The preceding result was obtained in [8] for the classical case $F = L^1(\mu)$, $F' = F_n^* = L^\infty(\mu)$.

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Fixed Points of Holomorphic Mappings

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1. Introduction

Let D be a bounded domain in $\mathbb{C}^n$ (or, more generally, in a complex Banach space E). Let $f: D \to D$ be a holomorphic mapping. The set

$$\operatorname{Fix} f = \{x \in D \mid f(x) = x\}$$

has been studied by many people. Let us recall first the following theorem proved by E. Vesentini [8 and 9]:

Theorem 1.1: *Let B be the open unit ball of a complex Banach space E. Suppose that every point x belonging to the boundary ∂B of B is a complex extreme point of $\overline{B}$. Let $f: B \to B$ be a holomorphic mapping such that $f(0) = 0$. Then*

$$\operatorname{Fix} f = B \cap F,$$

where

$$F = \{v \in E \mid f'(0) \cdot v = v\}$$

is the eigenspace of the derivative $f'(0)$ of f at the origin for the eigenvalue 1. Moreover, if E is reflexive, there exists a projection $p: E \to F$ of norm 1. So, $B \cap F$ is the image of a linear retraction $B \to B \cap F$.

The proof is based on the notion of complex geodesic. In fact, E. Vesentini [9] proved that, given $x \in D$, there exists a unique complex geodesic through the origin and x, and, more or less, this argument concludes the proof. But, in general, complex geodesics are not unique, and Vesentini's proof cannot be generalized. For example, the case of the bidisc $\Delta \times \Delta$ has been studied by M. Hervé [6] and E. Vesentini [8], and they proved the following result:

Theorem 1.2: *Let $f: \Delta \times \Delta \to \Delta \times \Delta$ be a holomorphic mapping. The set* $\operatorname{Fix} f$ *is one of the following sets:*

1. *the empty set ϕ;*
2. *one point;*
3. *there exists a holomorphic mapping $\varphi: \Delta \to \Delta$ such that*

$$\operatorname{Fix} f = \{(\zeta_1, \zeta_2) \in \Delta \times \Delta \mid \zeta_2 = \varphi(\zeta_1)\}$$

or

$$\operatorname{Fix} f = \{(\zeta_1, \zeta_2) \in \Delta \times \Delta \mid \zeta_1 = \varphi(\zeta_2)\};$$

4. $\Delta \times \Delta$.

So, in this example, the set $\operatorname{Fix} f$ is not a linear subspace, but it is always a connected submanifold.

Now, we are going to give the results of this talk, and, first, we will begin with the finite-dimensional case.

2. Bounded domains in $\mathbb{C}^n$

We begin with the following result:

Theorem 2.1: ([13]) *Let D be a bounded domain in $\mathbb{C}^n$ and let $f: D \to D$ be a holomorphic mapping. Then* $\operatorname{Fix} f$ *is a complex submanifold of D. If $a \in \operatorname{Fix} f$, its tangent space $T_a(\operatorname{Fix} f)$ is equal to*

$$F = \{v \in \mathbb{C}^n \mid f'(a) \cdot v = v\}.$$

The proof of this result uses ideas of H. Cartan [3] and E. Bedford [1]. Let $a \in \operatorname{Fix} f$, and let us consider the sequence $f^p = f \circ \cdots \circ f$ (p times) of iterates of f. We can find a sequence of integers $p_j \to +\infty$ such that $q_j = p_{j+1} - p_j$ and $r_j = p_{j+1} - 2p_j$ converge to $+\infty$ and that f^{p_j} converges to a holomorphic map F (uniformly on compact subsets of D). Now, by taking subsequences of the sequences q_j and r_j, we can suppose that

$$f^{q_j} \to \rho, \qquad f^{r_j} \to G.$$

By shrinking D if necessary, we can suppose that ρ, F and G send D to D. Then, by composition, one proves easily the following relations:

$$\rho \circ F = F \circ \rho = F, \qquad F \circ G = G \circ F = \rho, \qquad f \circ \rho = \rho \circ f.$$

We deduce that

$$\rho^2 = \rho \circ \rho = \rho \circ F \circ G = F \circ G = \rho.$$

So, ρ is a holomorphic retraction, and, by a result of H. Cartan [4], there exists a local coordinate chart u defined on a neighbourhood U of a, such that $u(a) = 0$ and that $u \circ \rho \circ u^{-1}$ is a linear projection.

