuous. Therefore $A^*f(S_j) = Af(\omega_j) \to Af(\omega_0) = A^*f(S_0)$.

■

Remark 1.4. In terms of nets the above results can be formulated as follows. The couple (S, m) on the locally compact space Ω is admissible if and only if for every convergent net $\omega_j \to \omega$ in Ω and for every point $\omega \in \Omega$ along with a compact neighborhood $U(\subset \Omega)$ of ω such that $U \cap [S(\omega_0) \setminus \{\omega\}] = \emptyset$ we have

$$\sum_{\eta\in U\cap S(\omega_j)} m(\eta) \to m(\omega) \text{ and } S(\omega_j) \cap U \to \{\omega\} \text{ in Hausdorff topology}$$
$$\text{if } \omega \in S(\omega_0),\ U \cap S(\omega_j) = \emptyset \ \ (j > j_0) \ \text{ for some index } j_0 \ \text{ if } \omega \notin S(\omega_0).$$

2. Isomorphism of continuous Reinhardt domains

Throughout this section Ω and $\widetilde{\Omega}$ denote two locally compact Hausdorff spaces and let $D \subset \mathcal{C}_0(\Omega)$, $\widetilde{D} \subset \mathcal{C}_0(\widetilde{\Omega})$ be two bounded symmetric continuous Reinhardt domains given in the form

$$D = D(S,m) := \Big\{ f \in \mathcal{C}_0(\Omega) : \sup_{\omega\in\Omega} Q_{(S,m)} f < 1 \Big\}$$

where

$$Q_{(S,m)} f(\omega) := \sum_{\eta \in S(\omega)} m(\eta) |f(\eta)|^2,$$

respectively $\widetilde{D} = D_{(\widetilde{S},\widetilde{m})}$ with the (uniquely determined) admissible couples (S,m) resp. $(\widetilde{S},\widetilde{m})$.

It is well-known [7] that D and $\widetilde{D}$ are the unit balls of the norms $\|f\| := [\sup Q_{(S,m)} f]^{1/2}$, $\|g\|_\sim := [\sup Q_{(\widetilde{S},\widetilde{m})} g]^{1/2}$ on $\mathcal{C}_0(\Omega)$ and $\mathcal{C}_0(\widetilde{\Omega})$, respectively. By a theorem of Kaup–Upmeier [4] D is holomorphically equivalent to $\widetilde{D}$ iff there is a surjective linear isometry $L\colon E \leftrightarrow \widetilde{E}$ where E and $\widetilde{E}$ denote the Banach spaces $(\mathcal{C}_0(\Omega), \|\cdot\|)$ resp. $(\mathcal{C}_0(\widetilde{\Omega}), \|\cdot\|_\sim)$. We shall be interested in the fine description of such linear isomorphisms.

According to [7], by identifying the dual space E' with the set of all complex Radon measures of bounded total variation on Ω, the family $\mathbf{Ext}\, B(E')$ of the extreme points of the unit ball of E' can be written in the form

$$\mathbf{Ext}\, B(E') = \bigcup_{P \in S(\Omega)} \Phi_P$$

where

$$\Phi_P := \Big\{ \sum_{\omega\in P} \xi_\omega \delta_\omega \colon \sum_{\omega \in P} m(\omega)^{-1} |\xi_\omega|^2 = 1 \Big\}$$

in terms of the one point supported measures $\delta_\omega\colon X \mapsto \mathbf{card}(X \cap \{\omega\})$, and the sets Φ_P with $P \in S(\Omega)(:= \{S(\omega)\colon \omega \in \Omega\})$ are orthogonal to each other in ℓ_1-sense. In particular each Φ_P is a connected component of $\mathbf{Ext}\, B(E')$.

