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ON FUNCTIONAL EQUATIONS RELATED TO BICIRCULAR PROJECTIONS

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ABSTRACT. In this paper we prove the following result. Let R be a $2-$torsion free semiprime $^*-$ring. Suppose that $D, G : R \to R$ are additive mappings satisfying the relations $D(xyx) = D(x)yx + xG(y^*)^*x + xyD(x), G(xyx) = G(x)yx + xD(y^*)^*x + xyG(x)$, for all pairs $x, y \in R$. In this case D and G are of the form $8D(x) = 2(d(x)+g(x))+(p+q)x+x(p+q)$, $8G(x) = 2(d(x) - g(x)) + (q - p)x + x(q - p)$, for all $x \in R$, where d, g are derivations of R and p, q are some elements from symmetric Martindale ring of quotients of R. Besides, $d(x) = -d(x^*)^*$, $g(x) = g(x^*)^*$, for all $x \in R$, and $p^* = p$, $q^* = -q$.

This research has been motivated by the work of Fošner and Ilišević [3]. Throughout, R will represent an associative ring. Given an integer $n \geq 2$, a ring R is said to be n-torsion free if for $x \in R$, $nx = 0$ implies $x = 0$. An additive mapping $x \mapsto x^*$ on a ring R is called an involution if $(xy)^* = y^*x^*$ and $x^{**} = x$ hold for all $x, y \in R$. A ring equipped with an involution is called a ring with involution or a $^*-$ring. Recall that a ring R is semiprime if $aRa = (0)$ implies $a = 0$. An additive mapping $D : R \to R$ is called a derivation if $D(xy) = D(x)y + xD(y)$ holds for all pairs $x, y \in R$. An additive mapping $T : R \to R$ is called a left centralizer in case $T(xy) = T(x)y$ holds for all pairs $x, y \in R$. The concept appears naturally in C^*-algebras. In ring theory it is more common to work with module homomorphisms. Ring theorists would write that $T : R_R \to R_R$ is a homomorphism of the right R–module R into itself. For a semiprime ring R all such homomorphisms are of the form $T(x) = qx$ for all $x \in R$, where q is an element of Martindale right ring of quotients Q_r of R (see Chapter 2 in [1]). We denote by Q_s Martindale

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symmetric ring of quotients. In case R has the identity element $T : R \to R$ is a left centralizer iff T is of the form $T(x) = ax$ for all $x \in R$ and some fixed element $a \in R$. The definition of right centralizer should be self-explanatory. In case R is a semiprime $^*-$ring, then the involution can be uniquely extended to Q_s (see Proposition 2.5.4 in [1]).

Let X be a complex Banach space and let $L(X)$ be the algebra of all bounded linear operators on X. A projection $P \in L(X)$ is bicircular in case all mappings of the form $e^{i\alpha}P + e^{i\beta}(I - P)$, where I denotes the identity operator, are isometric for all pairs of real numbers α, β. Stachó and Zalar [4] investigated bicircular projections on the C^*-algebra $L(H)$, the algebra of all bounded linear operators on a Hilbert space H. According to Proposition 3.4 in [4], every bicircular projection $P : L(H) \to L(H)$ satisfies the functional equation

$$P(xyx) = P(x)yx - xP(y^*)^*x + xyP(x), \tag{1}$$

for all pairs $x, y \in L(H)$. Some results concerning bicircular projections can be found in [5]. Fošner and Ilišević [3] investigated the above functional equation in $2-$torsion free semiprime $^*-$rings. They expressed the solution of the equation (1) in terms of derivations and so-called double centralizers. It is our aim in this paper to consider more general situation.

