III. Approximation of Distributions of Sums of Weakly Dependent Random Variables by the Normal Distribution

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This article is a review of known results on approximating distributions of sums of weakly dependent random variables (r.v.'s) and random fields defined on the integer lattice Z^d by the normal distribution. It also contains some new results of the author. Attention is given mainly to methods of proof due to Stein, Tikhomirov and Heinrich.

§1. Weak Dependence Conditions and Covariance Inequalities

Let $X_1, X_2, \ldots$ be a sequence of real r.v.'s, $\mathcal{F}_a^b$ the σ-algebra generated by X_i, $a \le i \le b$, R the real line, and $N = \{1, 2, \ldots\}$.

Later on we shall assume that the sequence $X_1, X_2, \ldots$ satisfies one of the following weak dependence conditions:

1. *m-dependence*: $\mathcal{F}_1^r$ and $\mathcal{F}_{r'}^\infty$ are independent for all integers r and r' such that $1 \le r < r' < \infty$, $r' - r > m$;
2. *ψ-mixing*:
$$\psi(\tau) = \sup_{t \in N} \sup_{\substack{A \in \mathcal{F}_1^t,\ B \in \mathcal{F}_{t+\tau}^\infty \\ \mathbf{P}\{A\}>0,\ \mathbf{P}\{B\}>0}} |\mathbf{P}\{AB\} - \mathbf{P}\{A\}\mathbf{P}\{B\}| / \mathbf{P}\{A\}\mathbf{P}\{B\} \downarrow 0,\ \tau \to \infty;$$
3. *uniformly strong mixing* (u.s.m.):
$$\varphi(\tau) = \sup_{t \in N} \sup_{\substack{A \in \mathcal{F}_1^t,\ B \in \mathcal{F}_{t+\tau}^\infty \\ \mathbf{P}(A)>0}} |\mathbf{P}\{AB\} - \mathbf{P}\{A\}\mathbf{P}\{B\}| / \mathbf{P}\{A\} \downarrow 0,\ \tau \to \infty;$$
4. *absolute regularity* (a.r.):
$$\beta(\tau) = \sup_{t \in N} \mathbf{E}\left(\sup_{B \in \mathcal{F}_{t+\tau}^\infty} |\mathbf{P}\{B/\mathcal{F}_1^t\} - \mathbf{P}\{B\}| \right) \downarrow 0,\ \tau \to \infty;$$
5. *strong mixing* (s.m.):
$$\alpha(\tau) = \sup_{t \in N} \sup_{A \in \mathcal{F}_1^t,\ B \in \mathcal{F}_{t+\tau}^\infty} |\mathbf{P}\{AB\} - \mathbf{P}\{A\}\mathbf{P}\{B\}| \downarrow 0,\ \tau \to \infty.$$

Strong mixing was introduced by Rosenblatt (1956), absolute regularity by Kolmogorov (Volkonskii and Rozanov (1959)), uniformly strong mixing by Ibragimov (1962), ψ-mixing by Blum, Hanson and Koopmans (1963) and m-dependence by Hoeffding and Robbins (1948) (and goes back to Bernstein (1926)).

All of these conditions are requirements of weak dependence between the start and end of a sequence of r.v.'s.

Information about these and other measures of weak dependency (and the corresponding mixing coefficients) as well as appropriate references may be found, for example, in Ibragimov and Linnik (1965), Ibragimov and Rozanov (1970), Statulevičius (1974), Philipp and Stout (1975), Bradley, Bryc and Janson (1985), Bradley (1986) and Bulinskii (1987), (1989). The mixing coefficients defined above are interrelated by the following inequalities (Ibragimov and Linnik (1965), Hipp (1979a)):

$$2\alpha(\tau) \le \varphi^{1/2}(\tau),$$
$$\alpha(\tau) \le \beta(\tau) \le \varphi(\tau) \le \psi(\tau),$$
$$\alpha(\tau) \le 1/4,\ \varphi(\tau) \le 1.$$

Important refinements of these inequalities, a comparison of the mixing coefficients and other measures of dependency are found in Kolmogorov and Rozanov (1960), Ibragimov and Linnik (1965), Ibragimov and Rozanov (1970), Peligrad (1982), Bradley (1983), (1986), Bradley and Bryc (1985), Bradley, Bryc and Janson (1985), Bulinskii (1985b), (1989) and Liptser and Shiryaev (1986). We denote by $\operatorname{cov}(\xi, \eta) = \mathbf{E}\xi\eta - \mathbf{E}\xi\mathbf{E}\eta$ the covariance of real r.v.'s ξ and η. We shall subsequently need bounds for the covariance in terms of the mixing coefficients α, β, φ and ψ.

Lemma 1. *Suppose that ξ is $\mathcal{F}_1^t$-measurable and η is $\mathcal{F}_{t+\tau}^\infty$-measurable; $t, \tau \in N$.*

1. *If* $\mathbf{E}|\xi| < \infty$ *and* $\mathbf{E}|\eta| < \infty$, *then* (see Philipp (1969a)):
$$|\operatorname{cov}(\xi, \eta)| \le \mathbf{E}|\xi|\mathbf{E}|\eta|\psi(\tau); \tag{1}$$
2. *If* $\mathbf{E}|\xi| < \infty$ *and* $\mathbf{P}\{|\eta| > C\} = 0$, *then* (see Billingsley (1977))
$$|\operatorname{cov}(\xi, \eta)| \le 2C\mathbf{E}|\xi|\varphi(\tau); \tag{2}$$
3. *If* $\mathbf{E}|\xi|^q < \infty$ *and* $\mathbf{E}|\eta|^r < \infty$, $q, r > 1$, $q^{-1} + r^{-1} = 1$, *then* (see Ibragimov (1962))
$$|\operatorname{cov}(\xi, \eta)| \le 2\mathbf{E}^{1/q}|\xi|^q\mathbf{E}^{1/r}|\eta|^r\varphi^{1/q}(\tau); \tag{3}$$
4. *If* $\mathbf{P}\{|\xi| > C_1\} = \mathbf{P}\{|\eta| > C_2\} = 0$, *then* (see Volkonskii and Rozanov (1959), (1961), Ibragimov (1962))
$$|\operatorname{cov}(\xi, \eta)| \le 4C_1C_2\alpha(\tau); \tag{4}$$
5. *If* $\mathbf{E}|\xi|^r < \infty$, $r > 1$, *and* $\mathbf{P}\{|\eta| > C\} = 0$, *then* (see Davydov (1968), Hipp (1979a))
$$|\operatorname{cov}(\xi, \eta)| \le 4C\mathbf{E}^{1/r}|\xi|^r(\alpha(\tau))^{1-r^{-1}}; \tag{5}$$
6. *If* $\mathbf{E}|\xi|^q < \infty$ *and* $\mathbf{E}|\eta|^r < \infty$, $q, r > 1$, $q^{-1} + r^{-1} < 1$, *then* (see Davydov (1968), Hipp (1979a))
$$|\operatorname{cov}(\xi, \eta)| \le 6\mathbf{E}^{1/q}|\xi|^q\mathbf{E}^{1/r}|\eta|^r(\alpha(\tau))^{1-q^{-1}-r^{-1}}. \tag{6}$$

Inequalities (1–6) may be extended easily by induction to finitely many r.v.'s (see, for example, Volkonskii and Rozanov (1959), (1961), Roussas and Ionnides (1987)). We shall need a result that follows from (4) and (5).

Lemma 2. *Suppose that ξ_j is $\mathcal{F}_{s_j}^{t_j}$-measurable, $j = 1, 2, \ldots, k$, where $1 \le s_1 \le t_1 < s_2 \le t_2 < \ldots < s_k \le t_k < \infty$, and that $\tau = \min_{1 \le j < k}(s_{j+1} - t_j)$.*

1. *If $P\{|\xi_j| > C_j\} = 0$, $j = 1, 2, \ldots, k$, then for $k \ge 2$*
$$\left| \mathbf{E}\left(\prod_{j=1}^k \xi_j\right) - \prod_{j=1}^k \mathbf{E}\xi_j \right| \le 4(k-1)\prod_{j=1}^k C_j\alpha(\tau); \tag{7}$$

2. *If* $\mathbf{E}|\xi_1|^r < \infty$, $r > 1$, *and* $\mathbf{P}\{|\xi_j| > C_j\} = 0$, $j = 2, 3, \ldots, k$, *then for* $k \geq 2$

$$\left| \mathbf{E}\left(\prod_{j=1}^{k} \xi_j\right) - \prod_{j=1}^{k} \mathbf{E}\xi_j \right| \leq 4(k-1)\mathbf{E}^{1/r}|\xi_1|^r \prod_{j=2}^{k} C_j(\alpha(\tau))^{1-r^{-1}}. \quad (8)$$

Sometimes it is advantageous to have an estimate of the covariance of r.v.'s in terms of the mixing coefficients when the σ-algebras $\mathcal{F}_1^t$ (the "past") and $\mathcal{F}_{t+\tau}^{\infty}$ (the "future") are replaced by σ-algebras of a more complex structure (see, for example, Lemma 3).

Let $\sigma(\mathcal{G} \cup \mathcal{H})$ be the σ-algebra generated by the σ-algebras $\mathcal{G}$ and $\mathcal{H}$.

Lemma 3 (Takahata (1981)). *Let* ξ *be* $\mathcal{F}_{t+\tau}^{t+\tau+n-1}$*-measurable and* η *be* $\sigma\{\mathcal{F}_1^t \cup \mathcal{F}_{t+2\tau+n-1}^{\infty}\}$*-measurable,* t, τ, $n \in N$.

1. *If* $\mathbf{E}|\xi|^q < \infty$ *and* $\mathbf{E}|\eta|^r < \infty$, $q, r > 0$, $q^{-1} + r^{-1} = 1$, *then*

$$|\operatorname{cov}(\xi, \eta)| \leq 6\mathbf{E}^{1/q}|\xi|^q \mathbf{E}^{1/r}|\eta|^r \psi^{1/q}(\tau); \quad (9)$$

2. *If* $\mathbf{E}|\xi|^q < \infty$ *and* $\mathbf{E}|\eta|^r < \infty$, $q, r > 0$, $q^{-1} + r^{-1} < 1$, *then*

$$|\operatorname{cov}(\xi, \eta)| \leq 18\mathbf{E}^{1/q}|\xi|^q \mathbf{E}^{1/r}|\eta|^r (\beta(\tau))^{1-q^{-1}-r^{-1}}. \quad (10)$$

If ξ and η in inequalities (1)–(10) are complex r.v.'s, then the right-hand sides of these inequalities must be multiplied by 4 (see, for example, Ibragimov and Linnik (1965), Roussas and Ionnides (1987)).

§2. Estimation of the Rate of Convergence in the Central Limit Theorem for Weakly Dependent Random Variables

2.1. Introduction and Notation.

We shall concentrate on estimating the rate of convergence in the one-dimensional central limit theorem (CLT).

Conditions for the applicability of the CLT to weakly dependent r.v.'s have been studied by many authors.

Devoted specifically to m-dependent r.v.'s are the papers by Hoeffding and Robbins (1948), Diananda (1955), Kallianpur (1955), Orey (1958), Zaremba (1958), and Berk (1973); to functionals defined on Markov chains, the papers by Bernstein (1926), Sirazhdinov (1955), Dobrushin (1956a,b), Nagaev (1957), (1962), Statulevičius (1961), (1969a,b), (1970a), Gudynas (1977), and Lifshits (1984). Ibragimov (1962) is basic as far as strictly stationary sequences of weakly dependent r.v.'s are concerned. The results in that paper were subsequently refined and generalized by Rosén (1967), Serfling (1968), Philipp (1969a), Berk (1973), Ibragimov (1975), Bradley (1981), (1988), Peligrad (1982), (1985), (1986a,b), (1986), (1990), Herrndorff (1983a,b), (1985), Utev (1984), (1990a,b), Samur (1984), Grin' (1990a,b), etc. Bergström (1970), (1971), (1972) studied this problem by means of his comparison method. The latest results as well as a bibliography on this topic may be found, for example, in the above papers by Peligrad, Bradley, Utev and Grin'. The advances by Utev in (1990a,b) were made by taking advantage of Peligrad's bounds for large deviation probabilities (1985) which were also used in a modified form by Grin' (1990a) (see also (1990b)).

As of today, there exists much research devoted to studying the rate of convergence in the CLT for weakly dependent r.v.'s.

This question was first investigated by Petrov for m-dependent r.v.'s in (1960). He obtained a bound in the uniform metric of order $O(n^{-(s-2)/(6s-4)})$ under the assumption that the s-th absolute moments of the variables, $2 < s \leq 3$, are uniformly bounded and the variance of their sum increases linearly; here and elsewhere n is the number of terms. The problem was later investigated for m-dependent r.v.'s by Ibragimov (1967), Egorov (1970), Stein (1972), Erickson (1973)–(1975), Tikhomirov (1980), Shergin (1976), (1979), (1983), Maejima (1978), Yudin (1981), (1989a), Zuparov (1981), Heinrich (1982), (1984), (1985a,b,d), Sunklodas (1982), (1984), (1989), Rhee (1985), (1986a), Zuev (1986) and others. The first optimal estimate, that is, of order $O(n^{-(s-2)/2})$, in the uniform metric for m-dependent r.v.'s in the CLT was obtained by Ibragimov (1967). He considered a sequence of r.v.'s of the form $f(\varepsilon_i, \varepsilon_{i+1}, \ldots, \varepsilon_{i+m-1})$, $i = 1, 2, \ldots$, where $\varepsilon_1, \varepsilon_2, \ldots$ are independent and identically distributed r.v.'s with $\mathbf{E}|\varepsilon_1|^s < \infty$, $2 < s \leq 3$.

The rate of convergence in the one-dimensional CLT has been estimated under other weak dependence conditions by Statulevičius (1962), (1974), (1977a,b), Iosifescu (1968), Philipp (1969b), Stein (1972), Dubrovin (1971), Bulinskii (1977), Yoshihara (1978a), Negishi (1977), Tikhomirov (1976), (1980), Dubrovin and Moskvin (1979), Schneider (1981), Zuparov (1981), Yudin (1984), (1987), (1989b), (1990), (1991), Sunklodas (1977a), (1982), (1984), (1989), Lappo (1986) and others.

These papers made use of diverse methods of proof.

The method of cumulants (semi-invariants) involving the logarithmic derivatives of the characteristic function (c.f.) of the original sum was developed by Statulevičius in (1961), (1969a,b), (1970a) in which he found exact bounds for nonhomogeneous Markov chains. This method was investigated under other weak dependence conditions in (1962), (1974), (1977a,b). Many of the papers mentioned above (see, for example, Petrov (1960), Iosifescu (1968), Philipp (1969b), Egorov (1970) and Bulinskii (1977)) employed methods of proof based on Bernstein's fruitful idea of splitting the sum Z_n of r.v.'s into two sums $Z_n = U_n + V_n$. Then by virtue of the weak dependency, U_n behaves as if its terms are independent and V_n has a relatively small variance. For example, when Z_n is centered and normalized, this was accomplished as a rule via the following inequality (Petrov (1960)): For any positive ε and $x \in R$,

$$|\mathbf{P}\{U_n + V_n < x\} - \Phi(x)| \le \sup_x |\mathbf{P}\{U_n < x\} - \Phi(x)| + \frac{\varepsilon}{\sqrt{2\pi}} + \mathbf{P}\{|V_n| \ge \varepsilon\}, \qquad (11)$$

where $\Phi(x)$ is the standard normal distribution function. However the estimates obtained in this way for the one-dimensional case are not optimal and the order is no better than $O(n^{-1/4})$ in the uniform metric.

Nevertheless, Bernstein's classical method has received further development in recent years in the estimation of the rate of convergence in the CLT for weakly dependent n-dimensional and infinite-dimensional r.v.'s. It is precisely in the infinite-dimensional case that this method has manifested its generality. But since we are confining ourselves to only the one-dimensional (that is, the R-valued) case, the reader interested in Bernstein's method in the n-dimensional and infinite-dimensional cases is referred to the papers by Hipp (1979b) and Zuparov (1983), (1984) (see also Dubrovin (1974), Lapinskas (1976), Gabbasov (1977), Sunklodas (1978)). Multi-dimensional and infinite-dimensional analogues of (11) may be found in Lapinskas (1976), Sunklodas (1978), Hipp (1979b) and Zuparov (1984). The passage to independent infinite-dimensional r.v.'s under absolute regularity can be accomplished by means of appropriate approximating inequalities in Gudynas (1989) and Eberlein (1984) (see also Yoshihara (1976), Hipp (1979b)), Zuparov (1983), (1984)).

In (1972), Stein gave a new way of estimating the rate of convergence in the CLT for weakly dependent r.v.'s. Stein's method involves using a linear differential equation in terms of the difference between the distribution function (d.f.) of the sum of weakly dependent r.v.'s and the normal distribution. By means of this method, Stein was able to derive a uniform bound for the rate of convergence in the CLT of order $O(n^{-1/2}\ln^2 n)$ under complete regularity (this is weaker than u.s.m.), and the optimal order $O(n^{-1/2})$ for m-dependency. However, he required the eight-order moments of the terms to be finite and that the original sequence of r.v.'s be strictly stationary.

Developing Stein's idea further, Tikhomirov (1980) constructed a similar differential equation for the c.f. of a sum of weakly dependent one-dimensional r.v.'s. Using this, he succeeded in obtaining the wanted bound for how close this c.f. is to the c.f. of the normal distribution. Finally, under minimal restrictions on the moments (the finiteness of the s-th order absolute moments, $2 < s \le 3$), in particular Tikhomirov derived the following uniform estimates for the rate of convergence in the CLT:

(a) when the s.m. coefficient decreases exponentially, the order is $O(n^{-(s-2)/2}\,(\ln n)^{s-1})$;
(b) for m-dependent r.v.'s, the optimal estimate is of order $O(n^{-(s-2)/2})$.

Tikhomirov (1980) also obtained a nonuniform estimate for strongly mixing r.v.'s (Theorem 11).

However the class of r.v.'s considered in (1980) is narrowed down by the requirement of strict stationarity. Schneider (1981) considered a uniformly strong mixing sequence and he replaced strict stationarity and finiteness of the s-th order absolute moments of the variables, $2 < s \le 3$, by uniform boundedness of the third-order absolute moments of the variables and linear growth of the variance of their sum. Then assuming the exponential decay of the u.s.m. coefficient, he obtained a uniform estimate for the rate of convergence in the CLT of order $O(n^{-1/2}\ln n)$.

Specifically devoted to the m-dependency case are the papers by Shergin (1976), (1979), (1983) and Heinrich (1982), (1984), (1985a,b,d). They employ the c.f. to work out differing ways of estimating the rate of convergence in the one-dimensional CLT to optimal order.

Yudin solved a more general problem in (1981)–(1991). He gives a general method of estimating the rate of convergence of the distributions of sums of weakly dependent r.v.'s to infinitely divisible distributions. His method is also applicable to the normal case. In addition, he studies how to approximate the distributions of sums of weakly dependent r.v.'s by distributions in the class L (1989a,b), (1991). We point out also that Yudin uses Bernstein's method of partitioning a sum and the basic idea behind Tikhomirov's method (1987), (1989)–(1991) when approximating the distributions of sums of weakly dependent r.v.'s by the distributions in class L. These studies are represented in detail in Yudin's book (1990).

Because they yield optimal estimates for the rate of convergence in the one-dimensional CLT for weakly dependent r.v.'s under minimal restrictions on the moments of the terms and because they are simple to use and are capable of solving other problems, the methods of Stein, Tikhomirov and Heinrich are currently some of the most extensively used techniques. Therefore we shall give a more detailed presentation below of precisely these three methods. Before doing this, we introduce some notation.

$\mathcal{N}$ will denote a real r.v. with standard normal d.f. $\Phi(x)$ and density $\varphi(x) = \Phi'(x)$.

Later on, we shall estimate the difference $\mathbf{E}h(Z_n) - \mathbf{E}h(\mathcal{N})$, where Z_n is a normalized sum of weakly dependent centered r.v.'s and h is either the indicator of an interval or satisfies a smoothness condition.

For any function $g : R \to R$, let

$$\mathcal{L}(g;p,\alpha) = \sup_{x\neq y} \frac{|g(x)-g(y)|}{|x-y|^\alpha(1+|x|^p+|y|^p)}, \quad ||g||_\infty = \sup_x |g(x)|,$$

and g' be the derivative of g.

In §2, 2.3. and §3, we shall assume that $h : R \to R$ satisfies $||h||_\infty < \infty$ (except in Theorems 3 and 16) and one of the conditions $H_1^{(p)} = \mathcal{L}(h;p,\alpha) < \infty$ or $H_2^{(p)} = \mathcal{L}(h';p+1,\alpha) < \infty$ with $p \ge 0$ and $0 < \alpha \le 1$.

We now single out the space $BL(R)$ of bounded functions $h : R \to R$ satisfying a Lipschitz condition, that is, such that

$$||h||_\infty < \infty \quad \text{and} \quad ||h||_L = \sup_{x\neq y} \frac{|h(x)-h(y)|}{|x-y|} < \infty.$$

Write
$$||h||_{BH_i^{(p)}} = ||h||_\infty + H_i^{(p)}, \quad ||h||_{BL} = ||h||_\infty + ||h||_L.$$

Let
$$X_1, X_2, \dots \tag{12}$$
be a sequence of real r.v.'s with $\mathbf{E}X_j = 0$ and $\mathbf{E}X_j^2 < \infty$, $j = 1, 2, \dots, n$. Let
$$S_n = \sum_{j=1}^n X_j, \quad B_n^2 = \mathbf{E}S_n^2, \quad Z_n = S_n/B_n, \quad F_n(x) = \mathbf{P}\{Z_n < x\},$$
$$A_j = X_j/B_n, \quad \overline{A}_j = A_j 1_{(|A_j|\le t)}, \quad \overline{\overline{A}}_j = A_j 1_{(|A_j|>t)},$$
$$L_{r,n} = \sum_{j=1}^n \mathbf{E}|A_j|^r, \quad \overline{L}_{r,n} = \sum_{j=1}^n \mathbf{E}|\overline{A}_j|^r, \quad \overline{\overline{L}}_{r,n} = \sum_{j=1}^n \mathbf{E}|\overline{\overline{A}}_j|^r,$$
$$L^*_{s,n} = nd_{s,n}, \quad d_{s,n} = \max_{1\le j\le n} \mathbf{E}|A_j|^s,$$
$$\Delta_n(x) = \mathbf{P}\{Z_n < x\} - \Phi(x), \quad \Delta_n = \sup_x |\Delta_n(x)|, \quad ||\cdot||_1 = \int_{-\infty}^{\infty} |\cdot|\, dx,$$
$$d_i^{(p)}(F_n, \Phi) = \sup_{h\in\mathcal{H}_i^{(p)}} |\mathbf{E}h(Z_n) - \mathbf{E}h(\mathcal{N})|/||h||_{BH_i^{(p)}},$$
$$d_{BL}(F_n, \Phi) = \sup_{h\in BL(R)} |\mathbf{E}h(Z_n) - \mathbf{E}h(\mathcal{N})|/||h||_{BL},$$
where $\mathcal{H}_i^{(p)} = \{h : ||h||_\infty < \infty,\ H_i^{(p)} < \infty\}$, $i = 1, 2$, and 1_A is the indicator of event A, $t > 0$ and $B_n > 0$.

