Bull. Sci. math. 127 (2003) 597–609

BULLETIN DES SCIENCES MATHEMATIQUES

www.elsevier.com/locate/bulsci

Norms and CB norms of Jordan elementary operators

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Received 19 May 2003; accepted 29 May 2003

Abstract

We establish lower bounds for norms and CB-norms of elementary operators on $\mathcal{B}(H)$. Our main result concerns the operator $T_{a,b}x = axb + bxa$ and we show $\|T_{a,b}\| \geqslant \|a\|\|b\|$, proving a conjecture of M. Mathieu. We also establish some other results and formulae for $\|T_{a,b}\|_{cb}$ and $\|T_{a,b}\|$ for special cases.

Keywords: Completely bounded; Hilbert–Schmidt norm; Numerical range

Our results are related to a problem of M. Mathieu [13,14] asking whether $\|T_{a,b}\| \geqslant c\|a\|\|b\|$ holds in general with $c = 1$. We prove this in Theorem 6 below.

In [14] the inequality is established for $c = 2/3$ and the best known result to date is $c = 2(\sqrt{2} - 1)$ as shown in [5,11,17]. There are simple examples which show that c cannot be greater than 1 in general and there are results which prove the inequality with $c = 1$ in special cases. The case $a^* = a$ and $b^* = b$ is shown in [12] where it is deduced from $\|T_{a,b}\|_{cb} = \|T_{a,b}\|$ under these hypotheses.

The equality of the the CB norm and the operator norm of $T_{a,b}$ also holds if a, b are commuting normal operators. See Section 3 below for references.

A result for $c = 1$ is shown in [2] under the assumption that $\|a + zb\| \geqslant \|a\|$ for all $z \in \mathbb{C}$. In more general contexts similar results (with varying values of c) are shown in [5,6].

As this manuscript was being written we learned of another proof of the main result [4], using rather different methods. Thanks are due to M. Mathieu for drawing our attention to this reference.

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0007-4497/$ – see front matter
doi:10.1016/S0007-4497(03)00046-0

1. Preliminaries

We call $T : \mathcal{B}(H) \to \mathcal{B}(H)$ an elementary operator if T has a representation

$$T(x) = \sum_{i=1}^{\ell} a_i x b_i$$

with $a_i, b_i \in \mathcal{B}(H)$ for each i. We cite [1] for an exposition of many of the known results on (more general) elementary operators and for other concepts we cite a number of treatises on operator spaces including [7,8,15]. In particular we will use the completely bounded (or CB) norm $\|T\|_{cb}$ of an elementary operator, the operator norm $\|T\|$ and the estimate in terms of the Haagerup tensor product norm $\|T\| \leqslant \|T\|_{cb} \leqslant \|\sum_{i=1}^{\ell} a_i \otimes b_i\|_h$.

We recall that the Haagerup norm of an element $w \in \mathcal{B}(H) \otimes \mathcal{B}(H)$ (of the algebraic tensor product) is defined by

$$\|w\|_h^2 = \inf \left\| \sum_{i=1}^{k} a_i a_i^* \right\| \left\| \sum_{i=1}^{k} b_i^* b_i \right\|$$

where the infimum is over all representations $w = \sum_{i=1}^{k} a_i \otimes b_i$. Moreover this infimum is achieved with both k-tuples $(a_1, a_2, \ldots, a_k)$ and $(b_1, b_2, \ldots, b_k)$ linearly independent.

Throughout H denotes a (complex) Hilbert space and $\mathcal{B}(H)$ the algebra of bounded linear operators on H. For x in the class of Hilbert–Schmidt operators on H we denote the Hilbert–Schmidt norm by $\|x\|_2$ (so that $\|x\|_2^2 = \operatorname{trace} x^* x$).

2. Lower bounds

Lemma 1. *Given linearly independent $a, b \in \mathcal{B}(H)$, we can find $c_1, c_2 \in \mathcal{B}(H)$, $\delta_1, \delta_2 > 0$ and $z \in \mathbb{C} \setminus \{0\}$ so that $a \otimes b + b \otimes a = c_1 \otimes c_1 + c_2 \otimes c_2$, $c_1 = (za + z^{-1}b)/\sqrt{2}$, $c_2 = i(za - z^{-1}b)/\sqrt{2}$ and*

$$\|a \otimes b + b \otimes a\|_h = \|\delta_1 c_1 c_1^* + \delta_2 c_2 c_2^*\| = \left\|\delta_1^{-1} c_1^* c_1 + \delta_2^{-1} c_2^* c_2\right\|.$$

Proof. We know from general facts cited above that the Haagerup norm infimum for $w = a \otimes b + b \otimes a$ is realised via a representation $w = a_1 \otimes b_1 + a_2 \otimes b_2$. Moreover, by scaling a_i to λa_i and b_i to $\lambda^{-1} b_i$ for a suitable λ we can arrange that

$$\|w\|_h = \|a_1 a_1^* + a_2 a_2^*\| = \|b_1^* b_1 + b_2^* b_2\|.$$

We adopt a convenient matrix notation

$$w = [a, b] \odot [b, a]^t = [a_1, a_2] \odot [b_1, b_2]^t$$

for the two tensor product expressions above (t for transpose) and note that all possible (linearly independent) representations of w take the form

$$w = [a_1', a_2'] \odot [b_1', b_2']^t = \big([a_1, a_2]\alpha\big) \odot \big(\alpha^{-1}[b_1, b_2]^t\big)$$

for a 2×2 invertible scalar matrix α. We use the transpose notation also for the linear operation on the tensor product that sends $a_1 \otimes b_1$ to $b_1 \otimes a_1$. Then we have

$$w = w^t = [b_1, b_2] \odot [a_1, a_2]^t = \big([a_1, a_2]\alpha\big) \odot \big([b_1, b_2]\big(\alpha^{-1}\big)^t\big)^t.$$

From $[b_1, b_2] = [a_1, a_2]\alpha$ and $[a_1, a_2]\alpha^t = [b_1, b_2]$ together with linear independence we get $\alpha = \alpha^t$ symmetric.