Lemma 2.1. *A linear mapping $L: \widetilde{E} \leftrightarrow E$ is a surjective linear isometry between $\widetilde{E}$ and E iff there exists a bijection $T: \Omega \leftrightarrow \widetilde{\Omega}$ along with a function $u: \Omega \times \widetilde{\Omega} \to \mathbb{C}$ supported on $\bigcup_{\omega\in\Omega} S(\omega) \times T(S(\omega))$ such that $\{\widetilde{S}(\eta)\colon \eta \in \widetilde{\Omega}\} = \{T(S(\omega))\colon \Omega \in \Omega\}$ and*

$$
\tag{2.2}
\begin{aligned}
&Lg(\omega) = \sum_{\eta\in T(S(\omega))} u(\omega,\eta)\, g(\eta) \qquad \text{for } g \in \widetilde{E} \text{ and } \omega \in \Omega,\\
&\sum_{\omega\in P} m(\omega)\Big|\sum_{\eta\in T(P)} u(\omega,\eta)\xi_\eta\Big|^2 = \sum_{\eta\in T(P)} \widetilde{m}(\eta)\big|\xi_\eta\big|^2
\end{aligned}
$$

for $\xi_\eta \in \mathbb{C}$ and $\eta \in T(P)$ whenever $P \in S(\Omega)$.

Proof. If $L: \widetilde{E} \to E$ is a surjective linear isometry then the dual operator L' maps $\mathbf{Ext}\, B(E')$ onto $\mathbf{Ext}\, B(\widetilde{E}')$. In particular, connected components are mapped onto connected components. Hence there is a bijection $\pi: S(\Omega) \leftrightarrow \widetilde{S}(\widetilde{\Omega})$ with $\widetilde{\Phi}_{\pi(P)} = L'\Phi_P$ $(P \in S(\Omega))$ and $\mathbf{card}\ P = \dim Span\ \Phi_P = \dim Span\ \widetilde{\Phi}_{\pi(P)} = \mathbf{card}\ \pi(P)$ $(P \in S(\Omega))$. This ensures the existence of a bijection $T: \Omega \leftrightarrow \widetilde{\Omega}$ with $T(P) = \pi(P)$ $(P \in S(\Omega))$. Then the existence of the function u with the stated identities is a routine consequence of the fact that L' is an isometry between the spaces $Span\ \Phi_P = \sum_{\omega\in P} \mathbb{C}\delta_\omega$ and $Span\ \widetilde{\Phi}_{\pi(P)} = \sum_{\eta\in\pi(P)} \mathbb{C}\widetilde{\delta}_\eta$.

The converse is immediate: a linear mapping $L: \widetilde{E} \to E$ satisfying the given identities is trivially an isometry between $\widetilde{E}$ and E.

∎

Remark 2.3. In terms of matrices, we can state the second condition as follows. Let us represent the sets $P \in S(\Omega)$, $T(P) = \pi(P) \in \widetilde{S}(\widetilde{\Omega})$ in any indexed form $P = \{\omega_k^{(P)}\colon k = 1, \dots, N_P\}$ and $T(P) = \{\eta_k^{(P)}\colon k = 1, \dots, N_P\}$, respectively. Then the matrix

$$
\mathbf{U}_P := \Big[m(\omega_k^{(P)})^{1/2} u(\omega_k^{(P)}, \eta_\ell^{(P)}) \widetilde{m}(\eta_\ell^{(P)})^{-1/2}\Big]_{k,\ell=1}^{N_P}
$$

is unitary. Hence its inverse is the Hermitian transpose

$$
\mathbf{U}_P^{-1} = \Big[m(\omega_k^{(P)})^{1/2} \overline{u(\omega_k^{(P)}, \eta_\ell^{(P)})} \widetilde{m}(\eta_\ell^{(P)})^{-1/2}\Big]_{\ell,k=1}^{N_P}.
$$

Thus if $L: \widetilde{E} \leftrightarrow E$ is a surjective isometry in the form (2.2), there exists a (unique) function $v: \widetilde{\Omega} \times \Omega \to \mathbb{C}$ supported on $\bigcup_{\omega\in\Omega} T(S(\omega)) \times S(\omega) = \bigcup_{P\in P} T(P) \times P$ such that $\mathbf{U}_P^{-1} = \Big[\widetilde{m}(\eta^{(P)})_\ell)^{1/2} v(\eta_\ell^{(P)}, \omega_k^{(P)}) m(\omega_k^{(P)})^{-1/2}\Big]_{\ell,k=1}^{N_i}$, and

$$
L^{-1} f(\eta) := \sum_{\omega\in P} v(\eta,\omega) f(\omega) \quad \text{for } f \in E,\ P \in S(\Omega) \text{ and } \eta \in T(P).
$$