The quantities $d_i^{(p)}(F_n, \Phi)$ and $d_{BL}(F_n, \Phi)$ may be expressed as follows:
$$d_i^{(p)}(F_n, \Phi) = \sup_{\substack{h\in\mathcal{H}_i^{(p)} \\ ||h||_{BH_i^{(p)}}\le 1}} |\mathbf{E}h(Z_n) - \mathbf{E}h(\mathcal{N})|,$$
$$d_{BL}(F_n, \Phi) = \sup_{\substack{h\in BL(R) \\ ||h||_{BL}\le 1}} |\mathbf{E}h(Z_n) - \mathbf{E}h(\mathcal{N})|.$$

The quantity $d_{BL}(F_n, \Phi)$ is known as the bounded Lipschitz distance between the d.f.'s F_n and Φ.

Later on, we shall omit the subscript n in the notation $L_{r,n}$, $\overline{L}_{r,n}$, $\overline{\overline{L}}_{r,n}$, $L^*_{s,n}$ and $d_{s,n}$ and instead of $d_i^{(p)}(F_n, \Phi)$ and $d_{BL}(F_n, \Phi)$ we shall write $d_i^{(p)}$ and d_{BL}.

The letter $C(\cdot)$ with or without a subscript will denote a finite positive constant (not always the same one) that depends on the quantities indicated in the parentheses. C is an absolute positive constant; Θ is a complex function not exceeding one in modulus; $0 < K < \infty$ and $\lambda > 0$ are constants.

We shall show later how the methods of Stein, Tikhomirov and Heinrich yield upper bounds for $||\Delta_n(x)||_1$, $d_i^{(p)}$, d_{BL} and Δ_n if the sequence (12) satisfies one of the above weak dependence conditions.

The bounds for $||\Delta_n(x)||_1$ (Theorem 1 and Corollary 1), $d_i^{(p)}$ (Theorems 3–8 and Corollaries 2, 3) and Δ_n (Theorem 10) for the mixing coefficients described above are new and are due to the author. More general results are contained in Propositions 2–4.

To describe the methods of Stein and Tikhomirov, we shall make use of the r.v.'s
$$Z_j^{(i)} = \sum_{|p-j|\le im} A_p \quad \text{and} \quad z_j^{(i)} = Z_n - Z_j^{(i)}$$
with $2mi+1 < n$ and $m = 1, 2, \dots$; in the case of m-dependence, $m = 0, 1, \dots$. The bounds derived by Stein's method make use of a subsidiary r.v. J which is uniformly distributed on the set $\{1, 2, \dots, n\}$ and independent of the r.v.'s $X_1, X_2, \dots, X_n$.

Let
$$c_* = \begin{cases} m+1 & \text{under } m\text{-dependency,} \\ 1+2\psi_n & \text{under } \psi\text{-mixing,} \\ 1+4\Phi_n & \text{under u.s.m.,} \\ 1+12B_{n,r} & \text{under a.r.,} \\ 1+12A_{n,r} & \text{under s.m.} \end{cases}$$
where $\psi_n = \sum_{\tau=1}^{n-1} \psi(\tau)$, $\Phi_n = \sum_{\tau=1}^{n-1} \varphi^{1/2}(\tau)$, $B_{n,r} = \sum_{\tau=1}^{n-1} (\beta(\tau))^{(r-2)/r}$, $A_{n,r} = \sum_{\tau=1}^{n-1} (\alpha(\tau))^{(r-2)/r}$ and $2 < r < \infty$.

Applying the respective inequalities (1), (3) and (6), we find when the sequence (12) is either m-dependent, ψ-mixing or u.s.m. that
$$\mathbf{E}Z_n^2 \le c_* L_2. \tag{13}$$
For a.r. or s.m.,
$$\mathbf{E}Z_n^2 \le c_* \sum_{j=1}^n \mathbf{E}^{2/r}|A_j|^r, \quad 2 < r < \infty. \tag{14}$$
However, for $2 < r < \infty$,
$$\sum_{j=1}^n \mathbf{E}^{2/r}|A_j|^r \le n^{(r-2)/r} L_r^{2/r}. \tag{15}$$
Therefore for the weakly dependent r.v.'s defined above in (12) with ψ_n, Φ_n, $B_{n,r}$ and $A_{n,r}$ respectively finite, the estimates (13)–(15) imply that
$$n^{-(r-2)/2} \le c_*^{r/2} L_r \tag{16}$$
for $2 < r < \infty$. Consequently if $C_1(\cdot)(m+1) \ge n$, then

$$1 \le (C_1(\cdot))^{(r-2)/2} c_*^{r/2} (m+1)^{(r-2)/2} L_r. \tag{17}$$

By virtue of (17), it suffices to consider the case where $C_1(\cdot)(m+1) < n$ when estimating $||\Delta_n(x)||_1$, $d_i^{(p)}$, d_{BL} and Δ_n.

2.2. Estimation of $||\Delta_n(x)||_1$.

Theorem 1. *Suppose that the sequence* (12) *is m-dependent. Then for* $m \ge 0$ *and* $t > 0$,

$$||\Delta_n(x)||_1 \le C\{\overline{\overline{L}}_1 + (m+1)\overline{\overline{L}}_2 + (m+1)^2\overline{L}_3 + (m+1)^3\overline{L}_4\}.$$

If the truncation level in this estimate is taken to be $t = 1$ and $\overline{\overline{L}}_1$ is replaced by $(m+1)\overline{\overline{L}}_1$, then one obtains Erickson's result (1974) (see also Sunklodas (1982)).

Corollary 1. *Let the hypotheses of Theorem 1 hold and let* $\mathbf{E}|X_j|^s < \infty$, $2 < s \le 3$ *and* $j = 1, 2, \dots, n$. *Then for* $m \ge 0$,

$$||\Delta_n(x)||_1 \le C(m+1)^{s-1} L_s.$$

By means of (24) below, similar bounds may be obtained for $||\Delta_n(x)||_1$ in the case of the mixing coefficients ψ, φ and β. Bounds were found by Sunklodas (1982), (1986) and Takahata (1983) for other mixing coefficients. For the sake of simplifying the presentation, we confine ourselves here to estimating $||\Delta_n(x)||_1$ just for m-dependent r.v.'s.

Heinrich's results for m-dependent r.v.'s (1985d) imply this particular one.

Theorem 2 (Heinrich (1985d)). *Suppose that the sequence* (12) *is m-dependent and that* $\mathbf{E}|X_j|^s < \infty$ *with* $2 < s < 3$ *and* $j = 1, 2, \dots, n$. *Then for* $1 \le p < \infty$ *and* $n \ge 1$,

$$\left(\int_{-\infty}^{\infty} (1 + |x|)^{sp-1} |\Delta_n(x)|^p \, dx \right)^{1/p} \le C(m+1)^{s-1} L_s^*.$$

Shergin (1983) considered m-dependent r.v.'s (12) with finite k-th absolute moments, $k \ge 2$, such that $\sum_{j=1}^n \mathbf{E}X_j^2 \le M_0 B_n^2$ for all $n \ge n_0$, where M_0 and n_0 are positive constants. He used Stein's technique to derive a bound for $\int_{-\infty}^{\infty} |x|^\ell |\Delta_n(x)| dx$ in terms of Lyapunov's quotients for $0 \le \ell \le k-1$.

Proof of Theorem 1.
Stein pointed out the following characterization of the standard normal law: If the r.v. $W = \mathcal{N}$, then $\mathbf{E}f'(W) - \mathbf{E}[Wf(W)] = 0$ for a fairly broad class of functions f, and conversely (see Stein (1972), (1981)). Consequently, the quantity $\mathbf{E}f'(W) - \mathbf{E}[Wf(W)]$ is a good measure of the proximity of the r.v. W to $\mathcal{N}$.

Therefore to estimate the difference $\mathbf{E}h(W) - \mathbf{E}h(\mathcal{N})$, it suffices to estimate the right-hand side of the relation

$$\mathbf{E}h(W) - \mathbf{E}h(\mathcal{N}) = \mathbf{E}f'(W) - \mathbf{E}[Wf(W)).$$

To this end, consider the linear differential equation

$$f'(y) - yf(y) = h(y) - \mathbf{E}h(\mathcal{N}). \tag{18}$$

Its solution is

$$f(y) = e^{y^2/2} \int_{-\infty}^{y} h_0(u) e^{-u^2/2} \, du = -e^{y^2/2} \int_{y}^{\infty} h_0(u) e^{-u^2/2} \, du, \tag{19}$$

where $h_0(y) = h(y) - \mathbf{E}h(\mathcal{N})$.

For instance, to estimate $\mathbf{P}\{W < x\} - \Phi(x)$, it is necessary to take h in (18) to be the indicator function of the interval, i.e., $h(y) = h_x(y) = 1_{(-\infty, x)}(y)$. Then (18) and (19) become respectively

$$f_x'(y) - yf_x(y) = 1_{(-\infty,x)}(y) - \Phi(x) \tag{20}$$

and

$$f_x(y) = \begin{cases} \Phi(-x)\Phi(y)/\varphi(y), & \text{if } y < x, \\ \Phi(x)\Phi(-y)/\varphi(y), & \text{if } y \ge x. \end{cases} \tag{21}$$

To estimate $||\Delta_n(x)||_1$, we make use of the following result.

Lemma 4 (Erickson (1974), Ho and Chen (1978)). *Let f_x be given by* (21). *Then for all real* y,

$$\int_{-\infty}^{\infty} |f_x(y)| dx = 1 \quad \textit{and} \quad \int_{-\infty}^{\infty} |f_x'(y)| dx \le 1.$$

In order that all of the r.v.'s occurring in (22) and (24) below should be well defined, it is assumed further that $6m + 1 < n$.

For brevity, put $Q_j = A_j Z_j^{(1)} - \mathbf{E}(A_j Z_j^{(1)})$.

It follows from (20) that

$$\begin{aligned} \Delta_n(x) &= \mathbf{E}\left\{ f_x'(Z_n) - \sum_{j=1}^n A_j f_x(Z_n) \right\} \\ &= \mathbf{E}\left\{ f_x'(Z_n) + \sum_{j=1}^n A_j \left[f_x(z_j^{(1)}) - f_x(Z_n) \right] - \sum_{j=1}^n A_j f_x(z_j^{(1)}) \right\}. \end{aligned}$$

Applying the Newton–Leibniz formula to the differences $f_x(z_j^{(1)}) - f_x(Z_n)$ and $f_x(u) - f_x(Z_n)$ and noting (20), we find that

$$\Delta_n(x) = \mathbf{E}f'_x(Z_n) + \sum_{j=1}^n \mathbf{E}\left\{A_j \int_{Z_n}^{z_j^{(1)}} u \int_{Z_n}^u f'_x(v)\,dv\,du\right\}$$
$$+\frac{1}{2}\sum_{j=1}^n \mathbf{E}\{A_j(Z_j^{(1)})^2 f_x(Z_n)\} - \sum_{j=1}^n \mathbf{E}\{A_j f_x(z_j^{(1)})\}$$
$$+\sum_{j=1}^n \mathbf{E}\left\{A_j \int_{Z_n}^{z_j^{(1)}} [1_{(-\infty,x)}(u) - 1_{(-\infty,x)}(Z_n)]du\right\}$$
$$-\sum_{j=1}^n \mathbf{E}\{A_j Z_j^{(1)} f'_x(Z_n)\}.$$

Similar reasoning leads to

$$\sum_{j=1}^n \mathbf{E}\{Q_j[f'_x(Z_n) - f'_x(z_j^{(2)})]\} = \sum_{j=1}^n \mathbf{E}\{Q_j Z_j^{(2)} f_x(Z_n)\}$$
$$+\sum_{j=1}^n \mathbf{E}\left\{Q_j z_j^{(2)} \int_{z_j^{(2)}}^{Z_n} f'_x(u)du\right\}$$
$$+\sum_{j=1}^n \mathbf{E}\{Q_j[1_{(-\infty,x)}(Z_n) - 1_{(-\infty,x)}(z_j^{(2)})]\}.$$

Since $\mathbf{E}Z_n^2 = 1$, we have

$$1 - \sum_{j=1}^n A_j Z_j^{(1)} = -\sum_{j=1}^n Q_j + \sum_{j=1}^n \mathbf{E}\{A_j z_j^{(1)}\}.$$

From the last three equalities, we conclude that (Sunklodas (1982))

$$\Delta_n(x) = \sum_{k=1}^9 E_k(x), \tag{22}$$

where

$$E_1(x) = \sum_{j=1}^n \mathbf{E}\left\{A_j \int_{Z_n}^{z_j^{(1)}} u \int_{Z_n}^u f'_x(v)\,dv\,du\right\},$$
$$E_2(x) = \sum_{j=1}^n \mathbf{E}\left\{Q_j z_j^{(2)} \int_{Z_n}^{z_j^{(2)}} f'_x(u)\,du\right\},$$
$$E_3(x) = \frac{1}{2}\sum_{j=1}^n \mathbf{E}\{A_j(Z_j^{(1)})^2 f_x(Z_n)\},$$
$$E_4(x) = -\sum_{j=1}^n \mathbf{E}\{Q_j Z_j^{(2)} f_x(Z_n)\},$$

$$E_5(x) = \sum_{j=1}^n \mathbf{E}\left\{A_j \int_{Z_n}^{z_j^{(1)}} [1_{(-\infty,x)}(u) - 1_{(-\infty,x)}(Z_n)]\,du\right\},$$
$$E_6(x) = -\sum_{j=1}^n \mathbf{E}\left\{Q_j[1_{(-\infty,x)}(Z_n) - 1_{(-\infty,x)}(z_j^{(2)})]\right\},$$
$$E_7(x) = -\sum_{j=1}^n \mathbf{E}\{A_j f_x(z_j^{(1)})\},$$
$$E_8(x) = -\sum_{j=1}^n \mathbf{E}\{Q_j f'_x(z_j^{(2)})\},$$
$$E_9(x) = \sum_{j=1}^n \mathbf{E}\{A_j z_j^{(1)}\}\mathbf{E}f'(Z_n).$$

It is precisely (22) that plays a basic role in estimating $||\Delta_n(x)||_1$ for weakly dependent r.v.'s. A similar relation was used by Erickson (1974) for m-dependent r.v.'s.

Applying Lemma 4 and the fact that

$$\int_{-\infty}^{\infty} |1_{(-\infty,x)}(u) - 1_{(-\infty,x)}(v)|dx = |u-v|, \tag{23}$$

for any $u, v \in R$, we can deduce the next result from (22).

Basic Inequality (Sunklodas (1982)):

$$||\Delta_n(x)||_1 \le I_1(m) + I_2(m) \tag{24}$$

with

$$I_1(m) = \frac{5}{6}n\mathbf{E}|A_J(Z_J^{(1)})^3| + n\mathbf{E}|A_J(Z_J^{(1)})^2| + 2n\mathbf{E}|Q_J Z_J^{(2)}| +$$
$$+\frac{n}{2}\mathbf{E}|A_J(Z_J^{(1)})^2(Z_J^{(2)} - Z_J^{(1)})| + n\mathbf{E}|Q_J Z_J^{(2)}(Z_J^{(3)} - Z_J^{(2)})|$$

and

$$I_2(m) = \frac{n}{2}\mathbf{E}|A_J(Z_J^{(1)})^2 z_J^{(2)}| + n\mathbf{E}|Q_J Z_J^{(2)} z_J^{(3)}| +$$
$$+\sum_{j=1}^n [||\mathbf{E}\{A_j f_x(z_j^{(1)})\}||_1 + ||\mathbf{E}\{Q_j f'_x(z_j^{(2)})\}||_1] + n|\mathbf{E}\{A_J z_J^{(1)}\}|.$$

Since $I_1(m)$ involves the moments of "close" r.v.'s, its estimation does not require the use of the weak dependence of $X_1, X_2, \ldots$

It is easy to see that for any real $r \ge 1$

$$\mathbf{E}|Z_J^{(i)}|^r \le (2mi+1)^r n^{-1} L_r, \quad i = 0, 1, \ldots, \tag{25}$$

and that

$$\mathbf{E}|Z_J^{(i)} - Z_J^{(i-1)}|^r \le (2m)^r n^{-1} L_r, \quad i = 2, 3, \dots \tag{26}$$

Therefore the application of Hölder's inequality and the estimates (25) and (26) yield

$$|I_1(m)| \le (2m+1)(18m+5)L_3 + \frac{1}{6}(2m+1)(128m^2+50m+5)L_4. \tag{27}$$

$I_2(m)$ has to be estimated separately for each weak dependence condition. Let the sequence (12) be m-dependent. Then

$$I_2(m) = \frac{n}{2}\mathbf{E}|A_J(Z_J^{(1)})^2 z_J^{(2)}| + n\mathbf{E}|Q_J Z_J^{(2)} z_J^{(3)}|.$$

Put

$$t_J^i = \mathbf{E}|Z_J^{(2)}|^i, \quad \delta_J^i = \mathbf{E}|Z_J^{(3)}|^i.$$

Then

$$\begin{aligned} \mathbf{E}|A_J(Z_J^{(1)})^2 z_J^{(2)}| &\le \mathbf{E}|A_J(Z_J^{(1)})^2| + \mathbf{E}|A_J(Z_J^{(1)})^2 t_J^1|, \\ \mathbf{E}|Q_J Z_J^{(2)} z_J^{(3)}| &\le \mathbf{E}|Q_J Z_J^{(2)}| + \mathbf{E}|Q_J Z_J^{(2)} \delta_J^1|. \end{aligned}$$

If $\xi_j = \sum_{p \in B_j} A_p$ and $\tau_j^i = \mathbf{E}|\xi_j|^i$, where $B_j \subset \{1, 2, \dots, n\}$ for any $j = 1, 2, \dots, n$, it is easy to see that

$$\mathbf{E}|\tau_J^i|^{j/i} \le \mathbf{E}|\xi_J|^j, \quad 0 < i \le j. \tag{28}$$

Then (25), (26), (28) and Hölder's inequality imply that

$$\begin{aligned} \mathbf{E}|A_J(Z_J^{(1)})^2| &\le (2m+1)^2 n^{-1} L_3, \\ \mathbf{E}|A_J(Z_J^{(1)})^2 t_J^1| &\le (2m+1)^2(4m+1) n^{-1} L_4, \\ \mathbf{E}|Q_J Z_J^{(2)}| &\le 2(2m+1)(4m+1) n^{-1} L_3, \\ \mathbf{E}|Q_J Z_J^{(2)} \delta_J^1| &\le 2(2m+1)(4m+1)(6m+1) n^{-1} L_4. \end{aligned}$$

Consequently,

$$I_2(m) \le (2m+1)[(9m+2.5)L_3 + (52m^2+23m+2.5)L_4). \tag{29}$$

Therefore substituting (27) and (29) in (24) and dropping the condition $6m+1 < n$ by means of (17), we obtain the following result.

Proposition 1. *Suppose that the sequence* (12) *is m-dependent and* $\mathbf{E}X_j^4 < \infty$, $j = 1, 2, \dots, n$. *Then for* $m \ge 0$,

$$||\Delta_n(x)||_1 \le (2m+1)[(27m+7.5)L_3 + (10/3)(22m^2+9.4m+1)L_4).$$

Truncation. To complete the proof of Theorem 1, we truncate the r.v.'s A_j, $j = 1, 2, \dots, n$, at the level $t > 0$ (§2, 2.1); in addition we put

$$\overline{A}_j^{(0)} = \overline{A}_j - \mathbf{E}\overline{A}_j, \quad \overline{\overline{A}}_j^{(0)} = \overline{\overline{A}}_j - \mathbf{E}\overline{\overline{A}}_j, \quad \overline{Z}_n^{(0)} = \sum_{j=1}^n \overline{A}_j^{(0)}.$$

Since any pair of r.v.'s ξ and η satisfies the inequality (see Erickson (1974), p. 527)

$$||\mathbf{P}\{\xi < x\} - \mathbf{P}\{\eta < x\}||_1 \le \mathbf{E}|\xi - \eta|, \tag{30}$$

it follows that

$$||\Phi(ax) - \Phi(x)||_1 \le \sqrt{2/\pi}\,|1 - a^{-1}| \tag{31}$$

for any positive a.

Therefore when $\mathbf{E}(\overline{Z}_n^{(0)})^2 > 0$, the sequence (12) satisfies

$$\begin{aligned} ||\Delta_n(x)||_1 \le\ & 2\overline{\overline{L}}_1 + \sqrt{2/\pi}\,|1 - \mathbf{E}(\overline{Z}_n^{(0)})^2| + \\ & + \mathbf{E}^{1/2}(\overline{Z}_n^{(0)})^2 ||\mathbf{P}\{\overline{Z}_n^{(0)} < x\mathbf{E}^{1/2}(\overline{Z}_n^{(0)})^2\} - \Phi(x)||_1. \end{aligned} \tag{32}$$

We note that the truncation inequality (32) holds for any dependency of the sequence (12).