We can now express $\alpha = u\Delta u^t$ where u is a unitary matrix and Δ is a diagonal matrix with positive diagonal entries $\delta_1^{-1}, \delta_2^{-1}$ ([10, Takagi's factorisation, 4.4.4] – see also the problems on pp. 212, 217 in [10]). Take $[a_1', a_2'] = [a_1, a_2]u$, $[b_1', b_2'] = [b_1, b_2](u^{-1})^t$ so that

$$w = [a_1', a_2'] \odot [b_1', b_2']^t,$$
$$\|w\|_h = \big\|(a_1')(a_1')^* + (a_2')(a_2')^*\big\| = \big\|(b_1')^*(b_1') + (b_2')^*(b_2')\big\|$$

and

$$[a_1', a_2']\Delta = [a_1, a_2]u\Delta = [a_1, a_2]\alpha\big(u^{-1}\big)^t = [b_1, b_2]\big(u^{-1}\big)^t = [b_1', b_2'].$$

In other words, $a_i'\delta_i^{-1} = b_i'$ $(i = 1, 2)$.

We now take $c_i = \sqrt{\delta_i}\, b_i'$ and we then have $w = c_1 \otimes c_1 + c_2 \otimes c_2$ together with

$$\|w\|_h = \|\delta_1 c_1 c_1^* + \delta_2 c_2 c_2^*\| = \big\|\delta_1^{-1} c_1^* c_1 + \delta_2^{-1} c_2^* c_2\big\|.$$

It remains to relate c_1, c_2 to a, b as claimed. If we put $a' = (c_1 - ic_2)/\sqrt{2}$ and $b' = (c_1 + ic_2)/\sqrt{2}$ we have

$$w = a' \otimes b' + b' \otimes a' = [a', b'] \odot [b', a']^t = [a, b] \odot [b, a]^t.$$

An easy argument shows that there is $z \in \mathbb{C}$ with either $a' = za$ and $b' = z^{-1}b$ or else $a' = z^{-1}b$ and $b' = za$. The first case is exactly as required but for the second case we need to swap the roles of c_1 and c_2. □

Theorem 2. *Assume that H is two-dimensional and $a, b \in \mathcal{B}(H)$. Let $T_{a,b}(x) = axb + bxa$. Then*

$$\|T_{a,b}\|_{cb} \geqslant \|a\|_2 \|b\|_2.$$

Proof. In the case where a, b are linearly dependent ($a = \lambda b$, say, $T_{a,b}x = 2\lambda axa$) we know $\|T\|_{cb} = \|T\| = 2\|a\|\|b\| \geqslant \|a\|_2\|b\|_2$. So we deal only with the case of independent a, b.

We first apply Lemma 1, $\|T_{a,b}\|_{cb} = \|a \otimes b + b \otimes a\|_h$ and the fact that the norm of a 2×2 positive matrix (the max of the eigenvalues) is at least half the trace to get

$$\|T_{a,b}\|_{cb} \geqslant \frac{1}{2}\big(\delta_1\|c_1\|_2^2 + \delta_2\|c_2\|_2^2\big),$$
$$\|T_{a,b}\|_{cb} \geqslant \frac{1}{2}\big(\delta_1^{-1}\|c_1\|_2^2 + \delta_2^{-1}\|c_2\|_2^2\big).$$

We deduce

$$
\begin{aligned}
\|T_{a,b}\|_{cb} &\geqslant \frac{1}{4}\bigl(\bigl(\delta_1+\delta_1^{-1}\bigr)\|c_1\|_2^2+\bigl(\delta_2+\delta_2^{-1}\bigr)\|c_2\|_2^2\bigr)\\
&\geqslant \frac{1}{2}\bigl(\|c_1\|_2^2+\|c_2\|_2^2\bigr)\\
&= \frac{1}{2}\,\mathrm{trace}(c_1^*c_1+c_2^*c_2)\\
&= \frac{1}{2}\,\mathrm{trace}\bigl((za)^*(za)+\bigl(z^{-1}b\bigr)^*\bigl(z^{-1}b\bigr)\bigr)\\
&= \frac{1}{2}\bigl(\|za\|_2^2+\bigl\|z^{-1}b\bigr\|_2^2\bigr)\\
&\geqslant \|za\|_2\bigl\|z^{-1}b\bigr\|_2=\|a\|_2\|b\|_2. \qquad \square
\end{aligned}
$$

Corollary 3 [11, Theorem 2.1]. *For $a, b \in \mathcal{B}(H)$ (H arbitrary)*

$$\|T_{a,b}\|_{cb} \geqslant \|a\|\,\|b\|.$$

Proof. We can reduce the proof to the case where H is two-dimensional by the argument given in [11, Theorem 2.1] (take unit vectors $\xi, \eta \in H$ where $\|a\xi\| \geqslant \|a\| - \varepsilon$ and $\|b\eta\| \geqslant \|b\| - \varepsilon$; consider $T_{qap,qbp}$ where p is a projection onto the span of ξ, η and q a projection onto the span of $a\xi, b\eta$). In two dimensions the result follows from Theorem 2. $\square$