Here necessarily

$$v(\eta,\omega) = \widetilde{m}(\eta)^{-1}\overline{u(\omega,\eta)}m(\omega) \quad \text{for } \omega \in P,\ \eta \in T(P) \text{ whenever } P \in S(\Omega).$$

Lemma 2.4. *Let $L{:}\,\widetilde{E} \to \mathbf{B}(\Omega) :=$ {bounded functions $\Omega \to \mathbb{C}$} be an operation satisfying (2.2) with a bijection $T{:}\,\Omega \leftrightarrow \widetilde{\Omega}$ and a function $u{:}\,\Omega \times \widetilde{\Omega} \to \mathbb{C}$ supported on $\bigcup_{P\in S(\Omega)} P \times T(P)$. Then in terms of the set-transformation T^* between $\mathcal{S} := \{S(\omega) \cup \{\infty\}{:}\ \omega \in \Omega\}$ and $\widetilde{\mathcal{S}} := \{\widetilde{S}(\eta) \cup \{\infty\}{:}\ \eta \in \widetilde{\Omega}\}$ defined by*

$$T^*(P \cup \{\infty\}) := T(P) \cup \{\infty\} \qquad \text{for } P \in S(\Omega) \cup \{\emptyset\},$$

we have the following relations.

(i) *If $L\widetilde{E} \subset E$ then T^* is Hausdorff continuous.*

(ii) *If T^* is Hausdorff continuous then we have $L\widetilde{E} \subset \mathcal{C}(\Omega)$ if and only if given any $\omega_0 \in \Omega$, $\eta_0 \in T(S(\omega_0))$ along with a compact neighborhood V of η_0 such that $V \cap \widetilde{S}(\eta_0)) = \{\eta_0\}$, for every $\varepsilon > 0$ there is a neighborhood U of ω_0 with*

$$(2.5) \qquad \Big|u(\omega_0,\eta_0) - \sum_{\eta\in V_0\cap T(S(\omega))} u(\omega,\eta)\Big| < \varepsilon \qquad \text{whenever } \omega \in U.$$

(iii) *If T^* is Hausdorff continuous, we have $\lim_{\omega\to\infty} Lg(\omega) = 0$ for all $g \in \widetilde{E}$ if and only if given any $\eta_0 \in \widetilde{\Omega}$ along with a compact neighborhood V of η_0 such that $V \cap T(\widetilde{S}(\eta_0)) = \{\eta_0\}$, for every $\varepsilon > 0$ there is a neighborhood $U \subset \Omega \cup \{\infty\}$ of ∞ with*

$$(2.6) \qquad \Big|\sum_{\eta\in V_0\cap T(S(\omega))} u(\omega,\eta)\Big| < \varepsilon \qquad \text{whenever } \omega \in U.$$

Proof. (i) Suppose L maps $\mathcal{C}_0(\widetilde{\Omega})$ into $\mathcal{C}_0(\Omega)$. To establish the Hausdorff continuity of $T^*{:}\,\mathcal{S} \leftrightarrow \widetilde{\mathcal{S}}$, observe that by assumption,

$$(2.7) \qquad \begin{aligned} A^*Lg(S) &:= \sum_{\omega\in S\setminus\{\infty\}} m(\omega)|Lg(\omega)|^2 = \sum_{\eta\in T^*(S)\setminus\{\infty\}} \widetilde{m}(\eta)|g(\eta)|^2 \\ &= A^*g(T^*(S)) \end{aligned}$$

for all $g \in \mathcal{C}_0(\widetilde{\Omega})$ and $S \in \mathcal{S}$. Recall also from elementary topology that a mapping between two compact spaces (cf. 1.3(ii)) is continuous if and only if its graph is closed. Consider a net $(S_j)_{j\in J}$ in $\mathcal{S}$ such that $S_j \to S_0$ and $T^*(S_j) \to R$ for some $S_0 \in \mathcal{S}$ and $R \in \widetilde{\mathcal{S}}$. By the previous observations, we have to show $R = T^*(S_0)$.