Since

$$|1 - \mathbf{E}(\overline{Z}_n^{(0)})^2| \le 9\overline{\overline{L}}_2 + \sum_{1 \le i \ne j \le n} |\mathbf{E}(\overline{\overline{A}}_i^{(0)} \overline{\overline{A}}_j^{(0)}) + 2\mathbf{E}(\overline{A}_i^{(0)} \overline{\overline{A}}_j^{(0)})|, \tag{33}$$

any m-dependent sequence (12) satisfies

$$|1 - \mathbf{E}(\overline{Z}_n^{(0)})^2| \le 9(2m+1)\overline{\overline{L}}_2 = \varepsilon_1. \tag{34}$$

To prove Theorem 1, it is assumed that $\varepsilon_1 \le 1/2$ (otherwise the estimates are trivial). From (34), we have that $\frac{1}{2} \le \mathbf{E}(\overline{Z}_n^{(0)})^2 \le \frac{3}{2}$ and consequently,

$$\sum_{j=1}^n \mathbf{E}|\overline{A}_j^{(0)} / \mathbf{E}^{1/2}(\overline{Z}_n^{(0)})^2|^r \le 2^{3r/2}\overline{L}_r \tag{35}$$

for positive r.

By virtue of the truncation inequality (32) and the bounds (34) and (35), to complete the proof of Theorem 1, it is sufficient to estimate $||\Delta_n(x)||_1$ for untruncated, r.v.'s. In other words, we make use of Proposition 1.

Corollary 1 follows from Theorem 1 with $t = (m+1)^{-1}$.

2.3. Estimation of $d_i^{(p)}$ and d_{BL}.

Theorem 3. *Suppose that the sequence* (12) *is m-dependent with* $\mathbf{E}|X_j|^{2+p+\alpha} < \infty$, $j = 1, 2, \dots, n$. *If* $6m + 1 < n$, *then*

$$|\mathbf{E}h(Z_n) - \mathbf{E}h(\mathcal{N})| \le C(p,\alpha)H_i^{(p)}\{(m+1)^{1+\alpha}L_{2+\alpha}(1 + \mathbf{E}|Z_n|^p) + (m+1)^{1+p+\alpha}L_{2+p+\alpha}\}.$$

Thus by Theorem 3, the estimation of $|\mathbf{E}h(Z_n) - \mathbf{E}h(\mathcal{N})|$, where h satisfies either the condition $H_1^{(p)} < \infty$ or $H_2^{(p)} < \infty$, reduces to estimating the absolute moment $\mathbf{E}|Z_n|^p$, $p \ge 0$.

Bounds for the absolute moments of a sum of weakly dependent r.v.'s were obtained by Utev (1984). For ψ-mixing or absolutely regular r.v.'s, estimating $d_i^{(p)}$, $i = 1, 2$, can also be reduced to estimating the absolute moments of Z_n (although of a higher order than p). However, for the sake of simplicity, we shall only consider here the estimation of $d_i^{(0)}$, $i = 1, 2$.

Theorem 4. *Let the sequence* (12) *be ψ-mixing with* $\psi(\tau) \le K\tau^{-\mu}$, *where* $\mu \ge (r-1)r$ *and let* $\mathbf{E}|X_j|^r < \infty$, $4 \le r < \infty$ *and* $j = 1, 2, \dots, n$. *Then*

$$d_i^{(0)} \le C(K, \mu, r, \alpha)\{n^{(1+\alpha)(r-1)/\mu}L_{2+\alpha} + L_r\}.$$

Theorem 5. *Let the sequence* (12) *be absolutely regular and let* $\mathbf{E}|X_j|^r < \infty$, $4 \le r < \infty$ *and* $j = 1, 2, \dots, n$.

1. *If* $\beta(\tau) \le Ke^{-\lambda\tau}$, *then*

$$d_i^{(0)} \le C(K, \lambda, r, \alpha)\{L_{2+\alpha}\ln^{1+\alpha}(n+1) + L_r\}.$$

2. *If* $\beta(\tau) \le K\tau^{-\mu}$ *with* $\mu \ge 2(r-1)r$, *then*

$$d_i^{(0)} \le C(K, \mu, r, \alpha)\{n^{2(1+\alpha)(r-1)/\mu}L_{2+\alpha} + L_r\}.$$

Truncation of the r.v.'s leads, for example, to the following results.

Theorem 6. *Let the sequence* (12) *be m-dependent. Then for* $m \ge 0$ *and* $t > 0$

$$d_1^{(0)} \le C(\alpha)\{\overline{\overline{L}}_1^{\alpha} + (m+1)^{\alpha}\overline{\overline{L}}_2^{\alpha} + (m+1)^{1+\alpha}\overline{L}_{2+\alpha}\}.$$

Corollary 2. *Let the hypotheses of Theorem* 6 *hold.*

1. *If* $\mathbf{E}|X_j|^{2+\alpha} < \infty$, $j = 1, 2, \dots, n$, *then for* $m \ge 0$

$$d_1^{(0)} \le C(\alpha)(m+1)^{\alpha(1+\alpha)}L_{2+\alpha}^{\alpha}.$$

2. *If* $\mathbf{E}|X_j|^s < \infty$, $2 < s \le 3$ *and* $j = 1, 2, \dots, n$, *then for* $m \ge 0$

$$d_{BL} \le C(m+1)^{s-1}L_s.$$

Theorem 7. *Suppose that* (12) *is ψ-mixing with* $\psi(\tau) \le Ke^{-\lambda\tau}$. *Then*

$$d_1^{(0)} \le C(K, \lambda, r, \alpha)\{\overline{\overline{L}}_1^{\alpha} + \overline{\overline{L}}_2^{\alpha} + \overline{L}_{2+\alpha}\ln^{1+\alpha}(n+1) + \overline{L}_r\}$$

for $t > 0$ *and* $4 \le r < \infty$.

Corollary 3. *Suppose that the hypotheses of Theorem* 7 *hold.*

1. *If* $\mathbf{E}|X_j|^{2+\alpha} < \infty$, $j = 1, 2, \dots, n$, *then*

$$d_1^{(0)} \le C(K, \lambda, \alpha)L_{2+\alpha}^{\alpha}\ln^{\alpha(1+\alpha)}(n+1).$$

2. *If* $\mathbf{E}|X_j|^s < \infty$, $2 < s \le 3$ *and* $j = 1, 2, \dots, n$, *then*

$$d_{BL} \le C(K, \lambda)L_s\ln^{s-1}(n+1).$$

Theorem 8. *Suppose that* (12) *is ψ-mixing with* $\psi(\tau) \le K\tau^{-\mu}$, *where* $\mu \ge 12$.

1. *If* $\mathbf{E}|X_j|^{2+\alpha} < \infty$, $j = 1, 2, \dots, n$, *then*

$$d_1^{(0)} \le C(K, \mu, \alpha)n^{3(1+\alpha)/\mu}L_{2+\alpha}^{\alpha}.$$

2. *If* $\mathbf{E}|X_j|^s < \infty$, $2 < s \le 3$ *and* $j = 1, 2, \dots, n$, *then*

$$d_{BL} \le C(K, \mu)n^{3(s-1)/\mu}L_s.$$

The reader can learn more details about estimating with the bounded Lipschitz metric d_{BL} by consulting Sunklodas (1989).

Proofs of Theorems 3–8.
The following result is true.

Lemma 5. *Let f and f' be given by* (18) *and* (19).

1. *If* $||h||_\infty < \infty$, *then* (see Erickson (1974), Barbour and Eagleson (1985))

$$|f(y)| \le c_1||h_0||_\infty \quad \text{and} \quad |f'(y)| \le 2||h_0||_\infty \tag{36}$$

for $\forall y \in R$, *where* $c_1 = \sup_{x \ge 0}\Xi(x)$ *and* $\Xi(x) = \Phi(-x)/\varphi(x)$.

2. *If* $H_1^{(p)} < \infty$ *or* $H_2^{(p)} < \infty$, *then* (Barbour (1986))

$$\mathcal{L}(f'; p, \alpha) \le C_i(p, \alpha)H_i^{(p)}, \tag{37}$$

where $i = 1$ *if* $H_1^{(p)} < \infty$ *and* $i = 2$ *if* $H_2^{(p)} < \infty$.

Furthermore, let $6m+1<n$. Then (18) implies that

$$\mathbf{E}h_0(Z_n)=\sum_{j=1}^{n}\{\mathbf{E}(A_jZ_j^{(1)})\mathbf{E}f'(Z_n)-\mathbf{E}[A_jf(Z_n)]+\mathbf{E}(A_jz_j^{(1)})\mathbf{E}f'(Z_n)\}.$$

In the summation, we first add and subtract $\mathbf{E}(A_jZ_j^{(1)})\mathbf{E}f'(z_j^{(1)})$, $\mathbf{E}[A_jf(z_j^{(1)})]$ and $\mathbf{E}[A_jZ_j^{(1)}f'(z_j^{(1)})]$ and then $\mathbf{E}[Q_jf'(z_j^{(2)})]$. Then using the identity

$$f(Z_n)-f(z_j^{(1)})-f'(z_j^{(1)})Z_j^{(1)}=\int_0^{Z_j^{(1)}}[f'(z_j^{(1)}+u)-f'(z_j^{(1)})]du,$$

we find that

$$\mathbf{E}h(Z_n)-\mathbf{E}h(\mathcal{N})=I_1+I_2, \tag{38}$$

where

$$\begin{aligned} I_1 &= \sum_{j=1}^{n}\mathbf{E}(A_jZ_j^{(1)})\mathbf{E}[f'(Z_n)-f'(z_j^{(1)})] \\ &\quad -\sum_{j=1}^{n}\mathbf{E}\{Q_j[f'(z_j^{(1)})-f'(z_j^{(2)})]\} \\ &\quad -\sum_{j=1}^{n}\mathbf{E}\left\{A_j\int_0^{Z_j^{(1)}}[f'(z_j^{(1)}+u)-f'(z_j^{(1)})]du\right\} \end{aligned}$$

and

$$\begin{aligned} I_2 &= -\sum_{j=1}^{n}\mathbf{E}\{Q_jf'(z_j^{(2)})\}+\sum_{j=1}^{n}\mathbf{E}(A_jz_j^{(1)})\mathbf{E}f'(Z_n) \\ &\quad -\sum_{j=1}^{n}\mathbf{E}[A_jf(z_j^{(1)})]. \end{aligned}$$

Put

$$\begin{aligned} \nu_j^i &= \mathbf{E}|Z_j^{(1)}|^i,\ \gamma_j^i=\mathbf{E}|Z_j^{(2)}-Z_j^{(1)}|^i,\ t_j^i=\mathbf{E}|Z_j^{(2)}|^i,\ \delta_j^i=\mathbf{E}|Z_j^{(3)}|^i, \\ \omega_j &= \mathbf{E}|(Z_j^{(1)})^\alpha(Z_j^{(2)}-Z_j^{(1)})^p|,\ w_j=\mathbf{E}|(Z_j^{(2)}-Z_j^{(1)})^\alpha(Z_j^{(3)}-Z_j^{(2)})^p|. \end{aligned}$$

Estimating $|I_1|$ with the help of (37), we obtain

$$|I_1|\le C(p,\alpha)H_i^{(p)}(nI_1'+I_1''), \tag{39}$$

where

$$\begin{aligned} I_1' &= \mathbf{E}|A_JZ_J^{(1)}\nu_J^\alpha|+\mathbf{E}|A_JZ_J^{(1)}\nu_J^{p+\alpha}|+\mathbf{E}|A_JZ_J^{(1)}\omega_J| \\ &\quad +\mathbf{E}|A_JZ_J^{(1)}\gamma_J^\alpha|+\mathbf{E}|A_JZ_J^{(1)}\gamma_J^{p+\alpha}|+\mathbf{E}|A_JZ_J^{(1)}w_J| \\ &\quad +\mathbf{E}|A_J(Z_J^{(1)})^{1+\alpha}|+\mathbf{E}|A_J(Z_J^{(1)})^{1+p+\alpha}| \\ &\quad +\mathbf{E}|A_J(Z_J^{(1)})^{1+\alpha}(Z_J^{(2)}-Z_J^{(1)})^p|+\mathbf{E}|A_JZ_J^{(1)}(Z_J^{(2)}-Z_J^{(1)})^\alpha| \\ &\quad +\mathbf{E}|A_JZ_J^{(1)}(Z_J^{(2)}-Z_J^{(1)})^{p+\alpha}| \\ &\quad +\mathbf{E}|A_JZ_J^{(1)}(Z_J^{(2)}-Z_J^{(1)})^\alpha(Z_J^{(3)}-Z_J^{(2)})^p| \end{aligned}$$

and

$$\begin{aligned} I_1'' &= \sum_{j=1}^{n}\mathbf{E}|A_jZ_j^{(1)}|[\mathbf{E}|(Z_j^{(1)})^\alpha(z_j^{(2)})^p|+\mathbf{E}|(Z_j^{(2)}-Z_j^{(1)})^\alpha(z_j^{(3)})^p|] \\ &\quad +\sum_{j=1}^{n}[\mathbf{E}|A_j(Z_j^{(1)})^{1+\alpha}(z_j^{(2)})^p|+\mathbf{E}|A_jZ_j^{(1)}(Z_j^{(2)}-Z_j^{(1)})^\alpha(z_j^{(3)})^p|]. \end{aligned}$$

Applying (25), (26), (28) and Hölder's inequalities, we find that $\mathbf{E}|A_JZ_J^{(1)}\nu_J^\alpha|$, $\mathbf{E}|A_JZ_J^{(1)}\gamma_J^\alpha|$, $\mathbf{E}|A_J(Z_J^{(1)})^{1+\alpha}|$ and $\mathbf{E}|A_JZ_J^{(1)}(Z_J^{(2)}-Z_J^{(1)})^\alpha|$ are bounded above by $(2m+1)^{1+\alpha}n^{-1}L_{2+\alpha}$ and the remaining terms of I_1' are bounded above by $(2m+1)^{1+p+\alpha}n^{-1}L_{2+p+\alpha}$. Therefore

$$I_1'\le C(p,\alpha)n^{-1}[(m+1)^{1+\alpha}L_{2+\alpha}+(m+1)^{1+p+\alpha}L_{2+p+\alpha}). \tag{40}$$

We point out that it would have been possible to manage without the weak dependency of the r.v.'s (12) because I_1' involves moments of close r.v.'s.

The quantities I_1'' and $|I_2|$ will be estimated separately for each weak dependency.

Let the sequence (12) be m-dependent. Since

$$\mathbf{E}|z_j^{(i)}|^p\le(1\vee 2^{p-1})(\mathbf{E}|Z_n|^p+\mathbf{E}|Z_j^{(i)}|^p),\quad i=1,2,$$

the m-dependence implies that

$$\begin{aligned} I_1'' &\le (1\vee 2^{p-1})n\big([\mathbf{E}|A_JZ_J^{(1)}\nu_J^\alpha|+\mathbf{E}|A_JZ_J^{(1)}\gamma_J^\alpha|+\mathbf{E}|A_J(Z_J^{(1)})^{1+\alpha}| \\ &\quad +\mathbf{E}|A_JZ_J^{(1)}(Z_J^{(2)}-Z_J^{(1)})^\alpha|]\mathbf{E}|Z_n|^p+\mathbf{E}|A_JZ_J^{(1)}\nu_J^\alpha t_J^p| \\ &\quad +\mathbf{E}|A_JZ_J^{(1)}\gamma_J^\alpha\delta_J^p|+\mathbf{E}|A_J(Z_J^{(1)})^{1+\alpha}t_J^p| \\ &\quad +\mathbf{E}|A_JZ_J^{(1)}(Z_J^{(2)}-Z_J^{(1)})^\alpha\delta_J^p|\big). \end{aligned} \tag{41}$$

The expression multiplying $\mathbf{E}|Z_n|^p$ has already been estimated. The estimates (25), (26), (28) and Hölder's inequality can be used to show that each of the remaining terms in (41) is bounded above by at least $(6m+1)^{1+p+\alpha}n^{-1}L_{2+p+\alpha}$. Consequently,

$$I_1''\le C(p,\alpha)[(m+1)^{1+\alpha}L_{2+\alpha}\mathbf{E}|Z_n|^p+(m+1)^{1+p+\alpha}L_{2+p+\alpha}). \tag{42}$$

$I_2 = 0$ by virtue of the m-dependence. Therefore to complete the proof of Theorem 3, it suffices to substitute (40) and (42) in (39).

When $p = 0$,

$$I_1'' = n[\mathbf{E}|A_J Z_J^{(1)} \nu_J^\alpha| + \mathbf{E}|A_J Z_J^{(1)} \gamma_J^\alpha| + \mathbf{E}|A_J (Z_J^{(1)})^{1+\alpha}| + \mathbf{E}|A_J Z_J^{(1)} (Z_J^{(2)} - Z_J^{(1)})^\alpha|]$$

occurs in the expression for nI_1'. Thus from (39) and (40), we find that

$$|I_1| \le C(\alpha) H_i^{(0)} m^{1+\alpha} L_{2+\alpha}. \tag{43}$$

If $H_1^{(0)} < \infty$ or $H_2^{(0)} < \infty$, it follows from (38) and (43) that any weakly dependent sequence (12) satisfies

$$|\mathbf{E}h(Z_n) - \mathbf{E}h(\mathcal{N})| \le C(\alpha) H_i^{(0)} m^{1+\alpha} L_{2+\alpha} + |I_2|. \tag{44}$$

Suppose that (12) is ψ-mixing. Then estimating with (9) and using (36), we obtain

$$|I_2| \le 6||h_0||_\infty n[2\mathbf{E}^{1/2}(A_J Z_J^{(1)})^2 + (2^{3/2} + c_1)\mathbf{E}^{1/2} A_J^2 + 2^{3/2}\mathbf{E}^{1/2}(A_J^2 \nu_J^2)]\psi^{1/2}(m+1). \tag{45}$$

By virtue of (25), (28) and Hölder's inequality,

$$|I_2| \le C||h||_\infty n^{1/2}[L_2^{1/2} + mL_4^{1/2}]\psi^{1/2}(m+1). \tag{46}$$

The estimates (44) and (46) yield the following result.

Proposition 2. *Let sequence* (12) *be ψ-mixing and let* $\mathbf{E}X_j^4 < \infty$, $j = 1, 2, \ldots, n$. *Then*

$$d_i^{(0)} \le C(\alpha)\{m^{1+\alpha} L_{2+\alpha} + n^{1/2}[L_2^{1/2} + mL_4^{1/2}]\psi^{1/2}(m+1)\}$$

providing $6m + 1 < n$.

Under a.r., the quantity $|I_2|$ can be estimated with the help of (10) and in similar fashion we deduce the following

Proposition 3. *Let sequence* (12) *be absolutely regular and let* $\mathbf{E}X_j^4 < \infty$, $j = 1, 2, \ldots, n$. *Then*

$$d_i^{(0)} \le C(\alpha)\{m^{1+\alpha} L_{2+\alpha} + [n^{3/4} L_4^{1/4} + mn^{1/2} L_4^{1/2}]\beta^{1/4}(m+1)\}.$$

Theorems 4 and 5 are consequences of the respective Propositions 2 and 3.

Truncation. As in the estimation of $||\Delta_n(x)||_1$, the r.v.'s A_j, $j = 1, 2, \ldots, n$, are truncated at level $t > 0$ (§2, 2.1). Recall the subsidiary variables

$$\overline{A}_j^{(0)} = \overline{A}_j - \mathbf{E}\overline{A}_j \quad \text{and} \quad \overline{Z}_n^{(0)} = \sum_{j=1}^n \overline{A}_j^{(0)}.$$

Let $h \in \mathcal{H}_1^{(0)}$. Then any real r.v.'s ξ and η satisfy

$$|\mathbf{E}h(\xi) - \mathbf{E}h(\eta)| \le 3H_1^{(0)}\mathbf{E}|\xi - \eta|^\alpha. \tag{47}$$

It is easy to see that $\mathbf{E}|Z_n - \overline{Z}_n^{(0)}|^\alpha \le 2^\alpha \overline{\overline{L}}_1^\alpha$ and, when $\mathbf{E}(\overline{Z}_n^{(0)})^2 > 0$, that

$$\mathbf{E}|\overline{Z}_n^{(0)}(1 - \mathbf{E}^{-1/2}(\overline{Z}_n^{(0)})^2)|^\alpha \le |1 - \mathbf{E}(\overline{Z}_n^{(0)})^2|^\alpha.$$

Therefore estimating according to (47), we conclude that

$$|\mathbf{E}h(Z_n) - \mathbf{E}h(\mathcal{N})| \le 3H_1^{(0)}[2^\alpha \overline{\overline{L}}_1^\alpha + |1 - \mathbf{E}(\overline{Z}_n^{(0)})^2|^\alpha] + |\mathbf{E}h(\overline{Z}_n^{(0)}/\mathbf{E}^{1/2}(\overline{Z}_n^{(0)})^2) - \mathbf{E}h(\mathcal{N})| \tag{48}$$

when $h \in \mathcal{H}_1^{(0)}$ and $\mathbf{E}(\overline{Z}_n^{(0)})^2 > 0$.

The truncation inequality (48) clearly holds for any dependency of the sequence (12). Therefore, the proofs of Theorems 6–8 are completed on merely applying it to Theorems 3–5.

2.4. Estimation of Δ_n.

Theorem 9. *Let the sequence* (12) *be m-dependent and let* $\mathbf{E}|X_j|^s < \infty$, $2 < s \le 3$ *and* $j = 1, 2, \ldots, n$. *Then for* $m \ge 0$ *and* $n \ge 1$

$$\Delta_n \le C(m+1)^{s-1} L_s^*.$$

Theorem 10. *Let* (12) *be a strongly mixing sequence and let* $\mathbf{E}|X_j|^s < \infty$, $2 < s \le 3$ *and* $j = 1, 2, \ldots, n$.

1. *If* $\alpha(\tau) \le Ke^{-\lambda\tau}$, *then*

$$\Delta_n \le C(K, \lambda, s) L_s^* \ln^{s-1}(n+1).$$

2. *If* $\alpha(\tau) \le K\tau^{-\mu}$ *with* $\mu \ge \dfrac{2(s-1)}{s(s-2)^2}[\beta(5s-6) - (s-2)(3-s)]$ *and* $\beta > 1$, *then*

$$\Delta_n \le C(K, \beta, s) n^c d_s,$$

where $c = (4\beta + s^2 - 4)/(2(2\beta + s - 4))$.