Proposition 4. *If $a, b \in \mathcal{B}(\mathbb{C}^2)$ are symmetric matrices, then*

$$\|T_{a,b}\|_{cb} = \|T_{a,b}\| = \inf_{x>0}\bigl\|xaa^* + (1/x)bb^*\bigr\|.$$

Proof. Now c_1, c_2 obtained from Lemma 1 are symmetric matrices. Using $c_i^* = \bar{c}_i =$ the complex conjugate matrix we have

$$\bigl\|\delta_1^{-1}c_1^*c_1+\delta_2^{-1}c_2^*c_2\bigr\| = \bigl\|\delta_1^{-1}\bar{c}_1c_1+\delta_2^{-1}\bar{c}_2c_2\bigr\| = \bigl\|\delta_1^{-1}c_1\bar{c}_1+\delta_2^{-1}c_1\bar{c}_2\bigr\|.$$

Thus

$$
\begin{aligned}
\|T_{a,b}\|_{cb} &\geqslant \left\|\frac{\delta_1+\delta_1^{-1}}{2}c_1c_1^* + \frac{\delta_2+\delta_2^{-1}}{2}c_2c_2^*\right\|\\
&\geqslant \|c_1c_1^*+c_2c_2^*\| = \|c_1^*c_1+c_2^*c_2\|
\end{aligned}
$$

so that the infimum in the Haagerup tensor norm is attained with $\delta_1 = \delta_2 = 1$. We thus have

$$\|T_{a,b}\|_{cb} = \inf_z\bigl\||z|^2aa^* + |z|^{-2}bb^*\bigr\|$$

and the desired formula for $\|T_{a,b}\|_{cb}$ (taking $x = |z|^2$).

From [18] we know that the convex hulls of the following two sets of matrices intersect

$$W_l = \left\{\begin{bmatrix} \langle c_1c_1^*\xi,\xi\rangle & \langle c_2c_1^*\xi,\xi\rangle \\ \langle c_1c_2^*\xi,\xi\rangle & \langle c_2c_2^*\xi,\xi\rangle \end{bmatrix} : \xi \in H,\ \|\xi\| = 1,\right.$$
$$\left.\left\langle\left(\sum_{i=1}^{2} c_ic_i^*\right)\xi,\xi\right\rangle = \|T_{a,b}\|_{cb}\right\}, \tag{1}$$

$$W_r = \left\{ \begin{bmatrix} \langle c_1^* c_1 \eta, \eta \rangle & \langle c_2^* c_1 \eta, \eta \rangle \\ \langle c_1^* c_2 \eta, \eta \rangle & \langle c_2^* c_2 \eta, \eta \rangle \end{bmatrix} : \eta \in H, \|\eta\| = 1, \right.$$
$$\left. \left\langle \left(\sum_{i=1}^{2} c_i^* c_i \right) \eta, \eta \right\rangle = \|T_{a,b}\|_{cb} \right\}. \tag{2}$$

Moreover the equality $\|T_{a,b}\|_{cb} = \|T_{a,b}\|$ holds if and only if the sets themselves intersect. For either of the sets (say W_l) to consist of more than one element, the hermitian operator concerned must have a double eigenvalue of the maximum eigenvalue $\|T_{a,b}\|_{cb}$, which means that (taking the case W_l)

$$\sum_{i=1}^{2} c_i c_i^*$$

is a multiple of the 2×2 identity matrix. But then by complex conjugation and symmetry $\sum_{i=1}^{2} c_i^* c_i$ is the same multiple of the identity.

In the case when W_l (and W_r by the symmetry) are singletons, we have $\|T_{a,b}\|_{cb} = \|T_{a,b}\|$ and using the following lemma, we can complete the proof for the other case.

Lemma 5. *If $c_1, c_2 \in \mathcal{B}(\mathbb{C}^2)$ are symmetric and satisfy $c_1 c_1^* + c_2 c_2^* =$ a multiple of the identity matrix, there exists u unitary so that either $u c_1 u^t$ and $u c_2 u^t$ are both diagonal* (t *for transpose*) *or*

$$u c_1 u^t = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}, \qquad u c_2 u^t = \begin{pmatrix} \zeta\alpha & \zeta\beta \\ \zeta\beta & -\zeta\bar{\alpha} \end{pmatrix}$$

with $\lambda > 0$, $\beta > 0$, $|\zeta| = 1$.

Proof. We can find u so that $u c_1 u^t$ is diagonal (with positive entries, [10, 4.4.4]).

We can replace c_i by $u c_i u^t$ ($i = 1, 2$) and assume without loss of generality that c_1 is diagonal. Then $c_2 c_2^*$ is diagonal, which means that the rows of c_2 are orthogonal. An easy analysis shows that either c_2 is diagonal or is a multiple (of modulus one) of a matrix of the form

$$\begin{pmatrix} \alpha & \beta \\ \beta & -\bar{\alpha} \end{pmatrix}.$$

The relation satisfied by c_1 and c_2 dictates that c_1 is a multiple of the identity in the latter case. □

Proof of Proposition 4 (*completed*). Invoking the lemma and the fact that $S(x) = uT(u^t x u)u^t$ has the same norm as T, and the same CB norm, we can reduce to the case where c_1, c_2 generate a commutative C^* algebra. In this case the fact that $\|S\|_{cb} = \|S\|$ is known (see references in Section 3). □

Theorem 6. *If $a, b \in \mathcal{B}(H)$ and $T_{a,b}(x) = axb + bxa$. Then*

$$\|T_{a,b}\| \geqslant \|a\| \|b\|.$$

More generally, the same inequality holds if A is a prime C^*-algebra, a, b are in the multiplier algebra of A and $T_{a,b}: A \to A$ is $T_{a,b}(x) = axb + bxa$.