Assume $R \neq T^*(S_0)$. Then we can fix a function $g_0 \in \mathcal{C}_0(\widetilde{\Omega})$ vanishing on exactly one of the sets $R \setminus \{\infty\}$ respectively $S_0 \setminus \{\infty\}$. That is $A^*g_0(R) \neq A^*g_0(T^*(S_0))$ However, by Corollary 1.3(iii) and by (2.7), we have

$$A^*g(R) = \lim_{j\in J} A^*g(T^*(S_j)) = \lim_{j\in J} A^*Lg(S_j)$$
$$= A^*Lg(S_0) = A^*g(T^*(S_0))$$

for all $g \in \mathcal{C}_0(\widetilde{\Omega})$. This contradiction proves the continuity of the mapping T^* if $L\widetilde{E} \subset E$.

(ii) Let the transformation T^* be Hausdorff continuous. Suppose $L\widetilde{E} \subset \mathcal{C}(\Omega)$ and fix any couple of points $\omega_0 \in \Omega$, $\eta_0 \in T(S(\omega_0))$. Fix also a compact neighborhood V of η_0 such that $V \cap T^*(S(\omega_0)) = \{\eta_0\}$. We can choose a function $h \in \mathcal{C}_0(\widetilde{\Omega})$ assuming the value 1 on the set V and vanishing on some neighborhood V_1 of the set $[\widetilde{S}(\eta) \cup \{\infty\}] \setminus \{\eta_0\}$. Notice we have $Lh(\omega_0) = u(\omega_0, \eta_0)$. The family $\mathcal{V} := \{S \in \widetilde{\mathcal{S}}\colon\ S \subset V \cup V_1\}$ is a neighborhood of $\widetilde{S}(\eta_0)(= T^*(S(\omega_0)))$. Since, by assumption the function Lh is continuous at the point ω_0, also the set $U_1 := \{\omega \in \Omega\colon\ |Lh(\omega) - u(\omega_0, \eta_0)| < \varepsilon\}$ is a neighborhood of ω_0 in Ω. By the continuity of the maps $\omega \to S(\omega) \cup \{\infty\}$ (cf. Theorem 1.2) and $T^*\colon \mathcal{S} \to \widetilde{\mathcal{S}}$, there exists a neighborhood $U \subset U_1$ of ω_0 such that $T(S(U)) \subset \mathcal{V}$. Then for the points $\omega \in U$ we have

$$Lh(\omega) = \sum_{\eta\in T(S(\omega))} u(\omega,\eta)h(\eta) = \sum_{\eta\in T(S(\omega))\cap V_0} u(\omega,\eta)$$

whence (2.6) is immediate.

Conversely, fix any function $g \in \widetilde{E}$ and a point $\omega_0 \in \Omega$. Write $N :=$ $\mathbf{card}(S(\omega_0))$ and

$$S(\omega_0) = \{\omega_0, \dots, \omega_{N-1}\}, \quad \omega_N := \infty, \quad \eta_k := T(\omega_k) \quad (k = 0, \dots, N-1).$$

Assume for every $\varepsilon > 0$ there is a neighborhood U of ω_0 satisfying (2.6) and let the set transformation T^* be Hausdorff continuous. To show the continuity of Lg at ω_0, notice that, by Remark 2.3, necessarily $\sup|u| \leq \sup m^{1/2}/\inf \widetilde{m}^{1/2} < \infty$. Also $N^* := \max_{\omega\in\Omega} \mathbf{card}(S(\omega)) < \infty$. Fix any $\varepsilon > 0$ and let

$$\varepsilon^* := \varepsilon/[2N^*(\max|g| + \sup|u|)].$$

Since $g \in \mathcal{C}_0(\widetilde{\Omega})$, there are disjoint compact neighborhoods $V_0, \dots, V_{N-1}$ of the respective points $\eta_0, \dots, \eta_{N-1}$ and a neighborhood V_N of $\eta_N := \infty$ in $\widetilde{\Omega} \cup \{\infty\}$ such that

$$|g(\eta_\ell) - g(\eta)| < \varepsilon^* \quad \text{for } \eta \in V_k \quad (k = 0, \dots, N).$$