For the exponential s.m. coefficient, the power of the logarithm may be lowered but under more stringent conditions on the moments of the terms. We now give a nonuniform bound in the case of strict stationarity.

Theorem 11 (Tikhomirov (1980)). *Suppose that* (12) *is a strictly stationary and strongly mixing sequence with* $\alpha(\tau) \leq Ke^{-\lambda\tau}$ *and that* $\mathbf{E}|X_1|^{4+\nu} < \infty$ *for some positive* ν. *Then*

$$\Delta_n \leq C(K, \lambda, \nu) n^{-1/2} \ln n$$

and

$$|\Delta_n(x)| \leq C(K, \lambda, \nu) \frac{\ln^3 n}{\sqrt{n}(1 + |x|)^4}.$$

Heinrich (1985d) obtained a nonuniform bound for $|\Delta_n(x)|$ for m-dependent r.v.'s. In particular, he proved the following result.

Theorem 12 (1985d). *Suppose that the sequence* (12) *is* m*-dependent and that* $\mathbf{E}|X_j|^s < \infty$, $2 < s \leq 3$ *and* $j = 1, 2, \ldots, n$. *Then*

$$|\Delta_n(x)| \leq C \frac{(m+1)^{s-1} L_s^*}{(1 + |x|)^s}$$

for all real x *and* $n \geq 1$.

When $s = 3$, the rate of convergence in Theorem 12 cannot be improved in relation to n, x (see Petrov (1987)) and m (see Berk (1973)).

We shall limit ourselves to proving Theorems 9 and 10.

Shergin (1979), (1990) was able to replace L_s^* by Lyapunov's quotient L_s in Theorems 9 and 12.

Tikhomirov (1986) obtained an estimate for the rate at which the d.f. of the $\max_{1 \leq k \leq n}(X_1 + \ldots + X_k)$ for a strictly stationary and strongly mixing sequence converges to the d.f.

$$\sqrt{2/\pi} \int_0^{x^+} e^{-u^2/2}\, du$$

in the uniform metric, where $x^+ = \max(0, x)$.

Proofs of Theorems 9 *and* 10.

Tikhomirov's method (presented in detail in Tikhomirov (1980) for a strictly stationary sequence) consists in deriving a linear differential equation for the c.f. $f_n(t) = \mathbf{E}e^{itZ_n}$ of a centered and normalized sum Z_n. This differential equation is "close" to the homogeneous differential equation $f'(t) = -tf(t)$ whose solution is the c.f. of the standard normal law, that is, $f(t) = e^{-t^2/2}$.

It will be recalled that Theorems 9 and 10 do not assume the stationarity of the r.v.'s (12). Similar bounds for Δ_n were found by Sunklodas (1984) for a sequence of either m-dependent or strongly mixing r.v.'s but under the additional condition $B_n^2 \geq c_0 n$, $0 < c_0 < \infty$.

In addition to the notation of §2, 2.1, write

$$\xi_j^{(l)} = e^{it(z_j^{(l-1)} - z_j^{(l)})} - 1, \quad \eta_j^{(r)} = e^{-itZ_j^{(r)}} - 1,$$
$$a_j^{(r-1)} = \mathbf{E}\left(iA_j \prod_{l=1}^{r-1} \xi_j^{(l)}\right), \quad z_j^{(0)} = Z_n.$$

Since $f_n'(t) = \sum_{j=1}^n \mathbf{E}(iA_j e^{itZ_n})$, by successively adding and subtracting the quantities $\mathbf{E}(iA_j e^{itz_j^{(1)}})$, $\mathbf{E}(iA_j \xi_j^{(1)} e^{itz_j^{(2)}})$, $\ldots$, $\mathbf{E}(iA_j \prod_{l=1}^{k-1} \xi_j^{(l)} e^{itz_j^{(k)}})$ in the summation, one finds that (Tikhomirov (1980))

$$f_n'(t) = \sum_{j=1}^n \mathbf{E}(iA_j e^{itz_j^{(1)}}) + \sum_{j=1}^n \sum_{r=2}^k \mathbf{E}\left(iA_j \prod_{l=1}^{r-1} \xi_j^{(l)} e^{itz_j^{(r)}}\right)$$
$$+ \sum_{j=1}^n \mathbf{E}\left(iA_j \prod_{l=1}^k \xi_j^{(l)} e^{itz_j^{(k)}}\right).$$

The relation

$$\mathbf{E}e^{itz_j^{(r)}} = \mathbf{E}(\eta_j^{(r)} + 1) f_n(t) + \mathbf{E}[(\eta_j^{(r)} - \mathbf{E}\eta_j^{(r)}) e^{itZ_n}]$$

can be utilized to prove that the derivative of $f_n(t)$ with respect to t is (Sunklodas (1984))

$$f_n'(t) = (E_1 + E_2) f_n(t) + E_3 + E_4 + E_5 + E_6, \tag{49}$$

in which

$$E_1 = \sum_{j=1}^n a_j^{(1)}, \quad E_2 = \sum_{j=1}^n \left[a_j^{(1)} \mathbf{E}\eta_j^{(2)} + \sum_{r=3}^k a_j^{(r-1)} \mathbf{E}(\eta_j^{(r)} + 1)\right],$$
$$E_3 = \sum_{r=2}^k \sum_{j=1}^n a_j^{(r-1)} \mathbf{E}[(\eta_j^{(r)} - \mathbf{E}\eta_j^{(r)}) e^{itZ_n}],$$
$$E_4 = \sum_{j=1}^n \mathbf{E}\left(iA_j \prod_{l=1}^k \xi_j^{(l)} e^{itz_j^{(k)}}\right), \quad E_6 = \sum_{j=1}^n \mathbf{E}(iA_j e^{itz_j^{(1)}}),$$
$$E_5 = \sum_{r=2}^k \sum_{j=1}^n \mathbf{E}\left[iA_j \prod_{l=1}^{r-1} \xi_j^{(l)} (e^{itz_j^{(r)}} - \mathbf{E}e^{itz_j^{(r)}})\right].$$

It is assumed here that $2km + 1 < n$.

The linear differential equation (49) is the starting point for proving Theorems 9–11. The passage from (49) to the difference of the c.f.'s is accomplished by means of the following lemma.

Lemma 6 (Sunklodas (1984)). *Let the linear differential equation*

$$f'(t) = (-t + \theta a(t)) f(t) + \theta b(t), \quad f(0) = 1, \tag{50}$$

be given for $|t| \le T_1$, *where*

$$a(t) = a^{(0)} + a^{(1)}|t| + a^{(2)}t^2 + a^{(3)}|t|^{s-1}, \quad 2 < s \le 3,$$
$$b(t) = b^{(0)} + b^{(2)}t^2.$$

Here the coefficients $a^{(i)} \ge 0$ $(i = 0,1,2,3)$ *and* $b^{(j)} \ge 0$ $(j = 0,2)$ *are independent of* t*; and* θ *is a complex function such that* $|\theta| \le 1$.

If $a^{(1)} \le 1/6$, *then*

$$|f(t) - e^{-t^2/2}| \le C[a^{(0)}|t| + a^{(1)}t^2 + a^{(2)}|t|^3 + a^{(3)}|t|^s]e^{-t^2/4} + C[b^{(0)} \min(|t|^{-1}, |t|) + b^{(2)}|t|] \tag{51}$$

for $|t| \le \min(T_1, T_2)$, *where*

$$T_2 = \min\left\{\frac{1}{a^{(0)}}, \frac{1}{6a^{(2)}}, \left(\frac{1}{6a^{(3)}}\right)^{1/(s-2)}\right\}.$$

Lemma 6 and Essen's inequality (see, for example, Petrov (1987), p. 154) lead easily to the next assertion:

If the c.f. $f(t) = \mathbf{E}e^{it\xi}$ of a real r.v. ξ satisfies the linear differential equation (50) for $|t| \le T_1$, then

$$\sup_x |\mathbf{P}\{\xi < x\} - \Phi(x)| \le C(a^{(0)} + a^{(1)} + a^{(2)} + a^{(3)} + b^{(0)} + b^{(2)}T_1 + T_1^{-1}). \tag{52}$$

Therefore to estimate Δ_n, all attention will be directed to deriving linear differential equation (50) for $f_n(t)$ so that its coefficients and T_1 assure a bound for Δ_n as close as possible to being optimum.

Let us proceed to prove Theorem 9 and 10.

Assume that $\mathbf{E}|X_j|^s < \infty$, $2 < s \le 3$ and $j = 1, 2, \ldots, n$. For any dependency,

$$E_1 = -t \sum_{j=1}^{n} \mathbf{E}(A_j Z_j^{(1)}) + \theta(2^{3-s}/(s-1))(2m+1)^{s-1} L_s |t|^{s-1}. \tag{53}$$

Now let the sequence (12) be m-dependent. Then

$$E_1 = -t + \theta(2^{3-s}/(s-1))(2m+1)^{s-1} L_s |t|^{s-1}. \tag{54}$$

It is easily seen that

$$\mathbf{E}^{1/2}|\xi_j^{(l)}|^2 \le \sqrt{2}(m+1) d_s^{1/s}|t| \tag{55}$$

and that

$$\mathbf{E}^{1/2}|\eta_j^{(r)}|^2 \le \sqrt{2r}(m+1) d_s^{1/s}|t|. \tag{56}$$

By virtue of the m-dependence and (55),

$$|a_j^{(r-1)}| \le \mathbf{E}^{1/2} A_j^2 \prod_{l=1}^{r-1} \mathbf{E}^{1/2}|\xi_j^{(l)}|^2 \le d_s^{1/s}(\sqrt{2}(m+1) d_s^{1/s}|t|)^{r-1} = a^{(r-1)}. \tag{57}$$

Since

$$|E_2| \le \sum_{j=1}^{n}\left[|a_j^{(1)}|\mathbf{E}|\eta_j^{(2)}| + \sum_{r=3}^{k}|a_j^{(r-1)}|\right), \tag{58}$$

(56) and (57) can be used to show that

$$|E_2| \le C(m+1)^2 n d_s^{3/s} t^2 \tag{59}$$

for $|t| \le (\sqrt{2}e^2(m+1)d_s^{1/s})^{-1} = T_1$.

By the m-dependence and the estimates (56) and (57), we find for $|t| \le T_1$ that

$$|E_3| \le \sum_{r=2}^{k} a^{(r-1)} \left(\sum_{j=1}^{n} \sum_{|p-j| \le 3rm} \mathbf{E}^{1/2}|\eta_j^{(r)}|^2 \mathbf{E}^{1/2}|\eta_p^{(r)}|^2\right)^{1/2} \le C(m+1)^{5/2} n^{1/2} d_s^{3/s} t^2. \tag{60}$$

By virtue of (57),

$$|E_4| \le n a^{(k)} \le C(m+1)^2 n^{1/2} d_s^{3/s} t^2 \tag{61}$$

for $k \ge 2 + \frac{1}{4}\ln n$ and $|t| \le T_1$. Since $E_5 = E_6 = 0$ because of the m-dependence, on substituting (54), (59)–(61) in (49), we find that

$$f_n'(t) = (-t + \theta a_n(t)) f_n(t) + \theta b_n(t) \tag{62}$$

providing $2km + 1 < n$, $k \ge 2 + \frac{1}{4}\ln n$ and $|t| < T_1$, where

$$a_n(t) = a_n^{(2)} t^2 + a_n^{(3)}|t|^{s-1}, \qquad b_n(t) = b_n^{(2)} t^2,$$
$$a_n^{(2)} = C(m+1)^2 n d_s^{3/s}, \qquad a_n^{(3)} = C(m+1)^{s-1} L_s^*,$$
$$b_n^{(2)} = C(m+1)^{5/2} n^{1/2} d_s^{3/s}.$$

Relations (52) and (62) yield Theorem 9 if $m+1 < n/(C\ln(n+1))$, $C > 1$. Theorem 9 follows for all $m \ge 0$ from the third inequality in Lemma 7 of §2, 2.5 and Esseen's inequality.

Now suppose that (12) is a s.m. sequence. Write $z_j^{(i)} = \hat{z}_j^{(i)} + \hat{\hat{z}}_j^{(i)}$, where $\hat{z}_j^{(i)}$ is the sum of those A_p in $z_j^{(i)}$ for which $p < j - im$ and $\hat{\hat{z}}_j^{(i)}$ is the sum of A_p for which $p > j + im$.

Then according to (6),

$$\sum_{j=1}^{n} |\mathbf{E}(A_j z_j^{(1)})| \le \sum_{j=1}^{n} [|\mathbf{E}(A_j \hat{z}_j^{(1)})| + |\mathbf{E}(A_j \hat{\hat{z}}_j^{(1)})|]$$
$$\le 6n^2 d_s^{2/s} (\alpha(m+1))^{(s-2)/s}.$$

Therefore (53) yields

$$E_1 = -t + \theta(2^{3-s}/(s-1))(2m+1)^{s-1} L_s^* |t|^{s-1} + \theta 6n^2 d_s^{2/s} (\alpha(m+1))^{(s-2)/s} |t|. \tag{63}$$

Put $\hat{\xi}_j^{(l)} = e^{itx} - 1$, $\hat{\hat{\xi}}_j^{(l)} = e^{ity} - 1$, where $x = \sum_{p=j-lm}^{j-(l-1)m-1} A_p$, and $y = \sum_{p=j+(l-1)m+1}^{j+lm} A_p$. Then by Minkowski's inequality, for $1 \le \mu \le s$

$$\max\{\mathbf{E}^{1/\mu} |\hat{\xi}_j^{(l)}|^\mu,\ \mathbf{E}^{1/\mu} |\hat{\hat{\xi}}_j^{(l)}|^\mu\} \le (m+1) d_s^{1/s} |t| \tag{64}$$

and

$$\mathbf{E}^{1/\mu} |\eta_j^{(r)}|^\mu \le (2rm+1) d_s^{1/s} |t| = \eta^{(r)}. \tag{65}$$

Similarly,

$$a_j^{(1)} \le (2m+1) d_s^{2/s} |t|. \tag{66}$$

We next estimate $a_j^{(r-1)}$ for $r = 3, 4, \ldots, k$. Since $|\xi_j^{(l)}| \le |\hat{\xi}_j^{(l)}| + |\hat{\hat{\xi}}_j^{(l)}|$, we have

$$|a_j^{(r-1)}| \le \sum_{p=0}^{r-1} \sum{}^{*} \mathbf{E} \left| A_j \prod_{\nu=1}^{p} \hat{\xi}_j^{(l_\nu)} \prod_{\mu=p+1}^{r-1} \hat{\hat{\xi}}_j^{(l_\mu)} \right|, \tag{67}$$

in which $\sum^*$ denotes summation over all collections of indices $1 \le l_1 < l_2 < \ldots < l_p \le r-1$ and $1 \le l_{p+1} < l_{p+2} < \ldots < l_{r-1} \le r-1$ such that $l_\nu \ne l_\mu$ for $\nu \ne \mu$ (see Tikhomirov (1980)).

By Hölder's inequality,

$$\mathbf{E} \left| A_j \prod_{\nu=1}^{p} \hat{\xi}_j^{(l_\nu)} \prod_{\mu=p+1}^{r-1} \hat{\hat{\xi}}_j^{(l_\mu)} \right|$$
$$\le \mathbf{E}^{1/s} \left| \prod_\nu{}' \hat{\xi}_j^{(l_\nu)} \prod_\mu{}' \hat{\hat{\xi}}_j^{(l_\mu)} \right|^s \mathbf{E}^{(s-1)/s} \left| A_j \prod_\nu{}'' \hat{\xi}_j^{(l_\nu)} \prod_\mu{}'' \hat{\hat{\xi}}_j^{(l_\mu)} \right|^{s/(s-1)},$$

where $\prod''$ and $\prod'$ are respectively products over all even and all odd l from 1 up to $r-1$.

Let $r-1$ be even (if $r-1$ is odd, one proceeds in the same way). Put $(\cdot) = (m+1) d_s^{1/s} |t|$. Then by (7), (8) and (64),

$$\mathbf{E} \left| \prod_\nu{}' \hat{\xi}_j^{(l_\nu)} \prod_\mu{}' \hat{\hat{\xi}}_j^{(l_\mu)} \right|^s \le (\cdot)^{s(r-1)/2} + 8(r-3) 2^{s(r-1)/2} \alpha(m+1)$$

and

$$\mathbf{E} \left| A_j \prod_\nu{}'' \hat{\xi}_j^{(l_\nu)} \prod_\mu{}'' \hat{\hat{\xi}}_j^{(l_\mu)} \right|^{s/(s-1)} \le d_s^{1/(s-1)} \big[(\cdot)^{s(r-1)/(2(s-1))}$$
$$+ 8(r-1) 2^{s(r-1)/(2(s-1))} (\alpha(m+1))^{(s-2)/(s-1)} \big].$$

Therefore

$$\mathbf{E} \left| A_j \prod_{\nu=1}^{p} \hat{\xi}_j^{(l_\nu)} \prod_{\mu=p+1}^{r-1} \hat{\hat{\xi}}_j^{(l_\mu)} \right| \le 8 d_s^{1/s} (\alpha(m+1))^{(s-1)/s} (r-1) 2^{r-1}$$
$$+ d_s^{1/s} (\cdot)^{r-1} + 7 d_s^{1/s} (2(\cdot))^{(r-1)/2} (\alpha(m+1))^{(s-2)/s} (r-1).$$

Since there are at most 2^{r-1} terms in (67), for $|t| \le (32(m+1) d_s^{1/s})^{-1} = T_3$ and $r = 3, 4, \ldots, k$, we obtain

$$|a_j^{(r-1)}| \le C \left\{ m^2 d_s^{3/s} t^2 \left(\frac{1}{2}\right)^{4r} + d_s^{1/s} (\alpha(m+1))^{(s-2)/s} \left[r \left(\frac{1}{2}\right)^r + r 4^r (\alpha(m+1))^{1/s} \right] \right\} = a^{(r-1)}. \tag{68}$$

It is now possible to estimate $|E_4|$. For $k \ge \ln n / (8 \ln 2) \ge 3$ and $|t| \le T_3$, we find that

$$|E_4| \le n a^{(k)} \le C[m^2 n^{1/2} d_s^{3/s} t^2 + d_s^{1/s} n (\alpha(m+1))^{(s-2)/s}]. \tag{69}$$

We estimate the terms E_2, E_3 and E_5 subject to the additional condition

$$k^{3/2} 4^k (\alpha(m+1))^{1/s} \le 1. \tag{70}$$

Substituting (66), (65) and (68) in (58), we find for $|t| \le T_3$ that

$$|E_2| \le C[m^2 n d_s^{3/s} t^2 + n d_s^{1/s} (\alpha(m+1))^{(s-2)/s}] \tag{71}$$

under condition (70).

Consider $\operatorname{cov}(\xi, \eta) = \mathbf{E}(\xi - \mathbf{E}\xi)(\overline{\eta - \mathbf{E}\eta})$. By virtue of (6),

$$|E_3| \le \sum_{r=2}^{k} a^{(r-1)} \left(\sum_{j=1}^{n} \sum_{p=1}^{n} |\operatorname{cov}(\eta_j^{(r)}, \eta_p^{(r)})| \right)^{1/2}$$
$$\le \sum_{r=2}^{k} a^{(r-1)} \left(\sum_{j=1}^{n} \sum_{|p-j| \le 2rm} \mathbf{E}^{1/2} |\eta_j^{(r)}|^2 \mathbf{E}^{1/2} |\eta_p^{(r)}|^2 \right.$$
$$\left. + \sum_{j=1}^{n} \sum_{|p-j| > 2rm} 24 \mathbf{E}^{1/s} |\eta_j^{(r)}|^s \mathbf{E}^{1/s} |\eta_p^{(r)}|^s (\alpha(|p-j| - 2rm))^{(s-2)/s} \right)^{1/2}.$$

Noting (66), (65) and (68), we obtain

$$|E_3| \le C[m^{1/2} + A_{n,s}^{1/2}][m^2 n^{1/2} d_s^{3/s} t^2 + n^{1/2} d_s^{1/s} (\alpha(m+1))^{(s-2)/s}] \qquad (72)$$

for $|t| \le T_3$ under assumption (70).

The definition of $z_j^{(r)}$ shows that $\hat{z}_j^{(r)}$ and $\hat{\hat{z}}_j^{(r)}$ cannot vanish simultaneously. We shall consider that both do not vanish (otherwise the computations merely become simpler). Observe that

$$\begin{aligned}
&\left|\mathbf{E}\left[iA_j \prod_{l=1}^{r-1} \xi_j^{(l)} (e^{tiz_j^{(r)}} - \mathbf{E}e^{itz_j^{(r)}})\right]\right| \\
&\le \left|\mathbf{E}\left(iA_j \prod_{l=1}^{r-1} \xi_j^{(l)} e^{itz_j^{(r)}}\right) - \mathbf{E}\left(iA_j \prod_{l=1}^{r-1} \xi_j^{(l)} e^{it\hat{z}_j^{(r)}}\right) \mathbf{E}e^{it\hat{\hat{z}}_j^{(r)}}\right| \\
&\quad + \left|\mathbf{E}\left[iA_j \prod_{l=1}^{r-1} \xi_j^{(l)} (e^{it\hat{z}_j^{(r)}} - \mathbf{E}e^{it\hat{z}_j^{(r)}})\right]\right| \\
&\quad + \left|\mathbf{E}e^{it\hat{z}_j^{(r)}} \mathbf{E}e^{it\hat{\hat{z}}_j^{(r)}} - \mathbf{E}e^{itz_j^{(r)}}\right| \cdot \left|\mathbf{E}\left(iA_j \prod_{l=1}^{r-1} \xi_j^{(l)}\right)\right|.
\end{aligned}$$

The right-hand side of this last inequality can be estimated by means of inequalities (5) and (4) and one can show that it does not exceed $48 \cdot 2^{r-1} d_s^{1/s} (\alpha(m+1))^{(s-1)/s}$. Summing the resultant inequalities over all j and r, we conclude that

$$|E_5| \le 48 n d_s^{1/s} (\alpha(m+1))^{(s-2)/s} \qquad (73)$$

under condition (70).