Proof. As shown in [14] and [11, Theorem 2.1], the essential case is the case where $A = \mathcal{B}(H)$ and $H = \mathbb{C}^2$ is 2-dimensional. We show in this case that $\|T_{a,b}\| \geqslant \|a\|\|b\|_2 \geqslant \|a\|\|b\|$ and so we can assume $\|a\| = \|b\|_2 = 1$ ($a, b \in \mathcal{B}(\mathbb{C}^2)$).

There exists u, v unitary so that uav is a diagonal matrix with diagonal entries $1, \lambda$, $0 \leqslant |\lambda| \leqslant 1$. Replacing T by $S(x) = uT(vxu)v$ we can assume that

$$a = \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix}, \qquad b = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}.$$

By multiplying b by a scalar of modulus 1 we can assume that $b_{12} = |b_{12}|$. Multiplying both a and b by a diagonal unitary u with diagonal entries 1 and $\bar{b}_{21}/|b_{21}|$ (that is, replacing T by $S(x) = uT(xu)$) we can assume also that $b_{21} = |b_{21}|$.

Now consider $T_t(x) = T(x^t)^t = axb^t + b^txa$ and

$$T_s(x) = \frac{1}{2}\big(T(x) + T_t(x)\big) = axb_s + b_sxa$$

with

$$b_s = \frac{1}{2}\big(b + b^t\big) = \begin{pmatrix} b_{11} & s_{12} \\ s_{12} & b_{22} \end{pmatrix}, \qquad s_{12} = \frac{b_{12} + b_{21}}{2}.$$

We claim that $\|T_s\| \geqslant 1$ and this will prove the theorem because $\|T_t\| = \|T\|$ and so $\|T_s\| \leqslant \|T\|$.

To show $\|T_s\| \geqslant 1$ we invoke Proposition 4 and show $\|T_s\|_{cb} \geqslant 1$. Note

$$\frac{1}{2} \leqslant \|b_s\|_2^2 = \|b\|_2^2 - \frac{1}{2}(b_{12} - b_{21})^2 \leqslant 1,$$
$$b_s b_s^* = \begin{pmatrix} |b_{11}|^2 + s_{12}^2 & s_{12}(b_{11} + \bar{b}_{22}) \\ s_{12}(\bar{b}_{11} + b_{22}) & |b_{22}|^2 + s_{12}^2 \end{pmatrix}$$

and write $\mu_i^2 = |b_{ii}|^2 + s_{12}^2$ ($i = 1, 2$) for the diagonal entries.

Now consider a unit vector $\xi = (\xi_1, \xi_2) \in \mathbb{C}^2$. Then

$$\begin{aligned} \big\|xaa^* + (1/x)b_sb_s^*\big\| &\geqslant \big\langle (xaa^* + (1/x)b_sb_s^*)\xi, \xi\big\rangle \\ &= x\langle aa^*\xi, \xi\rangle + (1/x)\langle b_sb_s^*\xi, \xi\rangle \\ &\geqslant 2\sqrt{\langle aa^*\xi, \xi\rangle\langle b_sb_s^*\xi, \xi\rangle} \end{aligned}$$

and we claim that there is a point in the joint numerical range

$$W = \big\{(x, y) = \big(\langle aa^*\xi, \xi\rangle, \langle b_sb_s^*\xi, \xi\rangle\big) \colon \|\xi\| = 1\big\} \subseteq \mathbb{R}^2$$

which is also on (or above) the hyperbola $xy = 1/4$. Verifying the claim will complete the proof.

We assume from now on that $\lambda = 0$, as this is the hardest case (smallest $\langle aa^*\xi, \xi\rangle$).

Being the joint numerical range of two hermitian operators (or the numerical range of the single operator $aa^* + ib_sb_s^*$), W is a convex set in the plane. In fact, because the space is 2-dimensional, W is either a straight line (in the case where the two operators commute, that is $s_{12}(b_{11}+\bar{b}_{22})=0$) or else an ellipse (together with its interior) [3, I.6.2]. The ellipse touches the vertical lines $x=0$ and $x=1$ at the points $(0,\mu_2^2)$ and $(1,\mu_1^2)$. Hence the centre of the ellipse is at the midpoint $(x_0,y_0)=(1/2,(1/2)(\mu_1^2+\mu_2^2))=(1/2,(1/2)(|b_{11}|^2+|b_{22}|^2)+s_{12}^2)=(1/2,(1/2)\|b_s\|_2^2)$.

In the case where we have a line and not a genuine ellipse, either $s_{12}=0$ (then the midpoint is $(1/2,1/2)$ and so on the hyperbola) or $b_{11}=-\bar{b}_{22}$ and the line is horizontal (at $y=(1/2)\|b_s\|_2^2\geqslant 1/4$ and so also meets the hyperbola). If $|b_{11}|\geqslant|b_{22}|$, then the point $(x,y)=(1,\mu_1^2)$ on the ellipse already satisfies $4xy\geqslant 1$ and so we assume that $|b_{22}|>|b_{11}|$.