By assumption, there exist ω_0-neighborhoods $U_0, \ldots, U_N$ such that

$$T(S(U_k)) \subset V_0 \cup \ldots \cup V_N ,$$

$$\Big| u(\omega_0, \eta_k) - \sum_{\eta \in V_k \cap T(S(\omega))} u(\omega, \eta) \Big| < \varepsilon^* \quad \text{for } \omega \in U_k \quad (k = 0, \ldots, N-1).$$

Define $U := U_0 \cap \cdots \cap U_N$ and consider any point $\omega \in U$. We have

$$\begin{aligned}
\Big| Lg(\omega_0) - Lg(\omega) \Big| &= \Big| \sum_{k<N} u(\omega_0, \eta_k) g(\eta_k) - \sum_{k=0}^{N} \sum_{\eta \in V) k \cap T(S(\omega))} u(\omega, \eta) g(\eta) \Big| \\
&\leq \sum_{k<N} \Big| u(\omega_0, \eta_j) - \sum_{\eta \in V_k \cap T(S(\omega))} u(\omega, \eta) g(\eta) \Big| + \\
&\qquad + \sum_{\eta \in V_N \cap T(S(\omega))} |u(\omega, \eta) g(\eta)| \\
&\leq \sum_{k<N} \Big| u(\omega_0, \eta_k) - \sum_{\eta \in V_k \cap T(S(\omega))} u(\omega, \eta) \Big| g(\eta_k) + \\
&\qquad + \sum_{k<N} \sum_{\eta \in V_k \cap T(S(\omega))} |u(\omega, \eta)| |g(\eta_k) - g(\eta)| + \\
&\qquad + \sum_{\eta \in T(S(\omega))} |u(\omega, \eta)| \sup_{\eta \in V_N} |g(\eta)| \\
&\leq \sum_{k<N} \varepsilon^* \max |g| + \sum_{k<N} \sup |u| \varepsilon^* + N^* \sup |u| \varepsilon^* \\
&\leq 2N^* \varepsilon^* [\max |g| + \sup |u|] = \varepsilon.
\end{aligned}$$

The continuity of Lg at ω_0 is established.

(iii) The proof is done by a straightforward modification of the estimates in (ii) where the point ∞ plays a similar role than ω_0 and formally we write $u(\infty, \eta) := 0$. ∎

Remark 2.8. In terms of nets the statement in (ii) can be formulated as follows.

The operator $Lg(\omega) := \sum_{\eta \in T(S(\omega))} u(\omega, \eta) g(\eta)$ with Hausdorff continuous T^* transforms continuous functions vanishing at infinity on $\widetilde{\Omega}$ into continuous functions on Ω if and only if for every convergent net $\omega_j \to \omega_0$ in Ω and for every point $\eta_0 \in T(S(\omega_0))$ along with a compact neighborhood $V \subset \widetilde{\Omega}$ of η_0 such that $\{\eta_0\} = V \cap T(S(\omega_0))$ $(= V \cap \widetilde{S}(\eta_0))$ we have $\sum_{\eta \in V \cap T(S(\omega_j))} u(\omega_j, \eta) \to u(\omega_0, \eta_0)$.

In the case of compact spaces Ω and $\widetilde{\Omega}$, in the light of Remark 2.3, the results of this section yield immediately the following description of the linear automorphisms $L: \widetilde{E} \leftrightarrow E$.

Theorem 2.9. *If Ω and $\widetilde{\Omega}$ are compact spaces, (S,m) resp. $(\widetilde{S},\widetilde{m})$ are admissible couples on them, then a linear operator $L\colon \mathcal{C}(\widetilde{\Omega}) \to \mathcal{C}(\Omega)$ maps the domain $\widetilde{D} := D_{(\widetilde{S},\widetilde{m})}$ onto $D := D_{(S,m)}$ if and only if $Lg(\omega) = \sum_{\eta\in T(S(\omega))} u(\omega,\eta)g(\eta)$ $(\omega \in \Omega,\ g \in \mathcal{C}_0(\widetilde{\Omega}))$ with a bijection $T\colon \Omega \leftrightarrow \widetilde{\Omega}$ such that $\{\widetilde{S}(\eta)\colon \eta \in \widetilde{\Omega}\} = \{T(S(\omega))\colon \omega \in \Omega\}$ and the map $S(\omega) \mapsto T(S(\omega))$ is Hausdorff (bi)continuous along with a function $u\colon \Omega \times \widetilde{\Omega} \to \mathbb{C}$ supported in $\bigcup_{\omega\in\Omega} S(\omega) \times T(S(\omega))$ such that the matrices $\left[m(\omega')^{1/2}u(\omega',\eta')\widetilde{m}(\eta')^{-1/2}\right]_{\omega'\in S(\omega),\eta'\in\widetilde{S}(\eta)}$ $(\omega \in \Omega,\ \eta \in T(S(\omega)))$ are all unitary and*