$|E_6|$ may be estimated in similar fashion with the result

$$|E_6| \le 32 n d_s^{1/s} (\alpha(m+1))^{(s-1)/s}. \qquad (74)$$

Put $A_s = \sum_{\tau=1}^{\infty} (\alpha(\tau))^{(s-2)/s}$. The substitution of (63), (69), (71)–(74) in (49) yields this: If $A_s < \infty$, $k^{3/2} 4^k (\alpha(m+1))^{1/s} \le 1$, $3 \le \ln n/(8 \ln 2) \le k$ and $2k(m+1) \le n$, then

$$f_n'(t) = (-t + \theta a_n(t)) f_n(t) + \theta b_n(t) \qquad (75)$$

for $|t| \le T_3$, where

$$\begin{aligned}
a_n(t) &= a_n^{(0)} + a_n^{(1)} |t| + a_n^{(2)} t^2 + a_n^{(3)} |t|^{s-1}, \\
b_n(t) &= b_n^{(0)} + b_n^{(2)} t^2,
\end{aligned}$$

$$\begin{aligned}
a_n^{(0)} &= C n d_s^{1/s} (\alpha(m+1))^{(s-2)/s}, & a_n^{(1)} &= C n^2 d_s^{2/s} (\alpha(m+1))^{(s-2)/s}, \\
a_n^{(2)} &= C m^2 n d_s^{3/s}, & a_n^{(3)} &= C m^{s-1} L_s^*, \\
b_n^{(0)} &= C(A_s) n d_s^{1/s} (\alpha(m+1))^{(s-2)/s}, & b_n^{(2)} &= C(A_s) m^{5/2} n^{1/2} d_s^{3/s}.
\end{aligned}$$

The differential equations (50) and (75) are of the same form and so (52) yields the following

Proposition 4. *Suppose that* (12) *is a s.m. sequence,* $\mathbf{E}|X_j|^s < \infty$, $2 < s \le 3$ *and* $j = 1, 2, \ldots, n$, $A_s < \infty$ *and* $k^{3/2} 4^k (\alpha(m+1))^{1/s} \le 1$, *with the positive integers* k *and* m *satisfying* $3 \le \ln n/(8 \ln 2) \le k$ *and* $2k(m+1) \le n$. *Then*

$$\Delta_n \le C(A_s)[m^{s-1} L_s^* + n^2 d_s^{2/s} (\alpha(m+1))^{(s-2)/s}).$$

It merely remains to select k and m depending on the rate of decay of the s.m. coefficient.

If $n \ge C(K, \lambda, s)$, part 1 of Theorem 10 follows from Proposition 4 with $m = \left[\frac{3s}{\lambda(s-2)} \ln(n+1)\right]$ and $k = \left[\frac{1}{(s-2)} \ln(n+1)\right]$.

If $n > C(K, \beta, s)$, part 2 of Theorem 10 follows from Proposition 4 with $m = [n^\varepsilon]$, $\varepsilon = s(s-2)/(2(s-1)(2\beta + s - 2))$ and $k = \left[\frac{5\beta}{8(2\beta+1)} \ln(n+1)\right]$.

For small n, Theorem 10 is a consequence of the estimate (17).

2.5. Heinrich's Method for m-Dependent Random Variables.

Heinrich's method (1982) is a fairly general way of deducing various limit theorems for the sums Z_n of m-dependent r.v.'s. It is based on the factorization of the c.f. $f_n(t) = \mathbf{E}e^{itZ_n}$ (or moment-generating function $\mathbf{E}e^{zZ_n}$, $z \in (C^1)$ in a neighborhood of $t = 0$ (or $z = 0$).

We shall only consider here the factorization of $f_n(t)$. By making use of the factorization of $f_n(t)$ once it has been found, one can obtain, for instance, these sorts of results for sums of m-dependent r.v.'s: convergence to unbounded distributions, uniform and non-uniform bounds for the rate of convergence in the CLT, asymptotic expansions, moderate and large deviations and so on.

The gist of Heinrich's method will be demonstrated by the proof of a single lemma.

Lemma 7. *Suppose that* (12) *is an* m*-dependent sequence and that* $\mathbf{E}|X_j|^s < \infty$, $2 < s \le 3$ *and* $j = 1, 2, \ldots, n$. *Then the following inequalities hold in the interval*

$$|t| \le \left(\frac{1-2c}{2C_1(s)} \cdot \frac{1}{(m+1)^{s-1} L_s^*}\right)^{1/(s-2)} = T_4$$

for $m \ge 0$*:*

1. $\left|\ln f_n(t) + \frac{t^2}{2}\right| \le C_1(s)(m+1)^{s-1} L_s^* |t|^s$,
2. $|f_n(t)| \le e^{-ct^2}$,
3. $|f_n(t) - e^{-t^2/2}| \le C_1(s)(m+1)^{s-1} L_s^* |t|^s e^{-ct^2}$,

where

$$C_1(s) = \left(149 + \frac{1}{s}\right) \frac{2^{4-s}}{s-1}, \quad 0 < c < 1/2.$$

When $s=3$, Lemma 7 follows (with different constants) from Corollary 3.2 in Heinrich (1982). Since $|e^z| \le e^{|z|}$ and $|e^z - 1| \le |z|e^{|z|}$ for any complex z, the second and third inequalities in Lemma 7 are consequence of the first one. To prove the first inequality, we need several preliminary assertions.

For any sequence of complex r.v.'s $\xi_1, \xi_2, \ldots$ such that $\mathbf{E}|\xi_j|^k < \infty$, $j = 1, 2, \ldots, k$, the symbol $\hat{\mathbf{E}}\xi_1\xi_2\ldots\xi_k$ means that $\hat{\mathbf{E}}\xi_1 = \mathbf{E}\xi_1$ and for $k \ge 2$, that (see Heinrich (1982))

$$\hat{\mathbf{E}}\xi_1\xi_2\ldots\xi_k = \mathbf{E}\xi_1\xi_2\ldots\xi_k - \sum_{j=1}^{k-1}\hat{\mathbf{E}}\xi_1\ldots\xi_j\mathbf{E}\xi_{j+1}\ldots\xi_k. \tag{76}$$

This symbol was first introduced in another way by Statulevičius (1970b) and is known as the k-th centered moment. Among the many interesting properties of centered moments are the following.

Lemma 8 (Heinrich (1982)). *Let $\xi_1, \xi_2, \ldots, \xi_k$ be 1-dependent complex r.v.'s.*

1. *If $\mathbf{E}|\xi_j|^k < \infty$, $j = 1, 2, \ldots, k$, then*

$$\hat{\mathbf{E}}(\xi_1 + a_1)(\xi_2 + a_2)\ldots(\xi_k + a_k) = \hat{\mathbf{E}}\xi_1\xi_2\ldots\xi_k, \tag{77}$$

where $a_1, a_2, \ldots, a_k$ are any complex numbers.

2. *If $\mathbf{E}|\xi_j|^2 < \infty$, $j = 1, 2, \ldots, k$, then*

$$|\hat{\mathbf{E}}\xi_1\xi_2\ldots\xi_k| \le 2^{k-1}\prod_{j=1}^{k}\mathbf{E}^{1/2}|\xi_j|^2. \tag{78}$$

Corresponding to a sequence (12) of m-dependent r.v.'s, we form new 1-dependent r.v.'s

$$Y_j = \sum_{p=(j-1)(m+1)+1}^{j(m+1)} A_p, \quad j = 1, 2, \ldots, N = [n/(m+1)),$$

$$Y_{N+1} = \begin{cases} \sum_{p=N(m+1)+1}^{n} A_p & \text{if } N(m+1) < n, \\ 0 & \text{if } N(m+1) = n, \end{cases}$$

where $[x]$ is the integer part of x.

Put $U_j = \sum_{i=1}^{j} Y_i$, $j = 1, 2, \ldots, N+1$, $w = \max_{1 \le j \le N+1}\mathbf{E}^{1/2}|e^{itY_j} - 1|^2$ and $u_j(t) = 2\mathbf{E}^{1/2}|e^{itY_{j-1}} - 1|^2\mathbf{E}^{1/2}|e^{itY_j} - 1|^2$.

Then the following is true.

Lemma 9 (Heinrich (1982)). *If $w \le 1/6$, then*

1.
$$f_n(t) = \prod_{j=1}^{N+1} g_j(t), \tag{79}$$

where $g_1(t) = \mathbf{E}e^{itY_1}$, and for $j = 2, 3, \ldots, N+1$,

$$g_j(t) = \frac{\mathbf{E}e^{itU_j}}{\mathbf{E}e^{itU_{j-1}}} = \mathbf{E}e^{itY_j} + \sum_{a=1}^{j-1}\frac{\hat{\mathbf{E}}(e^{itY_a} - 1)(e^{itY_{a+1}} - 1)\ldots(e^{itY_j} - 1)}{\prod_{p=a}^{j-1} g_p(t)}; \tag{80}$$

2. *for $j = 1, 2, \ldots, N+1$,*

$$|g_j(t) - 1| \le |\mathbf{E}e^{itY_j} - 1| + 3u_j(t) \tag{81}$$
$$\le 2w. \tag{82}$$

Lemma 9 is proved by induction using relations (76)–(78).

By virtue of (80),

$$\begin{aligned}\sum_{j=1}^{N+1}[g_j(t) - 1] &= \sum_{j=1}^{N+1}(\mathbf{E}e^{itY_j} - 1) \\ &+ \sum_{j=2}^{N+1}\sum_{a=1}^{j-1}\left(\frac{1}{g_a(t)\ldots g_{j-1}(t)} - 1\right)\hat{\mathbf{E}}(e^{itY_a} - 1)\ldots(e^{itY_j} - 1) \\ &+ \sum_{j=3}^{N+1}\sum_{a=1}^{j-2}\hat{\mathbf{E}}(e^{itY_a} - 1)\ldots(e^{itY_j} - 1) \\ &+ \sum_{j=2}^{N+1}\hat{\mathbf{E}}(e^{itY_{j-1}} - 1)(e^{itY_j} - 1).\end{aligned}$$

Since $Y_1, Y_2, \ldots, Y_{N+1}$ are 1-dependent r.v.'s with zero expectations,

$$\begin{aligned}\frac{t^2}{2} &= \sum_{j=1}^{N+1}\left[\mathbf{E}e^{itY_j} - 1 - \frac{(it)^2}{2}\mathbf{E}Y_j^2\right] \\ &+ \sum_{j=2}^{N+1}[\hat{\mathbf{E}}(e^{itY_{j-1}} - 1)(e^{itY_j} - 1) - (it)^2\mathbf{E}(Y_{j-1}Y_j)] \\ &- \sum_{j=1}^{N+1}(\mathbf{E}e^{itY_j} - 1) - \sum_{j=2}^{N+1}\hat{\mathbf{E}}(e^{itY_{j-1}} - 1)(e^{itY_j} - 1).\end{aligned}$$

Adding the last two relations, we find that if (12) is a sequence of m-dependent r.v.'s and $\mathbf{E}|X_j|^s < \infty$, $2 < s \le 3$ and $j = 1, 2, \ldots, n$, then

$$\ln f_n(t) + \frac{t^2}{2} = \Sigma_1 + \ldots + \Sigma_5, \tag{83}$$

where

$$\Sigma_1 = \sum_{j=1}^{N+1} [\ln g_j(t) - (g_j(t) - 1)],$$

$$\Sigma_2 = \sum_{j=2}^{N+1} \sum_{a=1}^{j-1} \left(\frac{1}{g_a(t) \ldots g_{j-1}(t)} - 1 \right) \hat{\mathbf{E}}(e^{itY_a} - 1) \ldots (e^{itY_j} - 1),$$

$$\Sigma_3 = \sum_{j=3}^{N+1} \sum_{a=1}^{j-2} \hat{\mathbf{E}}(e^{itY_a} - 1) \ldots (e^{itY_j} - 1),$$

$$\Sigma_4 = \sum_{j=1}^{N+1} \left[\mathbf{E}e^{itY_j} - 1 - \frac{(it)^2}{2} \mathbf{E}Y_j^2 \right],$$

$$\Sigma_5 = \sum_{j=2}^{N+1} \left[\hat{\mathbf{E}}(e^{itY_{j-1}} - 1)(e^{itY_j} - 1) - (it)^2 \mathbf{E}(Y_{j-1}Y_j) \right].$$

Everywhere below when estimating the right-hand side of (83), we are assuming that $w \le 1/6$.

By the simple inequality $|\ln z - (z-1)| \le |z-1|^2$, which is true for $|z-1| \le 1/2$, and the estimates (81)–(82), it follows that

$$|E_1| \le 2w \sum_{j=1}^{N+1} |\mathbf{E}e^{itY_j} - 1| + 6w \sum_{j=2}^{N+1} u_j(t). \tag{84}$$

According to (82),

$$\left| \frac{1}{g_a(t) \ldots g_{j-1}(t)} - 1 \right| \le 3(j-a)2^{j-a-1} w,$$

and the estimate (78) leads to

$$|\Sigma_2| \le 27w \sum_{j=2}^{N+1} u_j(t). \tag{85}$$

The estimate (78) also assures that

$$|\Sigma_3| \le 3w \sum_{j=3}^{N+1} u_j(t). \tag{86}$$

For $2 < s \le 3$,

$$|e^{ix} - 1| \le 2^{3-s}|x|^{s-2},$$
$$|e^{ix} - 1 - ix| \le (2^{3-s}/(s-1))|x|^{s-1},$$
$$\left| e^{ix} - 1 - ix - \frac{(ix)^2}{2} \right| \le (2^{3-s}/(s-1)s)|x|^s,$$

and so

$$|\Sigma_4| \le (2^{3-s}/(s-1)s)|t|^s \sum_{j=1}^{N+1} \mathbf{E}|Y_j|^s, \tag{87}$$

$$|\Sigma_5| \le 3(2^{3-s}/(s-1))|t|^s \sum_{j=1}^{N+1} \mathbf{E}|Y_j|^s, \tag{88}$$

and

$$w \sum_{j=1}^{N+1} |\mathbf{E}e^{itY_j} - 1| \le (2^{3-s}/(s-1))(N+1) \max_{1 \le j \le N+1} \mathbf{E}|Y_j|^s |t|^s. \tag{89}$$

Adding (84)–(88) and using (89) in conjunction with the fact that

$$\sum_{j=2}^{N+1} u_j(t) \le 4 \sum_{j=1}^{N+1} |\mathbf{E}e^{itY_j} - 1|,$$

we find for $w \le 1/6$ that

$$\left| \ln f_n(t) + \frac{t^2}{2} \right| \le C_1(s)(m+1)^{s-1} L_s^* |t|^s. \tag{90}$$

It remains to observe that $w \le 1/6$ when $|t| \le T_4$. Consequently, the first inequality of Lemma 7 has been proved and thereby all of Lemma 7.

From Lemma 9 it is seen that the functions $g_j(t)$ (whose product is the c.f. of Z_n) although not c.f.'s, behave primarily like the c.f.'s of the Y_i's. The next lemma, for example, underscores this fact.

Lemma 10 (Heinrich (1982)). *Let the sequence* (12) *be* m*-dependent and let* $\max_{1 \le j \le n} \mathbf{E}|X_j|^p < \infty$ *for some* $p = 1, 2, \ldots$. *Then when* $w \le 1/6$,

1. $\max_{1 \le j \le N+1} \left| \frac{d^p}{dt^p} g_j(t) \right| \le C(p) \max_{1 \le j \le N+1} \mathbf{E}|Y_j|^p,$
2. $\max_{1 \le j \le N+1} \left| \frac{d^p}{dt^p} \ln g_j(t) \right| \le C(p) \max_{1 \le j \le N+1} \mathbf{E}|Y_j|^p,$

where the constants $C(p)$ *can be determined explicitly.*

Lemma 9 therefore plays a fundamental role in the study of the limiting law for the distribution of the normalized sum Z_n of m-dependent r.v.'s (see Heinrich (1982), (1984), (1985a,b,c,d)).

§3. Estimation of the Rate of Convergence in the Central Limit Theorem for Weakly Dependent Random Fields

Let $Z^d = \{a = (a_1, \ldots, a_d) : a_i \in \{0, \pm 1, \ldots\},\ i = 1, 2, \ldots, d\}$, $||a|| = \max_{1 \le i \le d} |a_i|$ and $\mathcal{V} = \{V \subset Z^d : |V| < \infty\}$, where $|V| = \#\{a : a \in V\}$ is the number of elements in V. The distance between $V_1, V_2 \in \mathcal{V}$ is defined as follows: $d(V_1, V_2) = \min\{||a - b|| : a \in V_1, b \in V_2\}$. $\mathcal{F}_V$ denotes the σ-algebra of events generated by the r.v.'s $\{X_a, a \in V\}$.

In what follows, we shall consider a real random field $\{X_a, a \in Z^d\}$, $d \ge 1$, satisfying one of the following weak dependence conditions:

1. *m-dependence*: $\mathcal{F}_{V_1}$ and $\mathcal{F}_{V_2}$ are independent for $\forall V_1, V_2 \in \mathcal{V}$ with $d(V_1, V_2) > m$;

2. *strong mixing* (s.m.): if there exist functions $M : Z_+^2 \to [1, \infty)$ and $\alpha : N \to [0, \infty)$ such that M is nondecreasing in each argument, $\alpha(r) \downarrow 0$ as $r \to \infty$ and for $\forall V_1, V_2 \in \mathcal{V}$,

$$\sup_{\substack{A \in \mathcal{F}_{V_1} \\ B \in \mathcal{F}_{V_2}}} |\mathbf{P}\{AB\} - \mathbf{P}\{A\}\mathbf{P}\{B\}| \le M(|V_1|, |V_2|)\alpha(d(V_1, V_2)).$$

The definition of these and other mixing coefficients for random fields as well as references on the subject may be found, for instance, in Dobrushin (1968), Bulinskii (1987), (1989), Takahata (1983), (1984), Sunklodas (1986) and Nakhapetyan (1987). Let

$$\{X_a,\ a \in Z^d\}, \quad d \ge 1, \tag{91}$$

be a real random field with $\mathbf{E}X_a = 0$ and $\mathbf{E}X_a^2 < \infty$ for $a \in V$. For $V \in \mathcal{V}$, $V \ne \emptyset$, put

$$S_V = \sum_{a \in V} X_a, \quad B_V^2 = \mathbf{E}S_V^2, \quad Z_V = S_V / B_V,$$

$$F_V(x) = \mathbf{P}\{Z_V < x\}, \quad A_a = X_a / B_V, \quad L_r = \sum_{a \in V} \mathbf{E}|A_a|^r,$$

$$L_s^* = |V| d_s, \quad d_s = \max_{a \in V} \mathbf{E}|A_a|^s, \quad \Delta_V(x) = F_V(x) - \Phi(x),$$

$$\Delta_V = \sup_x |\Delta_V(x)|, \quad ||\Delta_V(x)||_1 = \int_{-\infty}^{\infty} |\Delta_V(x)| dx,$$

$$d_i^{(p)}(F_V, \Phi) = \sup_{h \in \mathcal{H}_i^{(p)}} |\mathbf{E}h(Z_V) - \mathbf{E}h(\mathcal{N})| / ||h||_{BH_i^{(p)}},$$

where $\mathcal{H}_i^{(p)}$ is the class of functions $h : R \to R$ with norm $||h||_{BH_i^{(p)}}$ defined in §2, 2.1, and $i = 1, 2$.

The rate of convergence in the CLT for weakly dependent random fields has been estimated by generalizing the methods developed for sequences of

weakly depenent r.v.'s. The specific difficulties that have to be overcome in estimating Δ_V for multi-indexed terms, the distinctive features of mixing fields, the limits of applicability of Bernstein's method to random fields and other related question are discussed in detail in Bulinskii's book (1989). Leonenko (1975) found a bound for Δ_V in the case of m-dependent random fields for integer parallelepipeds $V \subset Z^d$. When $d = 1$, it reduces to Petrov's result (1960) cited above. By generalizing Maejima's results (1978) for r.v.'s, Rao (1981) found a nonuniform bound for $|\Delta_V(x)|$ for integer parallelepipeds for m-dependent random fields. He conjectured particularly that it was impossible to obtain an estimate for Δ_V of order $O(|V|^{-\gamma})$, $0 < \gamma \le 1/2$, even for an m-dependent random field. This conjecture was disproved by Takahata (1983) and Guyon and Richardson (1984). A more precise uniform estimate (compared to those of Leonenko (1975) and Rao (1981)) for weakly dependent random additive functions (encompassing the class of m-dependent ones) was found by Bulinskii (1977). He subsequently strengthened this estimate (1987). The proofs by Rao, Leonenko and Bulinskii (1977) utilize Bernstein's method.

More exact estimates of the rate of convergence in the CLT for weakly dependent random fields have been found by means of the techniques of Stein and Tikhomirov.

Guyon and Richardson (1984) study the rate of convergence in the CLT for centered weakly dependent random fields $\{X_a, a \in Z^d\}$ (either m-dependent or s.m. with $M \equiv 1$) that satisfy $\sup_{a \in Z^d} \mathbf{E}|X_a|^{2+\delta} < \infty$, $\delta > 0$. The summation $S_{V_n} = \sum_{a \in V_n} X_a$ is over a strictly increasing sequence of sets $V_n \in \mathcal{V}$ such that $\liminf_{n \to \infty} B_{V_n}^2 / |V_n| > 0$, where $B_{V_n}^2 = \mathbf{E}S_{V_n}^2$.