For the genuine ellipse case we write its equation in the form

$$\alpha_{11}(x-x_0)^2+2\alpha_{12}(x-x_0)(y-y_0)+(y-y_0)^2+\beta=0. \tag{3}$$

Using the information that the ellipse has a vertical tangent at $(0,\mu_2^2)$ and its intersection with the line $x=1/2$ is the line segment $\{(1/2,y)\colon |y-y_0|\leqslant s_{12}|b_{11}+\bar{b}_{22}|\}$ (take ξ with $\xi_1=1/\sqrt{2}$), we can solve for the coefficients

$$\begin{aligned}
&\alpha_{12}=\mu_2^2-\mu_1^2=|b_{22}|^2-|b_{11}|^2,\\
&\beta=-s_{12}^2|b_{11}+\bar{b}_{22}|^2,\\
&\alpha_{11}=\bigl(|b_{11}|^2-|b_{22}|^2\bigr)^2+4s_{12}^2|b_{11}+\bar{b}_{22}|^2=\alpha_{12}^2-4\beta.
\end{aligned}\tag{4}$$

We can rewrite the equation in the form

$$\bigl(\alpha_{12}(x-x_0)+(y-y_0)\bigr)^2-4\beta(x-x_0)^2+\beta=0$$

and so we can parametrise the ellipse via

$$x=x_0+(1/2)\sin\omega, \tag{5}$$

$$\begin{aligned}
y&=y_0-(1/2)\alpha_{12}\sin\omega+\sqrt{-\beta}\cos\omega\\
&=(1/2)\bigl(|b_{11}|^2+|b_{22}|^2\bigr)+s_{12}^2-(1/2)\bigl(|b_{22}|^2-|b_{11}|^2\bigr)\sin\omega\\
&\quad+s_{12}|b_{11}+\bar{b}_{22}|\cos\omega
\end{aligned}\tag{6}$$

$(0\leqslant\omega\leqslant 2\pi)$. We look for $\omega\in[0,\pi/2]$ where $4xy\geqslant 1$. We use $|b_{11}+\bar{b}_{22}|\geqslant|b_{22}|-|b_{11}|=\varepsilon_{12}$ (say) and represent for convenience $|b_{11}|^2+|b_{22}|^2=\cos^2\theta$ $(0\leqslant\theta<\pi/2)$. Note $4s_{12}^2\geqslant(b_{12}-b_{21})^2$, $2s_{12}^2\geqslant(1/2)(b_{12}-b_{21})^2=1-\|b_s\|_2^2$, $4s_{12}^2\geqslant 1-\cos^2\theta$ and $s_{12}\geqslant(1/2)\sin\theta$. Moreover $|b_{22}|+|b_{11}|\leqslant\sqrt{2}\cos\theta$. Thus

$$2y\geqslant(1/2)+(1/2)\cos^2\theta+\varepsilon_{12}(\sin\theta\cos\omega-\sqrt{2}\cos\theta\sin\omega). \tag{7}$$

Choose $\omega=\tan^{-1}((1/\sqrt{2})\tan\theta)$, $\sin\omega=\sin\theta/\sqrt{\sin^2\theta+2\cos^2\theta}$ and

$$4xy\geqslant\left(1+\frac{\sin\theta}{\sqrt{1+\cos^2\theta}}\right)\bigl(1/2+(1/2)\cos^2\theta\bigr)\geqslant 1. \qquad\square$$

Remark 7. With some additional effort, we can adapt the proof above to establish the lower bound $\|T_{a,b}\|\geqslant\|a\|_2\|b\|_2$ for the case $a,b\in\mathcal{B}(\mathbb{C}^2)$ (and thus get a stronger result than Theorem 2).

It seems that this does not follow from the methods used in [4].

Proof. A sketch of the additional details follows. We assume by symmetry that $\|a\|_2/\|a\| \leqslant \|b\|_2/\|b\|$ and normalise $\|a\| = 1$, $\|b\|_2 = 1$ as before. This time we cannot assume $\lambda = 0$, but we note that $|\det b| \geqslant |\lambda|/(1 + |\lambda|^2)$ (for example, take $b = ub_0v$ where u, v are unitary and b_0 is diagonal with diagonal entries $1/\sqrt{1+\mu^2}$ and $\mu/\sqrt{1+\mu^2}$, $1 \geqslant \mu \geqslant |\lambda|$).

In this case the ellipse will have vertical tangents at $x = |\lambda|^2$ and $x = 1$ and will be centered at $(x_0, y_0) = ((1 + |\lambda|^2)/2, (1/2)\|b_s\|_2^2)$. Eq. (3) of the ellipse now has

$$\alpha_{12} = \frac{|b_{22}|^2 - |b_{11}|^2}{1 - |\lambda|^2},$$

β as in (4) and $\alpha_{11} = \alpha_{12}^2 - 4\beta/(1 - |\lambda|^2)^2$. We can rewrite the equation of the ellipse as

$$\bigl(\alpha_{12}(x - x_0) + (y - y_0)\bigr)^2 - \frac{4\beta}{(1 - |\lambda|^2)^2}(x - x_0)^2 + \beta = 0$$

and then we can parametrise via

$$x = (1/2)\bigl(1 + |\lambda|^2\bigr) + (1/2)\bigl(1 - |\lambda|^2\bigr)\sin\omega \tag{8}$$

(in place of (5)) and (6) as before.