Theorem 1. *Let R be a $2-$torsion free semiprime $^*-$ring. Suppose that $D, G : R \to R$ are additive mappings satisfying the relations*

$$\begin{aligned} D(xyx) &= D(x)yx + xG(y^*)^*x + xyD(x), \\ G(xyx) &= G(x)yx + xD(y^*)^*x + xyG(x), \end{aligned}$$

for all pairs $x, y \in R$. In this case D and G are of the form

$$\begin{aligned} 8D(x) &= 2(d(x) + g(x)) + (p + q)x + x(p + q), \\ 8G(x) &= 2(d(x) - g(x)) + (q - p)x + x(q - p), \end{aligned}$$

for all $x \in R$ where d and g are derivations and p and q are fixed elements from Q_s. Besides, $d(x) = -d(x^*)^*, g(x) = g(x^*)^*$, for all $x \in R$, and $p^* = p$, $q^* = -q$.

In a special case when $G = -D$ the theorem above reduces to a result which can be compared with Proposition 1.2 and Corollary 1.3 in [3]. In Corollary 1.3 there is an additional assumption that $R^2 = R$. Our approach differs from those used in [3] and seems to be more direct. In the proof of Theorem 1 we use a result proved by Brešar [2] (Theorem A) and a recent result proved by Vukman, Kosi-Ulbl and Eremita [6] (Theorem B).

Theorem A ([2], Theorem 4.3). *Let R be a $2-$torsion free semiprime ring. Suppose that $D : R \to R$ is an additive mapping satisfying the relation*

$$D(xyx) = D(x)yx + xD(y)x + xyD(x),$$

for all pairs $x, y \in R$. In this case D is a derivation.

THEOREM B ([6], Theorem 2.1). *Let R be a $2-$torsion free semiprime ring. Suppose that $T : R \to R$ is an additive mapping satisfying the relation*

$$T(xyx) = T(x)yx - xT(y)x + xyT(x),$$

for all pairs $x, y \in R$. In this case T is of the form $2T(x) = qx + xq$, for all $x \in R$ and some $q \in Q_s$.

PROOF OF THEOREM 1. The proof goes in several steps.

FIRST STEP. Let us first assume that $D = G$ and put $F = D$. In this case we have the relation

$$F(xyx) = F(x)yx + xF(y^*)^*x + xyF(x), \tag{2}$$

for all pairs $x, y \in R$. It is our aim to prove that F is of the form

$$F(x) = 2d(x) + qx + xq, \tag{3}$$

for all $x \in R$, where d is a derivation of R and q is a fixed element from Q_s. Besides, $d(x) = d(x^*)^*$, for all $x \in R$, and $q^* = -q$. Let us introduce mappings $d : R \to R$ and $f : R \to R$ by

$$d(x) = F(x) + F(x^*)^* \tag{4}$$

and $f(x) = F(x) - F(x^*)^*$. Now we have

$$d(x^*)^* = (F(x^*) + F(x)^*)^* = F(x) + F(x^*)^* = d(x), \quad x \in R.$$

From the relation (2) one obtains easily that

$$d(xyx) = d(x)yx + xd(y)x + xyd(x), \tag{5}$$

and

$$f(xyx) = f(x)yx - xf(y)x + xyf(x), \tag{6}$$

is fulfilled for all pairs $x, y \in R$. Now it follows from the relation (5) and Theorem A that d is a derivation. On the other hand one can conclude from the relation (6) applying Theorem B that f is of the form $2f(x) = qx + xq$, for all $x \in R$ and some $q \in Q_s$. We have therefore

$$2F(x) - 2F(x^*)^* = qx + xq, \tag{7}$$

for all $x \in R$. Putting in the above relation x^* for x we obtain $2F(x^*) - 2F(x)^* = qx^* + x^*q, x \in R$, which gives

$$2F(x^*)^* - 2F(x) = q^*x + xq^*, \quad x \in R.$$

Combining the above relation with the relation (7) we obtain $(q+q^*)x+x(q+q^*) = 0$, for all $x \in R$, whence it follows after some calculation because of semiprimeness that $q^* = -q$. Combining the relation (4) with the relation (7) we obtain $4F(x) = 2d(x) + qx + xq, x \in R$, which completes the proof of the first step.