In particular, they show that

1. for m-dependent fields

$$\Delta_{V_n} = \begin{cases} O(B_{V_n}^{-\delta}) & \text{if } 0 < \delta < 1; \quad (92) \\ O[B_{V_n}^{-1} (\log B_{V_n})^{(d-1)/2}) & \text{if } \delta \ge 1; \quad (93) \end{cases}$$

2. for s.m. random fields with $M \equiv 1$ and α an exponentially decreasing function (here $a \wedge b = \min(a, b)$)

$$\Delta_{V_n} = O[B_{V_n}^{-(\delta \wedge 1)} (\log B_{V_n})^{d(1+\delta) \wedge 2}). \tag{94}$$

If $\sup_{a \in Z^d} \mathbf{E}|X_a|^{4+\delta} < \infty$, $\delta > 0$, the last estimate can be improved to

$$\Delta_{V_n} = O[B_{V_n}^{-1} (\log B_{V_n})^d). \tag{95}$$

Guyon and Richardson (1984) also investigated the case where $M \not\equiv 1$ and α decreases like a power function. The proofs are carried out by Tikhomirov's method.

Takahata (1983) found upper bounds for Δ_{V_n} and $||\Delta_{V_n}(x)||_1$ when the random field is m-dependent or s.m. with $M \not\equiv 1$ and α exponentially decreasing; the summation $S_{V_n} = \sum_{a \in V_n} X_a$ is over a sequence of sets $V_n \in \mathcal{V}$

such that $|V_n| \to \infty$ $(n \to \infty)$ and $\liminf_{n\to\infty} B^2_{V_n}/|V_n| > 0$. Under these conditions, the following was shown:

1. for m-dependent random fields

$$\Delta_{V_n} = O(|V_n|^{-1/2}) \tag{96}$$

if $\sup_{a\in Z^d} \mathbf{E}X_a^8 < \infty$, and

$$||\Delta_{V_n}(x)||_1 = O(|V_n|^{-1/2}) \tag{97}$$

if $\sup_{a\in Z^d} \mathbf{E}X_a^4 < \infty$;

2. for s.m. random fields with $M(n,m) \le B(n+m)^k$ for some $k > 1$ and $\alpha(\tau) \le Ke^{-\lambda\tau}$

$$\Delta_{V_n} = O[|V_n|^{-1/2}(\log|V_n|)^d] \tag{98}$$

if $\sup_{a\in Z^d} \mathbf{E}|X_a|^{8+\delta} < \infty$, $\delta > 0$, and

$$||\Delta_{V_n}(x)||_1 = O[|V_n|^{-1/2}(\log|V_n|)^d] \tag{99}$$

if $\sup_{a\in Z^d} \mathbf{E}|X_a|^{4+\delta} < \infty$, $\delta > 0$.

The estimate (96) established by Takahata (1983) for an m-dependent random field refines Riauba's paper (1980). Takahata (1983) used Stein's technique to prove his result.

The methods of Stein (Takahata (1983), Sunklodas (1986)) and Tikhomirov (Guyon and Richardson (1984), Sunklodas (1986), Bulinski (1986a), (1987)) may be extended to weakly dependent nonstationary random fields whose terms have finite absolute moments of order s, $2 < s \le 3$. Without any assumptions about the linear growth of the variance B^2_V of the sum S_V, $V \in \mathcal{V}$, one may estimate $||\Delta_V(x)||_1$ (by Stein's method) and Δ_V (by Tikhomirov's method) in such a way that for $d = 1$ these estimates yield the best known estimates (or ones close to them) for a weakly dependent sequence of r.v.'s (Tikhomirov (1980), Erickson (1974), Sunklodas (1982)).

We state a number of results of this kind.

Theorem 13 (Sunklodas (1986)). *Suppose that the random field* (91) *is m-dependent and that $\mathbf{E}|X_a|^s < \infty$, $2 < s \le 3$ and $a \in V$. Then*

$$\Delta_V \le C(d)\{(m+1)^{d(s-1)}L_s^* + (m+1)^d d_s^{1/s}(\ln(|V|+1))^{(d-1)/2}\}$$

for $m+1 \le |V|^{1/d}/(C\ln(|V|+1))$, $C > 1$.

Theorem 14 (Sunklodas (1986)). *Suppose that* (91) *is a s.m. random field with $M(n,m) \le B(n+m)^p$, $\alpha(\tau) \le Ke^{-\lambda\tau}$ and that $\mathbf{E}|X_a|^s < \infty$, $2 < s \le 3$, with $a \in V$. If $0 \le p < \infty$, then*

$$\begin{aligned}\Delta_V \le\ & C(B,K,\lambda,d,p,s)\{L_s^*(\ln(|V|+1))^{d(s-1)} \\ & + |V|^{1/2} d_s^{2/s}(\ln(|V|+1))^{1+(dp(s-2)/(2s))}\}.\end{aligned}$$

Theorem 15 (Sunklodas (1986)). *Suppose that* (91) *is an m-dependent random field and $\mathbf{E}|X_a|^s < \infty$, $2 < s \le 3$, $a \in V$. Then for $m \ge 0$*

$$||\Delta_V(x)||_1 \le C(d)(m+1)^{d(s-1)}L_s.$$

The author has also found more precise estimates for $||\Delta_V(x)||_1$ when the terms have finite second moments.

It should be noted that for weakly dependent random fields (just as for a sequence of r.v.'s), Δ_V can be estimated in terms of L_s^* and d_s and $||\Delta_V(x)||_1$ can be estimated in terms of Lyapunov's quotient L_s.

Close results to Theorems 13 and 14 were obtained independently by Bulinskii (1986a), (1987). His mixing conditions are more general since they take into account a "geometric" aspect of selecting the sets used to define the mixing coefficient (see also Bulinskii (1989)).

Herrndorf (1983b) constructed an example of a s.m. strictly stationary sequence of r.v.'s whose s.m. coefficient decrease arbitrarily fast and whose partial sums have a variance increasing regularly. However, the CLT is not obeyed if just the second moments of the terms exist. It is therefore reasonable to estimate the rate of convergence in the CLT for weakly dependent random fields by imposing moment restrictions such as $\sup_{a\in Z^d} \mathbf{E}\mathcal{G}(|X_a|) < \infty$ on the terms, where $\mathcal{G}$ satisfies the condition $\lim_{x\to\infty} x^{-2}\mathcal{G}(x) = \infty$. Bulinskii and Doukhan (1990) obtained an estimate for the rate of convergence in the CLT for $\tilde{\alpha}$-mixing random fields (see (1990)) assuming the finiteness of the moments of the terms of "small" order (for example, of the type $\mathbf{E}X^2\ln_+^\delta(x) < \infty$). These estimates were derived by means of truncation as applied to Bulinskii's results (1986a). The author (1990) found estimates for $d_i^{(p)}$, $i = 1, 2$, for various types of mixing random fields. The problem is reduced to estimating an absolute moment of Z_V whose order depends on p and the type of mixing.

Here we state just one estimate which generalizes Theorem 3 to random fields.

Theorem 16 (Sunklodas (1990)). *Suppose that the function $h : R \to R$ satisfies the condition $H_1^{(p)} < \infty$ or $H_2^{(p)} < \infty$* (see §2, 2.1) *and that* (91) *is an m-dependent random field with $\mathbf{E}|X_a|^{2+p+\alpha} < \infty$ and $a \in V$. If $(6m+1)^d < |V|$, then*

$$\begin{aligned}|\mathbf{E}h(Z_n) - \mathbf{E}h(\mathcal{N})| \le\ & C(d,p,\alpha)H_i^{(p)}\{(m+1)^{d(1+\alpha)}L_{2+\alpha}(1+\mathbf{E}|Z_V|^p) \\ & + (m+1)^{d(1+p+\alpha)}L_{2+p+\alpha}\}.\end{aligned}$$

The boundedness of h is not required in Theorem 16.

Zuev (1989) found an estimate for Δ_V for $m(d)$-dependent random fields (for an exact definition, see that paper).

Shergin (1988), (1990) found estimates for Δ_V, $\Delta_V(x)$ and $\int_{-\infty}^{\infty}|x|^l|\Delta_V(x)|dx$ for finitely dependent r.v.'s (see Chen (1978)).

Mukhamedov (1987) obtained estimates for $\int_{-\infty}^{\infty}|x|^l|\Delta_V(x)|dx$ using Stein's technique for s.m. and u.s.m. random fields.

For weakly dependent stationary r.v.'s and random fields with a very slowly increasing mixing coefficient, Nakhapetyan (1987), (1989) applied his small blocking method to find estimates of the rate of convergence in the CLT that are sufficient for the law of the iterated logarithm.

Tikhomirov (1983) found a nonuniform estimate for $|\Delta_V(x)|$ for strongly mixing strictly stationary random fields defined on the integer lattice Z_+^d and taking values in a finite-dimensional Euclidean space R^k. Tikhomirov (1983) extended his method to the estimation of the rate of convergence in the CLT for strictly stationary Hilbert-valued r.v.'s.

Asymptotic expansions for weakly dependent r.v.'s and random fields may be found in the papers by Heinrich (1985a), (1986), (1990a), Rhee (1985) and Götze and Hipp (1983), (1989). Stein's method was investigated by Barbour (1990) in the context of functional approximation of Wiener and other Gaussian processes.

References*

Aminev, F.A., and Dubrovin, V.T. (1984): Estimation of the rate of convergence in the central limit theorem for homogeneous random fields. Issled. Prikl. Mat., Kazan Univ. Press, Kazan **10**, 3–12. English transl.: J. Sov. Math. **44**, No. 5, 559–567 (1989). Zbl. 594.60027

Babu, G.J. (1980): An inequality for moments of sums of truncated φ-mixing random variables and its applications. Sankhya, Ser. A **42**, No. 1–2, 1–8. Zbl. 486.60030

Babu, G.J., Ghosh, H., and Singh, K. (1978): On rates of convergence to normality for φ-mixing processes. Sankhya, Ser. A **40**, No. 3, 278–293. Zbl. 414.60025

Bakirov, N.K. (1987): The central limit theorem for weakly dependent variables. Mat. Zametki **41**, No. 1, 104–109. English transl.: Math. Notes. **41**, 63–67 (1987). Zbl. 645.60026

Barbour, A.D. (1986): Asymptotic expansions based on smooth functions in the central limit theorem. Probab. Theor. Relat. Fields **72**, No. 2, 289–303. Zbl. 572.60029

Barbour, A.D. (1990): Stein's method for diffusion approximation. Probab. Theor. Relat. Fields **84**, No. 3, 297–322. Zbl. 685.60009

Barbour, A.D., and Eagleson, G.K. (1985): Multiple comparisons and sums of dissociated random variables. Adv. Appl. Probab. **17**, No. 1, 147–162. Zbl. 559.60027

Basalykas, A. (1986): Estimation of the rate of convergence of some estimators of weakly dependent observations. Liet. Mat. Rink. **26**, No. 4, 607–615 (Russian). Zbl. 647.62028

Bergström, H. (1970): A comparison method for distribution functions of sums of independent and dependent random variables. Teoriya Veroyat. Primen. **15**, No. 3, 442–468. Also: Theory Probab. Appl. **15**, 430–457. Zbl. 209.20005

* For the convenience of the reader, references to reviews in Zentralblatt für Mathematik (Zbl.), compiled by means of the MATH database, and Jahrbuch über die Fortschritte der Mathematik (Jbuch) have, as far as possible, been included in this bibliography.

Bergström, H. (1971): Reduction of the limit problem for sums of random variables under a mixing condition. Proc. 4th Conf. Probab. Theory, Romania, pp. 107–120. Zbl. 305.60006

Bergström, H. (1972): On the convergence of sums of random variables in distribution under a mixing condition. Period. Math. Hung. **2**, No. 1–4, 173–190. Zbl. 252.60009

Berk, K.N. (1973): A central limit theorem for m-dependent random variables with unbounded m. Ann. Probab. **1**, No. 2, 352–354. Zbl. 263.60006

Bernstein, S.N. (1926): Sur l'extension du théorème limite du calcul des probabilités aux sommes de quantités dépendantes. Math. Ann. **97**, 1–59. Jbuch 52.0517.03

Bernstein, S.N. (1964): Collected Works, Vol. 4. Nauka, Moscow (Russian). Zbl. 198.50902

Bhattacharya, R.N., and Rao R. Ranga (1976): Normal Approximation and Asymptotic Expansions. John Wiley, New York. Zbl. 331.41023

Billingsley, P. (1968): Convergence of Probability Measures. John Wiley, New York. Zbl. 172.21201

Blum, J.R., Hanson, D.L., and Koopmans, L.H. (1963): On the strong law of large numbers for a class of stochastic processes. Z. Wahrscheinlichkeitstheorie Verw. Geb. **2**, No. 1, 1–11. Zbl. 117.35603

Bolthausen, E. (1982): On the central limit theorem for stationary mixing random fields. Ann. Probab. **10**, No. 4, 1047–1050. Zbl. 496.60020

Bolthausen, E. (1984): An estimate of the remainder in a combinatorial central limit theorem. Z. Wahrscheinlichkeitstheorie Verw. Geb. **66**, No. 3, 379–386. Zbl. 563.60026

Bradley, R.C. (1980a): A remark on the central limit question for dependent random variables. J. Appl. Probab. **17**, No. 1, 94–101. Zbl. 424.60025

Bradley, R.C. (1980b): On the φ-mixing condition for stationary random sequences. Duke Math. J. **47**, No. 2, 421–433. Zbl. 442.60035

Bradley, R.C. (1981): Central limit theorems under weak dependence. J. Multivariate Anal. **11**, No. 1, 1–16. Zbl. 453.60028

Bradley, R.C. (1983): Equivalent measures of dependence. J. Multivariate Anal. **13**, No. 1, 167–176. Zbl. 508.60004

Bradley, R.C. (1986): Basic properties of strong mixing conditions. Dependence in Probab. and Statistics. (Eds. E. Eberlein and M.S. Taqqu). Progress in Probability and Statistics, Birkhäuser, Boston **2**, 165–192. Zbl. 603.60034

Bradley, R.C. (1988): A central limit theorem for stationary ρ-mixing sequences with infinite variance. Ann. Probab. **16**, No. 1, 313–332. Zbl. 643.60018

Bradley, R.C. (1988): A caution on mixing conditions for random fields. Statist. Probab. Lett. **8**, No. 5, 489–491. Zbl. 697.60054

Bradley, R.C. (1990): On ρ-mixing except on small sets. Pac. J. Math. **146**, No. 2, 217–226. Zbl. 733.60059

Bradley, R.C., and Bryc, W. (1985): Multilinear forms and measures of dependence between random variables. J. Multivariate Anal. **16**, No. 3, 335–367. Zbl. 586.62086

Bradley, R.C., Bryc, W., and Janson, S. (1985): Remarks on the foundation of measures of dependence. [Center for Stoch. Proc. Univ. of North Carolina. Tech. Rep. 105.] New perspectives in theoretical and applied statistics, Selected Papers of the 3rd International Meeting of Statistics, Bilbao/Spain 1986, 421–437 (1987). Zbl. 619.60011

Bradley, R.C., Bryc, W., and Janson, S. (1987): On dominations between measures of dependence. J. Multivariate Anal. **23**, No. 2, 312–329. Zbl. 627.60009

Bradley, R.C., and Peligrad, M. (1986): Invariance principles under a two-part mixing assumption. Stoch. Proc. Appl. **22**, No. 2, 271–289. Zbl. 609.60048

Bulinskii, A.V. (1977): On the speed of convergence in the central limit theorem for additive random functions. Dokl. Akad. Nauk. SSSR **235**, No. 1, 741–744. English transl.: Sov. Math., Dokl. **18**, No. 4, 1009–1013. Zbl. 391.60025

Bulinskii, A.V. (1979): The central limit theorem for random fields with strong and weak dependence. Dokl. Akad. Nauk, SSSR **248**, No. 1, 17–19. English transl.: Sov. Math., Dokl. **20**, No. 5, 929–931. Zbl. 435.60047

Bulinskii, A.V. (1984): On measures of dependence close to the maximum correlation coefficient. Dokl. Adad. Nauk SSSR **277**, No. 6, 1296–1298. English transl.: Sov. Math., Dokl. **30**, No. 1, 249–252. Zbl. 584.62084

Bulinskii, A.V. (1985a): Asymptotic normality of mixing random fields. Dokl. Akad. Nauk. SSSR **284**, No. 5, 1044–1048. English transl.: Sov. Math., Dokl. **32**, No. 2, 523–527. Zbl. 597.60049

Bulinskii, A.V. (1985b): On mixing conditions of random fields. Teor. Veroyatn. Primen. **30**, No. 1, 200–201. English transl.: Theory Probab. Appl. **30**, No. 1, 219–220.

Bulinskii, A.V. (1986a): An estimate of the rate of convergence in the central limit theorem for random fields. Dokl. Akad. Nauk. SSSR **291**, No. 1, 22–25. English transl.: Soviet. Math. Dokl. **34**, No. 3 (1987), 416–419. Zbl. 665.60028

Bulinskii, A.V. (1986b): The central limit theorem and invariance principle for mixing random fields. First Int. Congr. of the Bernoulli Statist. Soc., Tashkent, Vol. 1, 105–107.

Bulinskii, A.V. (1987): Limit theorems under weak dependence conditions. Probab. Theory and Math. Statistic., Proceedings of the Fourth Vilnius Conf., VNU Sc. Press, Utrecht, Vol. 1, 307–326. Zbl. 648.60025

Bulinskii, A.V. (1988): On various mixing conditions and asymptotic normality of random fields. Dokl. Akad. Nauk. SSSR **229**, No. 4, 785–789. English transl.: Soviet Math. Dokl. **37**, No. 2, 443–448. Zbl. 689.60049

Bulinskii, A.V. (1989): Limit Theorems under Conditions of Weak Dependency. Moscow Univ. Press, Moscow (Russian).

Bulinskii, A.V., and Doukhan, P. (1990): Vitesse de convergence dans le théorème de limite centrale pour les champs mélangeants satisfaisant des hypothèses de moments faibles, C.R. Acad. Sci., Paris, Sér I **311**, No. 12, 801–805. Zbl. 719.60020

Bulinskii, A.V., and Zhurbenko, I.G. (1976a): The central limit theorem for random fields. Dokl. Akad. Nauk. SSSR **226**, No. 1, 23–25. English transl.: Sov. Math., Dokl. **17**, No. 1, 14–17.

Bulinskii, A.V., and Zhurbenko, I.G. (1976b): The central limit theorem for additive functions. Teoriya Veroyat. Primen. **21**, No. 4, 707–717. English transl.: Theory Probab. Appl. **21**, No. 4, 687–697. Zbl. 382.60025

Chen, L.H.Y. (1972): An elementary proof of the central limit theorem. Bull. Singapore Math. Soc. 1972, 1–12.

Chen, L.H.Y. (1978): Two central limit problems for dependent random variables. Z. Wahrscheinlichkeitstheorie Verw. Geb. **43**, No. 3, 223–243. Zbl. 372.60029

Chen, L.H.Y. (1979): Stein's method in limit theorems for dependent variables. SEA Bull. Math. (Special Issue), 36–50. Zbl. 419.60019

Cohn, H. (1965): On a class of dependent random variables. Rev. Roumaine Math. Pures Appl. **10**, No. 10, 1593–1606. Zbl. 203.19403

Dasgupta, R. (1988): Nonuniform rates of convergence to normality for strong mixing processes. Sankhya, Ser. A, **50**, No. 3, 436–451. Zbl. 679.60052

Davis, R. (1983): Stable limits for partial sums of dependent random variables. Ann. Probab. **11**, No. 2, 262–269. Zbl. 511.60021

Davydov, Yu.A. (1968): Convergence of distributions generated by stationary stochastic processes. Teor. Veroyatn. Primen. **13**, No. 4, 730–737. English transl.: Theory Probab. Appl. **13**, No. 4, 691–696. Zbl. 174.49201

Davydov, Yu.A. (1970): The invariance principle for stationary processes. Teor. Veroyatn. Primen. **15**, No. 3, 498–509. English transl.: Theory Probab. Appl. **15**, No. 3, 487–498. Zbl. 209.48904

Davydov, Yu.A. (1973): Mixing conditions for Markov chains. Teor. Veroyatn. Primen. **18**, No. 2, 321–338. English transl.: Theory Probab. Appl. **18**, No. 2, 312–328. Zbl. 297.60031

Dehling, H., Denker, M., and Philipp, W. (1986): Central limit theorem for mixing sequences of random variables under minimal conditions. Ann. Probab. **14**, No. 4, 1359–1370. Zbl. 605.60027

Denker, M. (1986): Uniform integrability and the central limit theorem. Dependence in Probab. and Statist (Eds. E. Eberlein and M.S. Taqqu). Birkhäuser, Boston **2**, 269–274. Zbl. 612.60028

Denker, M., and Jakubowski, A. (1989): Stable limit distributions for strongly mixing sequences. Statist. Probab. Lett. **8**, No. 5, 477–483. Zbl. 694.60017

Deo, C.M. (1973): A note on empirical processes of strong mixing conditions. Ann. Probab. **1**, No. 5, 870–875. Zbl. 281.60034

Deo, C.M., and Wong, H.S.-F. (1980): On Berry–Esseen approximation and a functional LIL for a class of dependent random fields. Pac. J. Math. **91**, No. 2, 269–275. Zbl. 476.60030

Diananda, P.H. (1955): The central limit theorem for m-dependent variables. Proc. Cambridge Philos. Soc. **51**, No. 1, 92–95. Zbl. 064.13104

Dobrushin, R.L. (1956a): Central limit theorem for nonstationary Markov chains, I. Teor. Veroyatn. Primen. **1**, No. 1, 72–89. English transl.: Theory Probab. Appl. **1**, No. 1, 65–80. Zbl. 093.15001

Dobrushin, R.L. (1956b): Central limit theorem for nonstationary Markov chains, II. Teor. Veroyatn. Primen. **1**, No. 4, 365–425. English transl.: Theory Probab. Appl. **1**, No. 4, 329–383. Zbl. 093.15001

Dubrovin, V.T. (1971): A central limit theorem for sums of functions of weakly dependent random variables. Probability Methods and Cybernetics, Kazan Univ. Press, No. 9, 21–33 (Russian). Zbl. 276.60025

Dubrovin, V.T. (1974): A multidimensional central limit theorem for number-theoretic endomorphisms. Probability Methods and Cybernetics, Kazan Univ. Press **10**, 17–29 (Russian). Zbl. 321.60017

Dubrovin, V.T., and Moskvin, D.A. (1979): The central limit theorem for sums of functions of mixing sequences. Teor. Veroyatn. Primen. **24**, No. 3, 553–564. English transl.: Theory Probab. Appl. **24**, No. 3, 560–571. Zbl. 408.60026

Dvoretzky, A. (1972): Asymptotic normality for sums of dependent random variables. Proc. Sixth Berkeley Symp. Math. Statist. Probab. **2**, 513–535. Zbl. 256.60009

Eberlein, E. (1979): An invariance principle for lattices of dependent random variables. Z. Wahrscheinlichkeitstheorie Verw. Geb. **50**, No. 2, 119–133. Zbl. 414.60036

Eberlein, E. (1984): Weak convergence of partial sums of absolutely regular sequences. Statist. Probab. Lett. **2**, No. 5, 291–293. Zbl. 564.60025

Egorov, V.A. (1970): Some limit theorems for m-dependent random variables. Liet. Mat. Rink. **10**, No. 1, 51–59 (Russian). Zbl. 213.20201

Erickson, R.V. (1973): On an L_p version of Berry–Esseen theorem for independent and m-dependent random variables. Ann. Probab. **1**, No. 3, 497–503. Zbl. 292.60040

Erickson, R.V. (1974): L_1 bounds for asymptotic normality of m-dependent sums using Stein's technique. Ann. Probab. **2**, No. 3, 522–529.