We have proved that $\rho(D)$ is a submanifold of D containing $\operatorname{Fix} f$, and it is easy to prove that f is a biholomorphic automorphism of $\rho(D)$. It is clear that $\rho(D)$ is a hyperbolic manifold ([5]), and we can apply the following result of H. Cartan [2]:

Theorem 2.2: *Let X be a complex hyperbolic manifold of finite dimension n, and let a be a point of X. Let $f \in \operatorname{Aut}(X)$ be a biholomorphic automorphism of X such that $f(a) = a$. Then there exists a local coordinate chart u defined in a neighbourhood U of a such that $u(a) = 0$ and that $u \circ f \circ u^{-1}$ is a linear automorphism of $\mathbb{C}^n$.*

This theorem applied to $f|_{\rho(D)}$ proves that $\operatorname{Fix} F$ is a submanifold of D.

Of course, $\operatorname{Fix} f$ is not connected in general; for example, consider the annulus

$$A = \{\zeta \in \mathbb{C} \mid 1/2 < |\zeta| < 2\},$$

and the automorphism f of A defined by $f(\zeta) = 1/\zeta$. In fact, as proved by P. Mazet and J.-P. Vigué [7], the components of $\operatorname{Fix} f$ do not always have the same dimensions.

3. Bounded convex domains in $\mathbb{C}^n$

Now, if we suppose that D is a bounded convex domain in $\mathbb{C}^n$, I can prove that the set $\operatorname{Fix} f$ is connected ([11 and 12]). In fact, we have the more precise result:

Theorem 3.1: ([12]) *Let D be a bounded convex domain in $\mathbb{C}^n$. Let $f: D \to D$ be a holomorphic mapping and let us assume that* $\operatorname{Fix} f$ *is not empty. Then there exists a holomorphic retraction $\psi: D \to \operatorname{Fix} f$.*

Idea of the proof. We consider φ_n defined by

$$\varphi_n(x) = \frac{1}{n} \sum_{p=0}^{n-1} f^p(x).$$

φ_n is a holomorphic mapping from D to D, and, by Montel's theorem, we can find a subsequence φ_{n_p} converging to φ (uniformly on compact subsets of D). φ is holomorphic, and, as D is taut, φ is a holomorphic mapping from D to D.

Let $a \in \operatorname{Fix} f$. By elementary linear algebra considerations, one proves that $\varphi'(a)$ is a linear projection onto

$$F = \{v \in \mathbb{C}^n \mid f'(a) \cdot v = v\}.$$

Now, let us define

$$\psi_n = \varphi^n.$$

Using Cauchy's inequalities, one proves that ψ_n converges uniformly on compact subsets of D to a holomorphic mapping ψ such that $\psi(D) \subset \operatorname{Fix} f$, and $\psi|_{\operatorname{Fix} f} = \operatorname{id}|_{\operatorname{Fix} f}$. The theorem is proved.

4. Bounded domains in reflexive Banach spaces

The results of this section have been proved in collaboration with P. Mazet [7].

The first idea we use to generalize these results to the case of bounded domains in reflexive Banach spaces is to consider weak topology and weak limits of sequences. However, it does not seem possible to generalize the proof of Theorem 2.1 for the following reason: if f_n (respectively, g_n) weakly converges to f (respectively, g), in general, $f_n \circ g_n$ does not converge to $f \circ g$. Fortunately, it is possible to generalize the proof I gave for bounded convex domains in $\mathbb{C}^n$, and we prove the following theorem:

Theorem 4.1: *Let D be a bounded convex domain in a reflexive Banach space E. Let $a \in D$, and let $f: D \to D$ be a holomorphic mapping such that $f(a) = a$. Then the set* $\operatorname{Fix} f$ *is a complex direct submanifold of D, tangent in a at*

$$F = \{v \in E \mid f'(a) \cdot v = v\},$$

and there exists a holomorphic retraction $\psi: D \to \operatorname{Fix} f$.