$$u(\omega_0,\eta_0) = \lim_j \sum_{\eta\in V\cap T(S(\omega_j))} u(\omega_j,\eta)\ ,$$

$$m(\omega_0)u(\omega_0,\eta_0)\widetilde{m}(\eta_0)^{-1} = \lim_j \sum_{\omega\in U\cap T^{-1}(\widetilde{S}(\eta_j))} m(\omega)u(\omega,\eta_j)\widetilde{m}(\eta_j)^{-1}$$

whenever $\omega_j \to \omega_0 \in \Omega$ resp. $\eta_j \to \eta_0 \in \widetilde{\Omega}$ are convergent nets such that $\eta_0 \in T(S(\omega_0))$ and U resp. V are compact neighborhoods of ω_0 resp. η_0 such that $U \cap S(\omega_0) = \{\omega_0\}$ and $V \cap \widetilde{S}(\eta_0) = \{\eta_0\}$.

Example 2.10. With the aid of the theorem we can check directly the somewhat surprising fact that *linearly equivalent symmetric Reinhardt domains may exist with topologically non-equivalent underlying spaces.* Consider the trivially admissible couples (S,m), $(\widetilde{S},\widetilde{m})$ defined by

$$\Omega := \{\omega^t\colon t\in\mathbb{R}\},\ S(\omega^t) := \{\pm\omega^t\}\ ,m(\omega^t) := 1, \quad \text{where} \quad \omega^t := e^{it},$$
$$\widetilde{\Omega} := \{\eta^{s,+},\eta^{s,-}\colon s\in\mathbb{R}\},\ \widetilde{S}(\eta^{s,\pm}) := \{\eta^{s,+},\eta^{s,-}\},\ \ \widetilde{m}(\eta^{s\pm}) := 1,$$
$$\text{where} \quad \eta^{s,\pm} := (e^{it},\pm1).$$

Thus $D := D_{S,m} = \{f \in \mathcal{C}(\Omega)\colon |f(\omega)|^2 + |f(-\omega)|^2 < 1 \ \ (|\omega| = 1)\}$ and $\widetilde{D} := D_{\widetilde{S},\widetilde{m}} = \{g \in \mathcal{C}(\widetilde{\Omega})\colon |g(\eta,1)|^2 + |g(\eta,-1)|^2 < 1 \ \ (|\eta| = 1)\}$ are symmetric continuous Reinhardt domains in $\mathcal{C}(\Omega)$ and $\mathcal{C}(\widetilde{\Omega})$, respectively. Notice that Ω is connected while $\widetilde{\Omega}$ is the disconnected union of two circles. However, the operator

$$Lg(\omega^t) := e^{it/2}[\cos\frac{t}{2}g(\eta^{2t,+}) + \sin\frac{t}{2}g(\eta^{2t,-})] \qquad (g\in\mathcal{C}(\widetilde{\Omega}),\ -\pi\le t<\pi)$$

is a bijection satisfying $L\widetilde{D} = D$. Indeed, we can write

$$Lg(\omega^t) = \sum_{\varepsilon=\pm} u(\omega^t,\eta^{t,\varepsilon})g(\eta^{t,\varepsilon}) = \sum_{\eta\in T(S(\omega^t))} u(\omega^t,\eta)g(\eta) \qquad \text{where}$$

$$T(\omega^s) := \eta^{2s,+}\ , \quad T(\omega^{s-\pi}) := \eta^{2s,-} \qquad \text{for} \ \ 0\le s<\pi\ ,$$