We now seek a point (x, y) on the ellipse where $4xy \geqslant 1 + |\lambda|^2$.

To dispose of the case $|b_{11}| \geqslant |b_{22}|$ we show $4y_0 \geqslant 1 + |\lambda|^2$ (and this also deals with the case where the ellipse degenerates into a line). Using $\|b\|_2 = 1$,

$$\begin{aligned} 4y_0 &= 2\|b_s\|^2 = 2 - (b_{12} - b_{21})^2 = 1 + \bigl(|b_{11}|^2 + |b_{22}|^2 + 2b_{12}b_{21}\bigr) \\ &\geqslant 1 + 2|b_{11}b_{22} - b_{12}b_{21}| \geqslant 1 + 2\frac{|\lambda|}{1 + |\lambda|^2} \geqslant 1 + |\lambda|^2. \end{aligned}$$

When $\varepsilon_{12} = |b_{22}| - |b_{11}| > 0$ we choose the same ω as before. From the lower bound (7) and (8) we get the desired $4xy \geqslant 1 + |\lambda|^2$ if we have $\cos^2\theta \geqslant 2|\lambda|^2/(1 + |\lambda|^4)$. For the remaining case note that

$$2y \geqslant |b_{11}|^2 + |b_{22}|^2 + 2s_{12}^2 = \frac{1}{2} + \frac{1}{2}\bigl(|b_{11}|^2 + |b_{22}|^2\bigr) + b_{12}b_{21} \geqslant \frac{1}{2} + |\det b|$$

and the resulting $2y \geqslant 1/2 + |\lambda|/(1 + |\lambda|^2)$ is a better lower bound that (7) when $\cos^2\theta < 2|\lambda|/(1+|\lambda|^2)$. In this situation we do get $4xy \geqslant 1+|\lambda|^2$. All eventualities are now covered because $2|\lambda|^2/(1 + |\lambda|^4) \leqslant 2|\lambda|/(1 + |\lambda|^2)$. □

3. Commuting cases

We consider now some cases where we can find relatively explicit formulae for $\|T_{a,b}\|$. These may shed some light on the difficulty of finding any explicit formula for the norm of a general elementary operator. One may consider the Haagerup formula for the CB norm as an explicit formula, though we shall observe that this is not so simple to compute even in the simplest cases.

The equality of the CB norm and the operator norm of $T_{a,b}$ holds if a, b are commuting normal operators. This appears already in the unpublished [9]. A significant part of the argument from [9] is published in [1, §5.4] and the remaining part uses the fact that all states on a commutative C^*-algebra are vector states. (By the Putnam–Fuglede theorem the C^*-algebra generated by commuting normal operators is commutative.) See also [16, Theorem 2.1] for a more general result on bimodule homomorphisms. Another proof (with slightly weaker hypotheses) is in [18].

We deal here only with H of dimension 2.

Proposition 8. *If H is two-dimensional and $a, b \in \mathcal{B}(H)$ commute, then $\|T_{a,b}\|_{cb} = \|T_{a,b}\|$.*

Proof. We can find an orthonormal basis of H so that a and b both have upper triangular (2×2) matrices. If a, b are diagonal, then they generate a commutative C^*-subalgebra of $\mathcal{B}(H)$ and in this case that $\|T_{a,b}\|_{cb} = \|a \otimes b + b \otimes a\|_h = \|T_{a,b}\|$ (see above).

Now c_1, c_2 obtained from Lemma 1 are also commuting upper triangular matrices. As used already in (1)–(2), from [18] we know that the convex hulls of the two sets of matrices intersect. In this case the sets are as not quite as before. Each c_i should be replaced by $\sqrt{\delta_i}\, c_i$ in the definition of W_l and by $1/\sqrt{\delta_i}\, c_i$ for W_r. Moreover the equality $\|T_{a,b}\|_{cb} = \|T_{a,b}\|$ holds if and only if the sets themselves intersect. For either of the sets (say W_l) to consist of more than one element, the hermitian operator concerned must have a double eigenvalue of the maximum eigenvalue $\|T_{a,b}\|_{cb}$, which means that (taking the case W_l)

$$\sum_{i=1}^{2} \delta_i c_i c_i^*$$

is a multiple of the 2×2 identity matrix. But the following lemma asserts that this cannot happen unless $\sqrt{\delta_1}\, c_1$ and $\sqrt{\delta_2}\, c_2$ are simultaneously diagonalisable (the case where we know the result). So W_l and W_r have one element each, they intersect and the result follows. $\square$

Lemma 9. *If a_1, a_2 are commuting elements of $\mathcal{B}(H)$ with H of dimension 2 and if $a_1 a_1^* + a_2 a_2^*$ is a multiple of the identity, then a_1, a_2 generate a commutative $*$-subalgebra of $\mathcal{B}(H)$.*

Proof. In a suitable orthonormal basis for H we can represent a_1, a_2 as upper triangular matrices

$$a_1 = \begin{bmatrix} x_1 & y_1 \\ 0 & z_1 \end{bmatrix}, \qquad a_2 = \begin{bmatrix} x_2 & y_2 \\ 0 & z_2 \end{bmatrix}$$

and then the condition for them to commute is $y_1(x_2 - z_2) = y_2(x_1 - z_1)$. (For later reference we call this value ρ.) So if $y_1 = 0$, then either y_2 also zero (both matrices diagonal and we are done) or else $x_1 = z_1$ and $a_1 = x_1 I_2$ is a multiple of the identity. But then $a_2 a_2^*$ is a multiple of the identity and this forces $y_2 = 0$ (both diagonal again).