Second step. Let us first assume that $D = -G$. Put $H = D$. We have the relation
$$H(xyx) = H(x)yx - xH(y^*)^*x + xyH(x),$$
for all pairs $x, y \in R$. In this case H is of the form
$$4H(x) = 2g(x) + px + xp, \tag{8}$$
for all $x \in R$, where g is a derivation of R and p is a fixed element from Q_s. Besides, $g(x) = -g(x^*)^*$, for all $x \in R$, and $p^* = p$. The proof of the second step will be omitted since it goes through using the same arguments as in the proof of the first step.

Third step. We are ready for the proof of general case. We have therefore the relations
$$D(xyx) = D(x)yx + xG(y^*)^*x + xyD(x), \quad x, y \in R \tag{9}$$
and
$$G(xyx) = G(x)yx + xD(y^*)^*x + xyG(x), \quad x, y \in R \tag{10}$$
Combining the relation (9) with the relation (10) we obtain
$$F(xyx) = F(x)yx + xF(y^*)^*x + xyF(x), \quad x, y \in R$$
where $F(x)$ stands for $D(x)+G(x)$. On the other hand subtracting the relation (10) from the relation (9) we arrive at
$$H(xyx) = H(x)yx - xH(y^*)^*x + xyH(x), \quad x, y \in R$$
where $H(x)$ denotes $D(x) - G(x)$. Now according to (3) and (8) we have
$$4D(x) + 4G(x) = 2d(x) + qx + xq, \quad x \in R \tag{11}$$
and
$$4D(x) - 4G(x) = 2g(x) + px + xp, \quad x \in R. \tag{12}$$
From (11) and (12) one obtains
$$8D(x) = 2(d(x) + g(x)) + (p + q)x + x(p + q), \quad x \in R$$
and
$$8G(x) = 2(d(x) - g(x)) + (q - p)x + x(q - p), \quad x \in R$$
which completes the proof of the theorem. □

Theorem 1 leads to the following conjecture.

Conjecture. Let R be a semiprime $^*-$ring with suitable torsion restrictions. Suppose that $D, G : R \to R$ are additive mappings satisfying the relations
$$\begin{aligned} D(x^3) &= D(x)x^2 + xG(x^*)^*x + x^2D(x), \\ G(x^3) &= G(x)x^2 + xD(x^*)^*x + x^2G(x), \end{aligned}$$

for all $x \in R$. In this case D and G are of the form

$$\begin{aligned} 8D(x) &= 2(d(x)+g(x)) + (p+q)x + x(p+q), \\ 8G(x) &= 2(d(x)-g(x)) + (q-p)x + x(q-p), \end{aligned}$$

for all $x \in R$, where d and g are derivations and p and q are fixed elements from Q_s. Besides, $d(x) = -d(x^*)^*, g(x) = g(x^*)^*$, for all $x \in R$, and $p^* = p$, $q^* = -q$.

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References

[1] K. I. Beidar, W. S. Martindale III and A. V. Mikhalev, Rings with generalized identities, Marcel Dekker Inc., New York, 1996.

[2] M. Brešar, *Jordan mappings of semiprime rings*, J. Algebra **127** (1989), 218-228.

[3] M. Fošner and D. Ilišević, *On a class of projections on $^*-$rings*, Comm. Algebra **33** (2005), 3293-3310.

[4] L. L. Stachó and B. Zalar, *Bicircular projections on some matrix and operator spaces*, Linear Algebra Appl. **384** (2004), 9-20.

[5] L. L. Stachó and B. Zalar, *Bicircular projections and characterization of Hilbert spaces*, Proc. Amer. Math. Soc. **132** (2004), 3019-3025.

[6] J. Vukman, I. Kosi-Ulbl and D. Eremita, *On certain equations in rings*, Bull. Austral. Math. Soc. **71** (2005), 53-60.

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