$$u(\omega^t,\eta^{2t,+}) := e^{it/2}\cos\frac{t}{2}\ , \quad u((\omega^t,\eta^{2t,-}) := e^{it/2}\sin\frac{t}{2} \qquad \text{for} \ \ -\pi\le t<\pi$$

and u vanishes elsewhere. Observe that the matrices

$$U^t := \begin{bmatrix} u(\omega^t, \eta^{2t,+}) & u(\omega^t, \eta^{2t,-}) \\ u(\omega^{t-\pi}, \eta^{2t,+}) & u(\omega^{t-\pi}, \eta^{2t,-}) \end{bmatrix} = e^{it/2} \begin{bmatrix} \cos(t/2) & \sin(t/2) \\ -\sin(t/2) & \cos(t/2) \end{bmatrix} \quad (-\pi \le t < \pi)$$

are unitary and $\lim_{t\uparrow\pi} U^t = \lim_{t\downarrow-\pi} U^t$. Hence it readily follows that the operator L suits the criteria for $L\widetilde{D} = D$ in Theorem 2.9.

Remark 2.11. In terms of nets, Lemma 2.4(iii) can be formulated as follows.

The operator $Lg(\omega) := \sum_{\eta\in T(S(\omega))} u(\omega,\eta)g(\eta)$ with Hausdorff continuous T^* transforms continuous functions vanishing at infinity on $\widetilde{\Omega}$ into functions vanishing at ∞ on Ω if and only if for every convergent net $\omega_j \to \infty$ in Ω and for every point $\eta_0 \in \widetilde{\Omega}$ along with a compact neighborhood V of η_0 such that $\{\eta_0\} = V \cap \widetilde{S}(\eta_0)$ we have $\sum_{\eta\in V\cap T(S(\omega_j))} u(\omega_j,\eta) \to 0$.

In the light of Remarks 2.8 and 2.11 we can describe the structure of linear isomorphisms of symmetric continuous Reinhardt domains as follows.

Theorem 2.12. *Let (S,m) and $(\widetilde{S},\widetilde{m})$ be admissible couples on the locally compact spaces Ω and $\widetilde{\Omega}$, respectively. Then the linear operator $L : \mathcal{C}_0(\widetilde{\Omega}) \to \mathcal{C}_0(\Omega)$ maps injectively the domain $D_{\widetilde{S},\widetilde{m}} (= \{g \in \mathcal{C}_0(\widetilde{\Omega}) : \sum_{\theta\in\widetilde{S}(\eta)} |g(\theta)|^2 < 1 \ \ (\eta\in\widetilde{\Omega})\})$ onto $D_{S,m} (= \{f \in \mathcal{C}_0(\Omega) : \sum_{\xi\in S(\omega)} |f(\xi)|^2 < 1 \ \ (\omega\in\Omega)\})$ if and only if there exists a bijection $T : \Omega \leftrightarrow \widetilde{\Omega}$ along with a function $u : \Omega\times\widetilde{\Omega} \to \mathbb{C}$ vanishing outside $\bigcap_{\omega\in\Omega} S(\omega)\times T(S(\omega))$ such that*

(i) $\widetilde{S}(T(\omega)) = T(S(\omega))$ $(\omega\in\Omega)$ and the mapping $T^* : P\cup\{\infty\} \mapsto T(P)\cup\{\infty\}$ is a Hausdorff continuous bijection between the families $\mathcal{S} := \{S(\omega)\cup\{\infty\} : \omega\in\Omega\}\cup\{\infty\}\}$ and $\widetilde{\mathcal{S}} := \{\widetilde{S}(\eta)\cup\{\infty\} : \eta\in\widetilde{\Omega}\}\cup\{\infty\}\}$;

(ii) the matrices $\left[m(\omega')^{1/2}u(\omega',\eta')\widetilde{m}(\eta')^{-1/2}\right]_{\omega'\in S(\omega),\eta'\in\widetilde{S}(\eta)}$ $(\omega\in\Omega,\ \eta\in T(S(\omega)))$ are all unitary;