In the case when y_1 and y_2 are both nonzero, we compute

$$a_1a_1^* + a_2a_2^* = \begin{bmatrix} |x_1|^2 + |y_1|^2 + |x_2|^2 + |y_2|^2 & y_1\bar{z}_1 + y_2\bar{z}_2 \\ \bar{y}_1 z_1 + \bar{y}_2 z_2 & |z_1|^2 + |z_2|^2 \end{bmatrix}.$$

Thus we have $y_1\bar{z}_1 + y_2\bar{z}_2 = 0$, which implies $(z_1, z_2) = \omega(\bar{y}_2, -\bar{y}_1)$ for some $\omega \in \mathbb{C}$. We also have equality of the two diagonal entries of the above matrix which gives us

$$|x_1|^2 + |x_2|^2 = \big(|\omega|^2 - 1\big)\big(|y_1|^2 + |y_2|^2\big).$$

Now $x_1 = \rho/y_2 + z_1 = \rho/y_2 + \omega\bar{y}_2$ and $x_2 = \rho/y_1 - \omega\bar{y}_1$, yielding

$$\left|\frac{\rho}{y_2} + \omega\bar{y}_2\right|^2 + \left|\frac{\rho}{y_1} - \omega\bar{y}_1\right|^2 = \big(|\omega|^2 - 1\big)\big(|y_1|^2 + |y_2|^2\big)$$

and hence the impossible condition

$$|\rho|^2\big(|y_1|^{-2} + |y_2|^{-2}\big) = -\big(|y_1|^2 + |y_2|^2\big). \qquad \square$$

Example 10. Consider $T_{a,b}$ acting on $\mathcal{B}(\mathbb{C}^2)$ with a, b diagonal 2×2 matrices. Then c_1, c_2 in Lemma 1 are also diagonal and we can see then directly that

$$\|c_1c_1^* + c_2c_2^*\| \leqslant \frac{1}{2}\big(\|\delta_1 c_1 c_1^* + \delta_2 c_2 c_2^*\| + \big\|\delta_1^{-1} c_1^* c_1 + \delta_2^{-1} c_2^* c_2\big\|\big)$$

so that the Haagerup norm is minimised with $\delta_1 = \delta_2 = 1$. Also $\|c_1c_1^* + c_2c_2^*\| = \||z|^2 aa^* + |z|^{-2} bb^*\|$ and so the Haagerup norm is the minimum of this.

Say the diagonal entries are λ_1, λ_2 for a and μ_1, μ_2 for b. Normalising a and b to have norm one, we can assume $\max(|\lambda_1|, |\lambda_2|) = 1$ and $\max(|\mu_1|, |\mu_2|) = 1$. If they both attain the maximum at the same index then it is easy to see that $\|T_{a,b}\| = 2 = 2\|a\|\|b\|$. If not, assume by symmetry that $|\lambda_1| = 1 = |\mu_2|$ and that $|\mu_1| \leqslant |\lambda_2|$. The Haagerup norm is then the minimum value of the maximum of two functions, and can be computed by elementary means. It gives the norm (the same as the CB norm in this case) as

$$\|T_{a,b}\| = \begin{cases} 2|\lambda_2| & \text{if } |\lambda_2| \geqslant 1/\sqrt{2} \text{ and } |\mu_1|^2 < 2 - |\lambda_2|^{-2}, \\ \dfrac{1 - |\mu_1|^2|\lambda_2|^2}{\sqrt{(1 - |\mu_1|^2)(1 - |\lambda_2|^2)}} & \text{otherwise.} \end{cases} \tag{9}$$

Summarising the calculation in a basis independent way, we can state the following.

Proposition 11. *Suppose that $a, b \in \mathcal{B}(\mathbb{C}^2)$ are commuting normal operators and that $\|a\|_2/\|a\| \geqslant \|b\|_2/\|b\|$. If a, b attain their norms at a common unit vector, then $\|T_{a,b}\| = 2\|a\|\|b\|$. If not*

$$\|T_{a,b}\| = \begin{cases} 2\|b\|\sqrt{\|a\|_2^2 - \|a\|^2} & \textit{if } \|a\|_2 \geqslant \sqrt{3/2}\|a\| \textit{ and } \|b\|_2^2 < \\ & \quad 3\|b\|^2 - (\|a\|^2\|b\|^2)/(\|a\|_2^2 - \|a\|^2) \\ \dfrac{\|a\|_2^2\|b\|^2 + \|a\|^2\|b\|_2^2 - \|a\|_2^2\|b\|_2^2}{\sqrt{(2\|a\|^2 - \|a\|_2^2)(2\|b\|^2 - \|b\|_2^2)}} & \textit{otherwise}. \end{cases} \tag{10}$$

Proof. Note that in a suitable orthonormal basis of $\mathbb{C}^2$, a, b will both be represented by diagonal matrices. □

4. A formula for self-adjoint operators

Our aim here is to present a proof of a formula from [12] that follows a similar approach to the one used in Section 2.

For a linear operator $T:\mathcal{B}(H) \to \mathcal{B}(H)$ we denote by T^* the associated operator defined by $T^*(x) = T(x^*)^*$. We call T self-adjoint if $T^* = T$.