Erickson, R.V. (1975): Truncation of dependent random variables. Teor. Veroyatn. Primen. **20**, No. 4, 892–900. Also: Theory Probab. Appl. **20**, No. 4, 873–880. Zbl. 362.60005

Gabbasov, F.G. (1977): A multidimensional central limit theorem for sums of functions of mixing sequences. Liet. Mat. Rink. **17**, No. 4, 83–98. English transl.: Lith. Math. J. **17**, No. 4, 494–505. Zbl. 373.60021

Gebelein, H. (1941): Das statistische Problem der Korrelation als Variations- und Eigenwertproblem und sein Zusammenhang mit der Ausgleichungsrechnung. Z. Angew. Math. Mech. **21**, 364–379. Zbl. 026.33402

Gnedenko, B.V., and Kolmogorov, A.N. (1949): Limit Distributions for Sums of Independent Random Variables. Gostekhizdat, Moscow. English transl.: Addison-Wesley, Reading (MA), 1954. Zbl. 056.36001

Goldie, C.M., and Greenwood, P.E. (1986a): Variance of set-indexed sums of mixing random variables under weak convergence of set-indexed processes. Ann. Probab. **14**, No. 3, 817–839. Zbl. 604.60032

Goldie, C.M., and Greenwood, P.E. (1986b): Central limit results for random fields. Proc. First Int. Congress Bernoulli Soc., Tashkent, 345–352. Zbl. 677.60027

Goldie, C.M., and Morrow, G.J. (1986): Central limit questions for random fields. Dependence in Probab. and Statist., Birkhäuser, Boston **2**, 275–289. Zbl. 605.60029

Gordin, M.I. (1969): The central limit theorem for stationary processes. Dokl. Akad. Nauk. SSSR **188**, No. 4, 739–741. English transl.: Sov. Math., Dokl. **10**, No. 4, 1174–1176. Zbl. 212.50005

Gorodetskii, V.V. (1982): The invariance principle for strongly mixing stationary random fields. Teor. Veroyat. Primen. **27**, No. 2, 358–364. English transl.: Theory Probab. Appl. **27**, No. 2, 380–385.

Gorodetskii, V.V. (1984): The central limit theorem and an invariance principle for weakly dependent random fields. Dokl. Akad. Nauk. SSSR **276**, No. 3, 528–531. English transl.: Sov. Math., Dokl. **29**, No. 3, 529–532. Zbl. 595.60025

Götze, F., and Hipp, C. (1983): Asymptotic expansions for sums of weakly dependent random vectors. Z. Wahrscheinlichkeitstheorie Verw. Geb. **64**, No. 2, 211–239. Zbl. 514.60027

Götze, F., and Hipp, C. (1989): Asymptotic expansions for potential functions of i.i.d. random fields. Probab. Theory Relat. Fields **82**, 3, 349–370. Zbl. 687.60022

Grin', A.G. (1982): On a condition for regularity of stationary sequences. Teor. Veroyatn. Primen. **27**, No. 4, 789–795. English transl.: Theory Probab. Appl. **27**, No. 4, 850–855. Zbl. 504.60043

Grin', A.G. (1990a): Domains of attraction of mixing sequences. Sib. Mat. Zh. **31**, 1. 53–65. English transl.: Sib. Math. J. **31**, No. 1, 43–52. Zbl. 714.60017

Grin', A.G. (1990b): On Domains of attraction for sums of dependent variables. Teor. Veroyatn. Primen. **35**, No. 2, 255–270. English transl.: Theory Probab. Appl. **35**, No. 2, 241–257. Zbl. 273.60024

Gudynas, P. (1977): The invariance principle for nonhomogeneous Markov chains. Liet. Mat. Rink. **17**, No. 2, 63–73. English transl.: Lith. Math. J. **17**, No. 4, 490–505. Zbl. 398.60033

Gudynas, P. (1983): On approximating distributions of sums of dependent Banach-valued random variables. Liet. Mat. Rink. **23**, No. 3, 3–21. English transl.: Lith. Math. J. **23**, No. 3, 251–263. Zbl. 539.60013

Gudynas, P. (1989): A generalization of an approximation inequality. Liet. Mat. Rink. **29**, No. 1, 27–34. English transl.: Lith. Math. J. **29**, No. 1, 17–22. Zbl. 682.60002

Guyon, X., and Richardson, S. (1984): Vitesse de convergence du théorème de la limite centrale pour des champs faiblement dépendants. Z. Wahrscheinlichkeitstheorie Verw. Geb. **66**, No. 2, 297–314. Zbl. 544.60031

Hall, P., and Heyde, C.C. (1980): Martingale Limit Theory and Its Application. Academic Press, New York. Zbl. 462.60045

Hegerfeldt, G.C., and Nappi, C. (1977): Mixing properties in lattice systems. Commun. Math. Phys. **53**, No. 1, 1–7. Zbl. 349.60105

Heinrich, L. (1982): A method of the derivation of limit theorems for sums of m-dependent random variables. Z. Wahrscheinlichkeitstheorie Verw. Geb. **60**, No. 4, 501–515. Zbl. 492.60028

Heinrich, L. (1984): Non-uniform estimates and asymptotic expansions of the remainder in the central limit theorem for m-dependent random variables. Math. Nachr. **115**, 7–20. Zbl. 558.60026

Heinrich, L. (1985a): Some remarks on asymptotic expansions in the central limit theorem for m-dependent random variables. Math. Nachr. **122**, 151–155. Zbl. 576.60019

Heinrich, L. (1985b): Stable limits for sums of m-dependent random variables. Serdica Bulgaricae Math. I Publ. **11**, No. 2, 189–199. Zbl. 582.60031

Heinrich, L. (1985c): Some estimates of the cumulant-generating function of a sum of m-dependent random vectors and their application to large deviations. Math. Nachr. **120**, 91–101. Zbl. 375.60028

Heinrich, L. (1985d): Non-uniform estimates, moderate and large deviation, in the central limit theorem for m-dependent random variables. Math. Nachr. **121**, 107–121. Zbl. 572.60031

Heinrich, L. (1986): Stable limit theorems for sums of multiply indexed m-dependent random variables. Math. Nachr. **127**, 193–210. Zbl. 609.60033

Heinrich, L. (1987): Asymptotic expansions in the central limit theorem for a special class of m-dependent random fields, I. Math. Nachr. **134**, 83–106. Zbl. 668.60026

Heinrich, L. (1990a): Asymptotic expansions in the central limit theorem for a special class of m-dependent random fields, II. Math. Nachr. **145**, 309–327. Zbl. 705.60025

Heinrich, L. (1990b): Non-uniform bounds for the error in the central limit theorem for random fields generated by functions of independent random variables. Math. Nachr. **145**, 345–364. Zbl. 705.60026

Heinrich, L., and Richter, W.-D. (1984): On moderate deviations of sums of m-dependent random vectors. Math. Nachr. **118**, 253–264. Zbl. 558.60027

Herrndorf, N. (1983a): The invariance principle for φ-mixing sequences. Z. Wahrscheinlichkeitstheorie Verw. Geb. **63**, No. 1, 97–108. Zbl. 506.60029

Herrndorf, N. (1983b): Stationary strongly mixing sequences not satisfying the central limit theorem. Ann. Probab. **11**, No. 3, 809–813. Zbl. 513.60033

Herrndorf, N. (1985): A functional central limit theorem for strongly mixing sequences of random variables. Z. Wahrscheinlichkeitstheorie Verw. Geb. **69**, No. 4, 541–550. Zbl. 558.60032

Hipp, C. (1979a): Convergence rates of the strong law for stationary mixing sequences. Z. Wahrscheinlichkeitstheorie Verw. Geb. **49**, No. 1, 49–62. Zbl. 398.60029

Hipp, C. (1979b): Convergence rates in the central limit theorem for stationary mixing sequences of random vectors. J. Multiv. Anal. **9**, 560–578. Zbl. 431.60021

Ho, S.-T., and Chen, L.H.Y. (1978): An L_p bound for the remainder in a combinatorial central limit theorem. Ann. Probab. **6**, No. 2, 231–249. Zbl. 375.60028

Hoeffding, W., and Robbins, H. (1948): The central limit theorem for dependent random variables. Duke Math. J. **15**, 773–780. Zbl. 031.36701

Ibragimov, I.A. (1962): Some limit theorems for stationary processes. Teor. Veroyatn. Primen. **7**, No. 4, 361–392. English transl.: Theory Probab. Appl. **7**, No. 4, 349–382. Zbl. 119.14204

Ibragimov, I.A. (1967): The central limit theorem for sums of functions of dependent variables and sums of the form $\sum f(2^k t)$. Teor. Veroyatn. Primen. **12**, No. 4, 655–665. English transl.: Theory Probab. Appl. **12**, No. 4, 596–607. Zbl. 217.49803

Ibragimov, I.A. (1975): A note on the central limit theorem for dependent random variables. Teor. Veroyatn. Primen. **20**, No. 1, 134–140. English transl.: Theory Probab. Appl. **20**, No. 2, 135–141. Zbl. 335.60023

Ibragimov, I.A., and Linnik, Yu.V. (1965): Independent and Stationary Sequences of Random Variables. Nauka, Moscow. English transl.: Wolters-Noordhoff, Groningen, 1971. Zbl. 219.60027

Ibragimov, I.A., and Rozanov, Yu.A. (1970): Gaussian Random Processes. Nauka, Moscow. English transl.: Springer-Verlag, New York (1978). Zbl. 392.60037

Iosifescu, M. (1968): La loi logarithme itéré pour une classe de variables aléatoires dépendantes. Teor. Veroyat. Primen. **13**, No. 2, 315–325. English transl.: Theory Probab. Appl. **13**, No. 2, 304–313. Zbl. 159.47302

Iosifescu, M. (1974): Limit theorems for φ-mixing sequences: a survey. Proc. Fifth Conf. on Probab. Theory, Sept. 1–6, 1974, Brasov, Romania. Editura Academiei Republicii Socialiste Romania, Bucharest, 51–57. Zbl. 376.60025

Iosifescu, M. (1980): Recent advances in mixing sequences of random variables. Third Int'l Summer School on Probab. Theory and Math. Statist., Varna, 1978. Bulgarian Academy of Sciences Publishing House, Sofia, 111–138. Zbl. 435.60024

Iosifescu, M., and Teodorescu, R. (1969): Random Processes and Learning. Springer-Verlag, Berlin. Zbl. 194.51101

Jakimavičius, D., and Statulevičius, V.A. (1987): Estimates of Cumulants and Centered Moments of Mixing Random Processes. Preprint No. 3, Acad. Sci. Lith. SSR, Inst. Math. and Cyb., Vilnius (Russian). Zbl. 661.60024

Janson, S. (1988): Normal convergence of higher semi-invariants with applications to sums of dependent random variables and random graphs. Ann. Probab. **16**, No. 1, 305–312. Zbl. 639.60029

Kallianpur, G. (1955): On a limit theorem for dependent random variables. Dokl. Akad. Nauk SSSR **101**, No. 1, 13–16 (Russian). Zbl. 067.10702

Kanagawa, S. (1981): Rates of convergence of the invariance principle for weakly dependent random variables. Keio Math. Seminar Rep. **6**, 23–25. Zbl. 464.60032

Kanagawa, S. (1982): On the rate of convergence of the invariance principle for stationary sequences. Keio Sci. Techn. Reports **35**, No. 3, 53–61. Zbl. 503.60045

Kesten, H., and O'Brien, G.L. (1976): Examples of mixing sequences. Duke Math. J. **43**, No. 2, 405–415. Zbl. 337.60035

Kolmogorov, A.N., and Rozanov, Yu.A. (1960): On strong mixing conditions for stationary Gaussian processes. Teor. Veroyatn. Primen. **5**, No. 2, 222–227. English transl.: Theory Probab. Appl. **5**, No. 2, 204–208. Zbl. 091.30001

Kolmogorov, A.N., and Sarmanov, O.V. (1960): S.N. Bernstein's work in probability theory. Teor. Veroyatn. Primen. **5**, No. 2, 215–221. English transl.: Theory Probab. Appl. **5**, No. 2, 197–203. Zbl. 087.33002

Kornfel'd, I.P., Sinai, Ya.G., and Fomin, S.V. (1980): Ergodic Theory. Nauka, Moscow. English transl.: Springer-Verlag, New York, 1982. Zbl. 493.28007

Krieger, H.A. (1984): A new look of Bergström's theorem on convergence in distribution for sums of dependent random variables. Israel J. Math. **47**, 32–64. Zbl. 536.60032

Lapinskas, R. (1976): On the rate of convergence for sums of infinite-dimensional random variables connected in a Markov chain. Liet. Mat. Rink. **16**, No. 4, 125–132. English transl.: Lith. Math. J. **16**, No. 4, 559–664. Zbl. 366.60093

Lapinskas, R. (1980): Limit theorems for weakly dependent random variables. Liet. Mat. Rink. **20**, No. 3, 91–97. English transl.: Lith. Math. J. **20**, No. 3, 244–249. Zbl. 468.60001

Lappo, P.M. (1986): Rate of convergence in the central limit theorem for a mixing sequence. Mat. Zametki **39**, No. 2, 295–299. English transl.: Math. Notes **39**, 160–162. Zbl. 619.60027

Leonenko, N.N. (1975): On estimating the rate of convergence in the central limit theorem for m-dependent random fields. Mat. Zametki **17**, No. 1, 129–132. English transl.: Math. Notes **17**, 76–78. Zbl. 347.60021

Leonenko, N.N. (1976): Limit theorems for additive random functions. Studies in Theory of Stochastic Processes. Math. Inst. Akad. Nauk Ukraine SSR, 94–105 (Russian).

Leonenko, N.N., Ivanov, A.V. (1986): Statistical Analysis of Random Fields. Vishcha Shkola, Kiev (Russian). English transl.: Kluwer, Dordrecht, 1989. Zbl. 713.62094

Leonenko, N.N., and Yadrenko, M.I. (1979): On the invariance principle for homogeneous random fields. Teor. Veroyatn. Primen. **24**, No. 1, 175–181. English transl.: Theory Probab. Appl. **24**, No. 1, 175–181. Zbl. 402.60049

Lifshits, B.A. (1978): On the central limit theorem for Markov chains. Teor. Veroyatn. Primen. **23**, No. 2, 295–312. English transl.: Theory Probab. Appl. **23**, No. 2, 279–296. Zbl. 389.60047

Lifshits, B.A. (1984): The invariance principle for weakly dependent variables. Teor. Veroyatn. Primen. **29**, No. 1, 33–40. English transl.: Theory Probab. Appl. **29**, No. 1, 33–40. Zbl. 535.60026

Lifshits, M.A. (1985): Partitioning of multidimensional sets. Rings and Modules. Limit Theorems of Probability Theory. Leningrad Univ. Press, Leningrad. No. 1, 175–178 (Russian). Zbl. 701.00008

Liptser, R.Sh., and Shiryaev, A.N. (1986): Theory of Martingales. Nauka, Moscow. English transl.: Kluwer, Dordrecht, 1989. Zbl. 728.60048

Maejima, M. (1978): A non-uniform estimate in the central limit theorem for m-dependent random variables. Keio Sci. Techn. Rep. **31**, No. 2, 15–20. Zbl. 451.60032

Malevich, T.L. (1980): On the rate of convergence in the central limit theorem for sums of m-dependent variables. Izv. Akad. Nauk. Uzb. SSR, Ser. Fiz.-Mat. Nauk. **2**, 25–29 (Russian). Zbl. 445.60016

Malyshev, V.A. (1975): The central limit theorem for Gibbsian random fields. Dokl. Adad. Nauk. SSSR **224**, No. 1, 35–38. English transl.: Sov. Math., Dokl. **16**, No. 5, 1141–1145.

Malyshev, V.A., and Minlos, R.A. (1985): Gibbs Random Fields. Nauka, Moscow. English transl.: Kluwer, Dordrecht-New York, 1991. Zbl. 731.60099

McLeish, D.L. (1974): Dependent central limit theorems and invariance principles. Ann. Probab. **2**, No. 4, 620–628. Zbl. 287.60025

McLeish, D.L. (1975): Invariance principles for dependent variables. Z. Wahrscheinlichkeitstheorie Verw. Geb. **32**, No. 3, 165–178. Zbl. 305.60010

Mukhamedov, A.K. (1987): On a global form of the central limit theorem for weakly dependent random fields. Math. Analysis, Algebra and Probability Theory. Tashkent Univ., Tashkent, 77–81. Zbl. 738.60045

Nagaev, S.V. (1957): Some limit theorems for stationary Markov chains. Teor. Veroyatn. Primen. **2**, No. 4, 389–416. English transl.: Theory Probab. Appl. **2**, No. 4, 378–406. Zbl. 078.03804

Nagaev, S.V. (1962): The central limit theorem for time-discrete Markov processes. Izv. Akad. Nauk. Uzb. SSR, Ser. Fiz.-Mat. Nauk **2**, 12–20 (Russian). Zbl. 122.13603

Nakhapetyan, B.S. (1978): The central limit theorem for strongly mixing random fields. Multidimensional Random Systems, Nauka, Moscow, 276–288. English transl.: Advances in Probability and Related Topics **6**, 531–547 (1980) Marcel Dekker. Zbl. 442.60053

Nakhapetyan, B.S. (1987): An approach to proving limit theorems for dependent random variables. Teor. Veroyatn. Primen. **32**, No. 3, 589–594. English transl.: Theory Probab. Appl. **32**, No. 3, 535–539. Zbl. 629.60045

Nakhapetyan, B.S. (1989): Weakly Dependent Random Fields and the Summation Problem for Gibbsian Fields in Large Volumes. Doctoral Dissertation, Erevan (Russian).

Neaderhouser, C.C. (1978a): Some limit theorems for random fields. Commun. Math. Phys. **61**, No. 3, 293–305. Zbl. 393.60019

Neaderhouser, C.C. (1978b): Limit theorems for multiply-indexed mixing random variables with applications to Gibbs random fields. Ann. Probab. **6**, No. 2, 207–215. Zbl. 374.60033

Negishi, H. (1977): The rate of convergence to normality for strong mixing sequences of random variables. Sci. Repts. Yokohama National Univ., Sec. **1**, 14, 17–25. Zbl. 431.60022

Oodaira, H., and Yoshihara, K. (1972): Functional central limit theorem for strictly stationary processes satisfying the strong mixing conditions. Kodai Math. Sem. Rep. **24**, No. 3, 259–269. Zbl. 245.60006

Orey, S. (1958): A central limit theorem for m-dependent random variables. Duke Math. J. **25**, No. 4, 543–546. Zbl. 107.13403

Paulauskas, V., and Račkauskas, A. (1987): Approximation Theory in the Central Limit Theorem. Exact Results in Banach Spaces, Mokslas, Vilnius. English transl.: Kluwer, Dordrecht (1989). Zbl. 715.60023

Peligrad, M. (1982): Invariance principles for mixing sequences of random variables. Ann. Probab. **10**, No. 4, 968–981. Zbl. 503.60044

Peligrad, M. (1983): A note on two measures of dependence and mixing sequences. Adv. Appl. Probab. **15**, 461–464. Zbl. 508.60033

Peligrad, M. (1985): An invariance principle for φ-mixing sequences. Ann. Probab. **13**, No. 4, 1304–1313. Zbl. 597.60018

Peligrad, M. (1986a): Recent advances in the central limit theorem and its weak invariance principle for mixing sequences of random variables (a survey). Dependence in Probab. and Statist. Birkhäuser, Basel-Stuttgart, 193–223. Zbl. 603.60022

Peligrad, M. (1986b): Invariance principles under weak dependence. J. Multivariate Anal. **19**, No. 2, 299–310. Zbl. 603.60023

Peligrad, M. (1987): On the central limit theorem for ρ-mixing sequences of random variables. Ann. Probab. **15**, No. 4, 1387–1394. Zbl. 638.60032

Peligrad, M. (1990): On Ibragimov-Iosifescu conjecture for φ-mixing sequences. Stoch. Proc. Appl. **35**, No. 2, 293–308. Zbl. 712.60020

Petrov, V.V. (1960): On the central limit theorem for m-dependent variables. Proc. All-Union Conf. on Probab. and Statist., 1958. Akad. Nauk. Armen. SSR Publ., Erevan (Russian). 38–44. Zbl. 100.34503

Petrov, V.V. (1987): Limit Theorems for Sums of Independent Random Variables. Nauka, Moscow. English transl.: Oxford, Clarendon Press 1995. Zbl. 621.60022 (826.60001)

Petrov, V.V. (1988): Sequences of m-orthogonal random variables. Zap. Nauchn. Semin. Leningrad. Otd. Mat. Inst. **119**, No. 7, 198–202. English transl.: J. Sov. Math. **27**, 3136–3139 (1984). Zbl. 494.60032

Philipp, W. (1969a): The central limit theorem for mixing sequences of random variables. Z. Wahrscheinlichkeitstheorie Verw. Geb. **12**, No. 2, 155–171. Zbl. 174.49904

Philipp, W. (1969b): The remainder in the central limit theorem for mixing stochastic processes. Ann. Math. Statist. **40**, No. 2, 601–609. Zbl. 179.23503

Philipp, W. (1980): Weak and L^p-invariance principles for sums of B-valued random variables. Ann. Probab. **8**, No. 1, 68–82. Zbl. 426.60033

Philipp, W., and Stout, W.F. (1975): Almost sure invariance principles for partial sums of weakly dependent random variables. Mem. Am. Math. Soc. **161**. Zbl. 361.60007

Philipp, W., and Webb, G.R. (1973): An invariance principle for mixing sequences of random variables. Z. Wahrscheinlichkeitstheorie Verw. Geb. **25**, No. 3, 223–237. Zbl. 259.60007

Prokhorov, Yu.V. (1956): Convergence of random processes and limit theorems in probability theory. Teor. Veroyatn. Primen. **1**, No. 2, 177–238. English transl.: Theory Probab. Appl. **1**, No. 2, 157–214.