Idea of the proof: As in the finite-dimensional case, we define

$$\varphi_n = \frac{1}{n} \sum_{p=0}^{n-1} f^p.$$

Let us consider on the set $H(D, \overline{D})$ of holomorphic functions from D to $\overline{D}$ the topology of uniform weak convergence on finite-dimensional compact subsets of D. It is more or less standard that $H(D, \overline{D})$ is compact, and so, we can find φ adherent to the sequence φ_n.

φ is a holomorphic mapping from D to D; it is clear that

$$\varphi \circ f = \varphi, \qquad \operatorname{Fix} f \subset \operatorname{Fix} \varphi,$$

and using the continuity of $f'(a)$ for the weak topology, we prove that

$$f'(a) \circ \varphi'(a) = \varphi'(a).$$

So, $\varphi'(a)$ is a projection onto F. Now, we consider the sequence of iterates

$$\psi_n = \varphi^n$$

of φ.

If g is holomorphic in a neighbourhood of a, we note

$$g = \sum_{p=0}^{\infty} P_p(g)$$

the development of g in series of homogeneous polynomials at a. We prove the following lemma.

Lemma 4.2: *For every $n \geq 0$ and $p \leq n$ we have*

$$P_p(f \circ \varphi^n) = P_p(\varphi^n) = P_p(\varphi^{n+1}).$$

We have already proved this lemma for $n = 1$, and the proof is by induction on n.

Using Cauchy's inequalities for bounded mappings, this lemma implies that ψ_n converges to a limit ψ uniformly on a ball of center a and of radius small enough. But, by [10], it implies that ψ_n converges to a limit ψ for the topology of local uniform convergence. So, $\psi \in H(D, D)$, and we have $f \circ \psi = \psi$, $\psi^2 = \psi$, $\operatorname{Fix} f = \operatorname{Fix} \psi$. As ψ is a holomorphic retraction, the theorem is a consequence of H. Cartan [4].

Now, if we do not suppose that D is convex, we can also define φ_n and φ. The only difference is that φ does not send D to D. However, if a is a fixed point of f, we prove in [7] that there exists a neighbourhood U of a such that $\varphi(U) \subset U$, and, with some small changes, we can generalize the proof to this case.

Theorem 4.3: (P. Mazet and J.-P. Vigué [7]) *Let D be a bounded domain in a reflexive Banach space E, and let $f: D \to D$ be a holomorphic mapping. Then the set* $\operatorname{Fix} f$ *is a complex direct submanifold of D.*

5. An example

To conclude this talk, I shall give an example [7] which proves that the conclusion of Theorems 4.1 and 4.3 is not true for every Banach space E.

Let $c_0(\mathbb{N})$ be the Banach space of sequences converging to 0 at infinity. Let B be the open unit ball of $c_0(\mathbb{N})$. Let $n \in \mathbb{N}$, and let f be a holomorphic mapping from the polydisk Δ^n into itself. Let us define $F: B \to B$ in the following way:

$$(Z_p)_{p\in\mathbb{N}} = F\big((z_p)_{p\in\mathbb{N}}\big),$$

where

$$(Z_0, \ldots, Z_{n-1}) = (z_0, \ldots, z_{n-1}),$$
$$(Z_n, \ldots, Z_{2n-1}) = f(z_0, \ldots, z_{n-1}),$$
$$Z_{2n+k} = z_{n+k}, \quad \forall k \geq 0.$$

It is easy to check that

$$\operatorname{Fix} F = Z(f) \times \{0\},$$

where

$$Z(f) = \big\{(z_0, \ldots, z_{n-1}) \in \Delta^n \mid f(z_0, \ldots, z_{n-1}) = 0\big\}$$

is the zero set of f. It is clear that $Z(f)$ and $\operatorname{Fix}(F)$ are not, in general, submanifolds.

There are also examples (P. Mazet and J.-P. Vigué [7]) in which E is a dual space.

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