Lemma 12 [18]. *For $T:\mathcal{B}(H) \to \mathcal{B}(H)$ a self-adjoint elementary operator, there is a representation $Tx = \sum_{i=0}^{\ell} \varepsilon_i c_i x c_i^*$ with $c_i \in \mathcal{B}(H)$, $\varepsilon_i \in \{-1, 1\}$ for each i and*

$$\|T\|_{cb} = \left\| \sum_{i=1}^{\ell} c_i c_i^* \right\|.$$

Lemma 13 [18]. *Let $T = T^*:\mathcal{B}(H) \to \mathcal{B}(H)$ be an elementary operator, $Tx = \sum_{i=1}^{k} c_i x c_i^* - \sum_{i=k+1}^{\ell} c_i x c_i^*$ with $0 \leqslant k \leqslant \ell$ and $(c_i)_{i=1}^{\ell}$ linearly independent.* (*We include $k = 0$ for the case where the first summand is absent and when $k = \ell$ the second summand is absent.*) *Then the ordered pair* $(k, \ell - k)$ (*which we could call the 'signature'*) *is the same for all such representations of T.*

Example 14 [12]. For $T:\mathcal{B}(H) \to \mathcal{B}(H)$ given by $Tx = axb^* + bxa^*$ with a, b linearly independent, we have

$$\|T\|_{cb} = \inf\bigl\{\|raa^* + sbb^* + 2t\Im(ab^*)\|\colon\ r > 0, s > 0, t \in \mathbb{R}, rs - t^2 = 1\bigr\}$$

(where $\Im(ab^*) = (ab^* - ba^*)/(2i)$ is the imaginary part).

Proof. We can rewrite $Tx = c_1 x c_1^* - c_2 x c_2^*$ if we take $c_1 = (a + b)/\sqrt{2}$ and $c_2 = (a - b)/\sqrt{2}$. Note for later use that we can undo this change by $a = (c_1 + c_2)/\sqrt{2}$, $b = (c_1 - c_2)/\sqrt{2}$.

According to Lemmas 12 and 13 we can find $\|T\|_{cb}$ as the infimum of $\|c_1'(c_1')^* + c_2'(c_2')^*\|$ where

$$[c_1', c_2'] = [c_1, c_2]\alpha$$

and α is an invertible 2×2 matrix with the property that

$$\alpha \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \alpha^* = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

As unitary diagonal α have no effect on the estimate $\|c_1'(c_1')^* + c_2'(c_2')^*\|$ we can work modulo these unitaries and then elementary analysis of the possibilities shows that we need only consider the cases

$$\alpha = \begin{bmatrix} p & \sqrt{p^2 - 1}\,\mathrm{e}^{i\theta} \\ \sqrt{p^2 - 1}\,\mathrm{e}^{-i\theta} & p \end{bmatrix}$$

(with $p \geqslant 1$, $\theta \in \mathbb{R}$). This leads us to consider only

$$[c_1', c_2'] = \left[pc_1 + \sqrt{p^2 - 1}\,\mathrm{e}^{-i\theta} c_2, \sqrt{p^2 - 1}\,\mathrm{e}^{i\theta} c_1 + pc_2\right].$$

Hence

$$\begin{aligned}
\|T\|_{cb} &= \inf_{p \geqslant 1,\ \theta \in \mathbb{R}} \left\| c_1'(c_1')^* + c_2'(c_2')^* \right\| \\
&= \inf \left\| \left(2p^2 - 1\right)(c_1 c_1^* + c_2 c_2^*) + 4p\sqrt{p^2 - 1}\,\Re\left(\mathrm{e}^{i\theta} c_1 c_2^*\right) \right\| \\
&= \inf \left\| \left(2p^2 - 1\right)(aa^* + bb^*) + 2p\sqrt{p^2 - 1}\cos\theta(aa^* - bb^*) \right. \\
&\quad \left. + 4p\sqrt{p^2 - 1}\sin\theta\,\Im(ab^*) \right\| \\
&= \inf_{p \geqslant 1,\ \theta \in \mathbb{R}} \left\| \left(2p^2 - 1 + 2p\sqrt{p^2 - 1}\cos\theta\right) aa^* \right. \\
&\quad \left. + \left(2p^2 - 1 - 2p\sqrt{p^2 - 1}\cos\theta\right) bb^* + 4p\sqrt{p^2 - 1}\sin\theta\,\Im(ab^*) \right\|.
\end{aligned}$$

The claimed formula follows by taking $r = 2p^2 - 1 + 2p\sqrt{p^2 - 1}\cos\theta$, $s = 2p^2 - 1 - 2p\sqrt{p^2 - 1}\cos\theta$ and $t = 2p\sqrt{p^2 - 1}\sin\theta$, noting that $rs - t^2 = 1$. We can recover p and $\cos\theta$ from r, s (with $r > 0$, $s > 0$, $rs \geqslant 1$) using $r + s = 2(2p^2 - 1)$, $r - s = 4p\sqrt{p^2 - 1}\cos\theta$. From the sign of $t = \pm\sqrt{rs - 1}$ we get $\sin\theta$ and so θ modulo 2π. □

Remark 15. In [12] it is also shown that, for T as in the example above, $\|T\|_{cb} = \|T\|$. A more general result can be found in [18].

Acknowledgements

Part of this work was done during a visit by the author to the University of Edinburgh in the autumn of 2002. A significant impetus to the work arose from discussions with Bojan Magajna and Aleksej Turnšek during a visit to Ljubljana in March 2003 and the author is very grateful to them for that. Thanks also to P. Legiša for finding the reference to [10].

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