(iii)

$$\lim_j \sum_{\eta\in V\cap T(S(\omega_j))} u(\omega_j,\eta) = u(\omega_0,\eta_0)$$

and

$$\lim_j \sum_{\omega\in U\cap T^{-1}(\widetilde{S}(\eta_j))} m(\omega)u(\omega,\eta_j)\widetilde{m}(\eta_j)^{-1} = m(\omega_0)u(\omega_0,\eta_0)\widetilde{m}(\eta_0)^{-1}$$

whenever $\omega_j \to \omega_0 \in \Omega$ resp. $\eta_j \to \eta_0 \in \widetilde{\Omega}$ are convergent nets such that $\eta_0 \in T(S(\omega_0))$ and U resp. V are compact neighborhoods of ω_0 resp. η_0 such that $U\cap S(\omega_0) = \{\omega_0\}$ and $V\cap\widetilde{S}(\eta_0) = \{\eta_0\}$.

(iv)

$$0 = \lim_j \sum_{\eta \in V \cap T(S(\omega_j))} u(\omega_j, \eta) = \lim_j \sum_{\omega \in U \cap T^{-1}(\widetilde{S}(\eta_j))} m(\omega) u(\omega, \eta_j) \widetilde{m}(\eta_j)^{-1} = 0$$

whenever $\omega_j \to \infty$ *resp.* $\eta_j \to \infty$ *are nets in* Ω *resp.* $\widetilde{\Omega}$, *furthermore* $\omega_0 \in \Omega$, $\eta_0 \in \widetilde{\Omega}$ *and* $U \subset \Omega$ *resp.* $V \subset \widetilde{\Omega}$ *are compact neighborhoods of* ω_0 *resp.* η_0 *such that* $U \cap S(\omega_0) = \{\omega_0\}$ *and* $V \cap \widetilde{S}(\eta_0) = \{\eta_0\}$.

Remark 2.13. We close this work by showing that the compactness of the underlying topological spaces is no holomorphic (and hence even no linear) invariant of symmetric continuous Reinhardt domains. Let

$$\begin{aligned} \Omega &:= \{(x,0),(x,1) :\ 0 \le x \le 1\} \setminus \{(0,0)\}, \\ \widetilde{S}(x,y) &:= \{(x,0),(x,1)\} \cap \widetilde{\Omega}, \quad \widetilde{m}(x,y) := 1, \\ \widetilde{\Omega} &:= \{(x,0),(x,x) :\ 0 \le x \le 1\}, \\ S(x,y) &:= \{(x,0),(x,x)\}, \quad m(x,y) := 2/\mathbf{card}(\widetilde{S}(x,y)). \end{aligned}$$

We regard Ω and $\widetilde{\Omega}$ with the topology inherited from $\mathbb{R}^2$. Then Ω is non-compact while $\widetilde{\Omega}$ is compact and we can identify the ideal point ∞ for Ω with $(0,0)$. So we have

$$D_{S,m} = \Big\{ f \in \mathcal{C}(\Omega) :\ \sum_{y=0}^{1} |f(x,y)|^2 < 1,\ f(0,0) = 0 \Big\},$$

$$D_{\widetilde{S},\widetilde{m}} = \Big\{ g \in \mathcal{C}(\widetilde{\Omega}) :\ \sum_{y=0}^{1} |g(x,xy)|^2 < 1 \Big\}.$$

The mapping

$$Lg(x,y) := 2^{-1/2} g(x,x) - (-1)^y 2^{-1/2} g(x,0) \qquad ((x,y) \in \Omega)$$

is a linear isomorphism with $LD_{\widetilde{S},\widetilde{m}} = D_{S,m}$. Namely, with the notations of Theorem 2.12, we have $Lg(\omega) = \sum_{\eta \in T(S(\omega))} u(\omega,\eta) g(\eta)$ where

$$\begin{aligned} T(x,y) &:= (x,xy), \quad u((x,y),(x,z)) \\ &:= -(-1)^{\mathrm{sgn}(y^2+z^2)} 2^{1/2} \quad (x \neq 0,\ (x,y) \in \Omega,\ (x,z) \in \widetilde{\Omega}), \end{aligned}$$

$T(0,0) := (0,0)$, $u((0,1),(0,0)) := 2^{3/2}$ and u vanishes elsewhere.

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