Prokhorov, Yu.V. (1972): Multidimensional distributions: inequalities and limit theorems. Itogi Nauki Tekhn. Ser. Teor. Veroyatn., Mat. Statist., Teor. Kibern. **10**, 5–24. English transl.: J. Sov. Math. **2**, 475–488 (1974). Zbl. 295.60013

Rao, B.L.S.P. (1975): Remark on the rate of convergence in the random central limit theorem for mixing sequences. Z. Wahrscheinlichkeitstheorie Verw. Gew. **31**, No. 2, 157–160. Zbl. 306.60011

Rao, B.L.S.P. (1981): A non-uniform estimate of the rate of convergence in the central limit theorem for m-dependent random fields. Z. Wahrscheinlichkeitstheorie Verw. Geb. **58**, No. 2, 247–256. Zbl. 465.60028

Renyi, A. (1959): On measures of dependence. Acta Math. Acad. Sci. Hung. **10**, 441–451. Zbl. 091.14403

Rhee Wan Soo (1985): An Edgeworth expansion for a sum of m-dependent random variables. Int. J. Math., Math. Sci. **8**, No. 3, 563–569. Zbl. 585.60032

Rhee Wan Soo (1986): On the characteristic function of a sum of m-dependent random variables valued in certain Banach spaces. Int. J. Math., Math. Sci. **9**, No. 2, 397–404. Zbl. 612.60026

Rhee Wan Soo, and Talagrand, M. (1981): On Berry–Esseen type bounds for m-dependent random variables valued in certain Banach spaces. Z. Wahrscheinlichkeitstheorie Verw. Geb. **58**, No. 4, 433–451. Zbl. 474.60006

Rhee Wan Soo, and Talagrand, M. (1986): Uniform bound in the central limit theorem for Banach-space-valued dependent random variables. J. Multivariate Anal. **20**, No. 2, 303–320. Zbl. 606.60010

Rio, E. (1996): Sur le theorème de Berry–Esseen pour les suites faiblement dépendantes. Probab. Theory Relat. Fields **104**, 255–282. Zbl. 838.60017

Riauba, B. (1980): On the rate of convergence in the central limit theorem for m-dependent random fields. Liet. Mat. Rink. **20**, No. 1, 157–163. English transl.: Lith. Math. J. **20**, No. 1, 71–75. Zbl. 428.60030

Riauba, B. (1988): Central limit theorem for stationary random fields. Liet. Mat. Rink. **28**, No. 4, 758–769. English transl.: Lith. Math. J. **28**, No. 4, 375–382. Zbl. 674.60055

Rosén, B. (1967): On the central limit theorem for sums of dependent random variables. Z. Wahrscheinlichkeitstheorie Verw. Geb. **7**, No. 1, 48–82. Zbl. 147.17001

Rosenblatt, M. (1956): A central limit theorem and a strong mixing condition. Proc. Nat. Acad. Sci. USA **42**, No. 1, 43–47. Zbl. 070.13804

Rosenblatt, M. (1971): Markov Processes. Structure and Asymptotic Behavior. Springer-Verlag, New York. Zbl. 236.60002

Rosenblatt, M. (1972): Central limit theorem for stationary processes. Proc. Sixth Berkeley Symp. Math. Statist. and Probab. **2**, 551–561. Zbl. 255.60002

Rosenblatt, M. (1984): Asymptotic normality, strong mixing and spectral density estimates. Ann. Probab. **12**, No. 4, 1167–1180. Zbl. 545.62058

Roussas, G.G., and Ionnides, D. (1987): Moment inequalities for mixing sequences of random variables. Stoch. Anal. Appl. **5**, No. 1, 61–120. Zbl. 619.60022

Rozanov, Yu.A. (1960): A central limit theorem for additive random functions. Teor. Veroyatn. Primen. **5**, No. 2, 243–246. English transl.: Theory Probab. Appl. **5**, No. 2, 221–223. Zbl. 091.30503

Rozanov, Yu.A. (1963): Stationary Random Processes. Fizmatgiz, Moscow. English transl.: Springer-Holden-Day, San Francisco (1967). Zbl. 152.16302

Rozanov, Yu.A. (1981): Markov Random Fields. Nauka, Moscow. English transl.: Springer-Verlag, New York, 1982. Zbl. 498.60057

Rychlik, Z., and Szyszkowski, I. (1987): The invariance principle for φ-mixing sequences. Teor. Veroyatn. Primen. **32**, No. 3, 616–619. Also: Theory Probab. Appl. **32**, No. 3, 559–562. Zbl. 626.60031

Samur, J. (1984): Convergence of sums of mixing triangular arrays of random vectors with stationary rows. Ann. Probab. **12**, No. 2, 390–426. Zbl. 542.60012

Sarmanov, I.O. (1964): On Lyapunov's theorem for sums of weakly dependent random variables. Izv. Vysh. Uch. Zaved. Mat., No. 3, 123–130 (Russian). Zbl. 123.36201

Sarmanov, O.V., and Zakharov, V.K. (1960): Measures of dependence between random variables and the spectra of stochastic matrices. Mat. Sb. **52**, No. 4, 953–990 (Russian). Zbl. 143.19903

Saulis, L., and Statulevičius, V. (1989): Limit Theorems on Large Deviations. Mokslas, Vilnius. English transl.: Kluwer Academic, Dordrecht-Boston, 1991. Zbl. 744.60028

Schneider, E. (1981): On the speed of convergence in the random central limit theorem for φ-mixing processes. Z. Wahrscheinlichkeitstheorie Verw. Geb. **58**, No. 1, 125–138. Zbl. 465.60027

Serfling, R.J. (1968): Contributions to central limit theorem for dependent variables. Ann. Math. Statist. **39**, No. 4, 1158–1175. Zbl. 176.48004

Serfling, R.J. (1980): Approximation Theorems of Mathematical Statistics. John Wiley, New York. Zbl. 538.62002

Shergin, V. (1976): Estimation of the remainder in the central limit theorem for m-dependent random variables. Liet. Mat. Rink. **16**, No. 4, 245–250. English transl.: Lith. Math. J. **16**, No. 4, 637–641. Zbl. 371.60028

Shergin, V. (1979): On the rate of convergence in the central limit theorem for m-dependent random variables. Teor. Veroyatn. Primen. **24**, No. 4, 781–794. English transl.: Theory Probab. Appl. **24**, No. 4, 782–796. Zbl. 437.60018

Shergin, V. (1983): On a global form of the central limit theorem for m-dependent random variables. Teor. Veroyatn. Mat. Statist. **29**, 122–128. English transl.: Theory Probab. Appl. Statist. **29** (1984), 149–156. Zbl. 577.60024

Shergin, V. (1988): On the central limit theorem for finitely-dependent random variables. Teor. Sluchajnykh Protsessov **16**, 93–97. English transl.: J. Sov. Math. **67**, No. 4, 3244–3248 (1993). Zbl. 793.60029

Shergin, V. (1990): The central limit theorem for finitely dependent random variables. Probab. Theory and Math. Statist. Proc. of the Fifth Vilnius Conf, 1989. VSP BV, Utrecht; Mokslas, Vilnius. 424–431. Zbl. 732.60028

Shiryaev, A.N. (1960): Some problems in the spectral theory of higher-order moments. Teor. Veroyatn. Primen. **5**, No. 3, 193–213. English transl.: Theory Probab. Appl. **5**, No. 3, 265–284. Zbl. 109.36001

Sirazhdinov, S.Kh. (1955): Limit Theorems for Homogeneous Markov Chains. Izd-vo Akad. Nauk Uzbek. SSSR, Tashkent (Russian).

Sirazhdinov, S.Kh., and Formanov, Sh.K. (1979): Limit Theorems for Sums of Random Vectors in a Markov Chain. Fan, Tashkent (Russian). Zbl. 482.60019

Statulevičius, V. (1961): Local limit theorems and asymptotic expansions for nonhomogeneous Markov chains. Liet. Mat. Rink. **1**, Nos 1–2, 231–314 (Russian). Zbl. 126.14401

Statulevičius, V. (1962): On refinements of limit theorems for weakly dependent random variables. Proc. Sixth All-Union Conf. on Probab. Theory and Math. Statist., Vilnius, 113–119 (Russian). Zbl. 128.38202

Statulevičius, V. (1969a): Limit theorems for sums of random variables related in a Markov chain, I. Liet. Mat. Rink. **9**, No. 2, 345–362 (Russian). Zbl. 203.50206

Statulevičius, V. (1969b): Limit theorems for sums of random variables related in a Markov chain, II. Liet. Mat. Rink. **9**, No. 3, 635–672 (Russian). Zbl. 203.50206

Statulevičius, V. (1970a): Limit theorems for sums of random variables related in a Markov chain, III. Liet. Mat. Rink. **10**, No. 1, 161–169 (Russian). Zbl. 203.50206

Statulevičius, V. (1970b): On limit theorems for random functions, I. Liet. Mat. Rink. **10**, No. 3, 582–592 (Russian). Zbl. 266.60014

Statulevičius, V. (1974): Limit theorems for dependent random variables under various regularity conditions. Proc. Int. Congr. Math. Vancouver **2**, 173–181. Zbl. 366.60027

Statulevičius, V. (1977a): Application of semi-invariants to asymptotic analysis of distributions of random processes. J. Multivar. Anal. **4**, 325–337. Zbl. 445.60019

Statulevičius, V. (1977b): On limit theorems for dependent random variables. Abstracts of Commun. in Second Vilnius Conf. on Probab. Theory and Math. Statist., Vilnius **3**, 212–215.

Statulevičius, V. (1983): On conditions for almost Markov regularity. Teor. Veroyatn. Primen. **28**, No. 2, 358–361. English transl.: Theory Probab. Appl. **28**, No. 2, 379–383. Zbl. 513.60064

Stein, C. (1972): A bound for the error in the normal approximation to the distribution of a sum of dependent random variables. Proc. Sixth Berkeley Symp. Math. Statist. and Probab. **2**, 583–602. Zbl. 278.60026

Stein, C. (1981): Estimation of the mean of a multivariate normal distribution. Ann. Statist. **9**, No. 6, 1135–1151. Zbl. 476.62035

Sunklodas, J. (1977): Estimation of the rate of convergence in the central limit theorem for weakly dependent random variables. Liet. Mat. Rink. **17**, No. 3, 41–51. English transl.: Lith. Math. J. **17**, No. 3, 313–320. Zbl. 404.60028

Sunklodas, J. (1978): Estimation of the rate of convergence in the central limit theorem for m-dependent random vectors. Liet. Mat. Rink. **18**, No. 4, 175–186. English transl.: Lith. Math. J. **18**, No. 4, 566–574. Zbl. 388.60027

Sunklodas, J. (1982): Distance in the L_1 metric between the distribution of a sum of weakly dependent random variables and the normal distribution function. Liet. Mat. Rink. **22**, No. 2, 171–188. English transl.: Lith. Math. J. **22**, No. 2, 177–189. Zbl. 495.60039

Sunklodas, J. (1984): On the rate of convergence in the central limit theorem for strongly mixing random variables. Liet. Mat. Rink. **24**, No. 2, 174–185. English transl.: Lith. Math. J. **24**, No. 2, 182–190. Zbl. 558.60023

Sunklodas, J. (1986): Estimation of the rate of convergence in the central limit theorem for weakly dependent random fields. Liet. Mat. Rink. **26**, No. 3, 541–559. English transl.: Lith. Math. J. **26**, No. 3, 272–287. Zbl. 628.60032

Sunklodas, J. (1989): Estimation of the bounded Lipschitz metric for sums of weakly dependent random variables. Liet. Mat. Rink. **29**, No. 2, 385–393. English transl.: Lith. Math. J. **29**, No. 2, 187–193. Zbl. 696.60027

Sunklodas, J. (1990): Approximation by the normal distribution. Liet. Mat. Rink. **30**, No. 2, 382–391 (Russian). Zbl. 709.60012

Sunklodas, J. (1998): On a lower bound of the rate of convergence in the central limit theorem for m-dependent random fields. Teor. Veroyatn. Primen. **43**, No. 1, 171–179. English transl.: Theory Probab. Appl. **43**, No. 1, 162–169. Zbl. 990.28431

Szweczak, Z.S. (1988): On a central limit theorem for m-dependent sequences. Bull. Polish Acad. Sci. Math. **36**, No. 5–6, 327–331. Zbl. 759.60022

Szyszkowski, I. (1990): An invariance principle for dependent random variables. Acta Math. Hung. **56**, No. 1–2, 45–51. Zbl. 737.60023

Takahata, H. (1981): L_∞-bound for asymptotic normality of weakly dependent summands using Stein's result. Ann. Probab. **9**, No. 4, 676–683. Zbl. 465.60033

Takahata, H. (1983): On the rates in the central limit theorem for weakly dependent random fields. Z. Wahrscheinlichkeitstheorie Verw. Geb. **64**, No. 4, 445–456. Zbl. 514.60028

Takahata, H. (1984): The central limit theorem problems for energy in the Gibbs random fields. A short survey. Bull. Tokyo Gakugei Univ., Sec. 4 **36**, 1–15. Zbl. 552.60017

Tikhomirov, A.N. (1976): On the rate of convergence in the central limit theorem for weakly dependent variables. Vestn. Leningr. Univ., No. 7, Mat. Mekh. Astron. No. 2, 158–159 (Russian). Zbl. 346.60016

Tikhomirov, A.N. (1980): On the convergence rate in the central limit theorem for weakly dependent variables. Teor. Veroyatn. Primen. **25**, No. 4, 800–818. English transl.: Theory Probab. Appl. **25**, No. 4, 790–809 (1981). Zbl. 448.60019

Tikhomirov, A.N. (1983): On normal approximation of sums of mixing random fields. Dokl. Akad. Nauk. SSSR **272**, No. 2, 312–314. English transl.: Sov. Math., Dokl. **28**, 396–397. Zbl. 546.60052

Tikhomirov, A.N. (1986): On the distribution of the maximum sum of weakly dependent variables. Teor. Veroyatn. Primen. **31**, No. 4, 829–834. English transl.: Theory Probab. Appl. **31**, No. 4, 733–738. Zbl. 658.60045

Tikhomirov, A.N. (1990): On the normal approximation of sums of weakly dependent Hilbert-valued random variables. Probab. Theory and Math. Statist. Proc. of the Fifth Vilnius Conf., 1989. VSP BV, Utrecht; Mokslas, Vilnius, 482–494. Zbl. 726.60007

Utev, S.A. (1984): Inequalities for sums of weakly dependent random variables and estimates of rate of convergence in the invariance principle. Limit Theorems for Sums of Random Variables. Nauka, Novosibirsk, Tr. Inst. Mat. **3**, 50–77. English transl.: Transl. Ser. Math. Eng., 73–114 (1986). Zbl. 541.60016

Utev, S.A. (1989): Sums of φ-mixing random variables. Asymptotic Analysis of Distributions of Random Processes. Nauka, Novosibirsk. 78–100 (Russian). Zbl. 711.60032

Utev, S.A. (1990a): Central limit theorem for dependent random variables. Probab. Theory and Math. Statist. Proc. of Fifth Vilnius Conference, 1989. VSP BV, Utrecht; Mokslas, Vilnius, 519–528. Zbl. 732.60029

Utev, S.A. (1990b): On the central limit theorem for a double array of φ-mixing random variables. Teor. Veroyatn. Primen. **35**, No. 1, 110–117. English transl.: Theory Probab. Appl. **35**, No. 1, 131–139. Zbl. 724.60028

Utev, S.A. (1991): On a method of studying sums of weakly dependent random variables. Sib. Mat. Zh. **32**, No. 4, 165–183. English transl.: Sib. Math. J. No. 4, 675–690 (1991). Zbl. 778.60014, Zbl. 769.60023

Volkonskii, V.A., and Rozanov, Yu.A. (1959): Some limit theorems for random functions, I. Teor. Veroyatn. Primen. **4**, No. 2, 186–207. English transl.: Theory Probab. Appl. **4**, No. 2, 178–197. Zbl. 092.33502

Volkonskii, V.A., and Rozanov, Yu.A. (1961): Some limit theorems for random functions, II. Teor. Veroyatn. Primen. **6**, No. 2, 202–215. English transl.: Theory Probab. Appl. **6**, No. 2, 186–198. Zbl. 108.31301

Withers, C.S. (1981): Central limit theorems for dependent random variables, I. Z. Wahrscheinlichkeitstheorie Verw. Geb. **57**, No. 4, 509–534. Zbl. 461.60036

Withers, C.S. (1987): Central limit theorems for dependent variables, II. Probab. Theory Related Fields **76**, No. 1, 1–13. Zbl. 608.60019

Yokoyama, R. (1979): Convergence of moments in the central limit theorem for stationary φ-mixing sequences. Tsukuba J. Math. **3**, No. 2, 1–6. Zbl. 434.60026

Yokoyama, R. (1980): Moment bounds for stationary mixing sequences. Z. Wahrscheinlichkeitstheorie Verw. Geb. **52**, No. 1, 45–57. Zbl. 416.60004

Yokoyama, R. (1983): The convergence of moments in the central limit theorem for weakly dependent random variables. Tsukuba Math. J. **7**, No. 1, 147–156. Zbl. 521.60025

Yoshihara, K. (1976): Limiting behavior of U-statistics for stationary absolutely regular processes. Z. Wahrscheinlichkeitstheorie Verw. Geb. **35**, No. 3, 237–252. Zbl. 336.60037

Yoshihara, K. (1977): Convergence rates for integral-type functionals of absolutely regular processes. Yokohama Math. J. **25**, No. 2, 145–153. Zbl. 373.60032

Yoshihara, K. (1978a): Moment inequalities for mixing sequences. Kodai Math. J. **1**, No. 2, 316–328. Zbl. 392.60020

Yoshihara, K. (1978b): Probability inequalities for sums of absolutely regular processes and their applications. Z. Wahrscheinlichkeitstheorie Verw. Geb. **43**, No. 4, 319–329. Zbl. 372.60035

Yoshihara, K. (1979a): Summability of random variables satisfying the strong mixing condition. Sci. Rep. Yokohama Nat. Univ. Sect. 1 **26**, 9–15. Zbl. 429.60023

Yoshihara, K. (1979b): Convergence rates of the invariance principle for absolutely regular sequences. Yokohama Math. J. **27**, No. 1, 49–55. Zbl. 429.60027

Yoshihara, K. (1985): Central limit theorems for stationary mixing sequences. Yokohama Math. J. **33**, No. 1–2, 131–137. Zbl. 601.60022

Yudin, M.D. (1981): On the rate of convergence of the distribution of a sum of $f(n)$-dependent random variables to the normal law. Vestsi Akad. Nauk BSSR, Ser. Fiz.-Mat. Navuk **3**, 57–60 (Russian). Zbl. 466.60050

Yudin, M.D. (1984): On the rate of convergence of the distribution of a sum of weakly dependent random variables. Vestsi Akad. Nauk BSSR, Ser. Fiz.-Mat. Navuk **2**, 44–52 (Russian). Zbl. 546.60024

Yudin, M.D. (1987): A note on approximating distributions of sums of dependent variables with infinitely divisible distributions. Vestsi Akad. Nauk BSSR, Ser. Fiz.-Mat. Navuk **2**, 38–41 (Russian). Zbl. 624.60027

Yudin, M.D. (1989a): On approximating distributions of sums of m_n-dependent variables by distributions in class L. Izv. VUZ. Mat. **4**, 83–88 (Russian). English transl.: Sov. Math. **33**, No. 4, 103–109. Zbl. 618.60027

Yudin, M.D. (1989b): Approximation of distributions of sums of mixing random variables by distributions in class L. Teor. Veroyatn. Mat. Statist. **41**, 120–125. English transl.: Theory Probab. Math. Statist. **41** (1990), 143–147. Zbl. 60024

Yudin, M.D. (1990): Convergence of Distributions of Sums of Random Variables. Universitetskoe, Minsk (Russian). Zbl. 745.60021

Yudin, M.D. (1991): On the approximation of distributions of sums of dependent random variables by distributions from class L. Vestsi Akad. Nauk BSSR, Ser. Fiz.-Mat. Navuk, 1991, No. 1, 35–41 (Russian).

Zaremba, S.K. (1958): Note on the central limit theorem. Math. Z. **69**, 295–298. Zbl. 081.35201

Zhurbenko, I.G. (1972): On strong estimate of mixed cumulants of stochastic processes. Sib. Mat. Zh. **13**, No. 2, 293–308. English transl.: Sib. Math. J. **13**, No. 2, 202–213. Zbl. 246.60040

Zolotarev, V.M. (1986): Modern Theory of Summation of Independent Random Variables. Nauka, Moscow. English transl.: VSP, Utrecht (1997). Zbl. 649.60016

Zolotukhina, L.A. (1978): The central limit theorem for discrete random fields. Izv. Akad. Nauk Uzb. SSR, Ser. Fiz.-Mat. Nauk, **3**, 15–19 (Russian). Zbl. 405.60023

Zolotukhina, L.A., and Chugueva, V.N. (1978): Sufficient conditions for asymptotic normality of sums of discrete random fields dependent in strips. Mat. Zametki **23**, No. 5, 725–732. English transl.: Math. Notes **23**, No. 5, 400–403. Zbl. 404.60056

Zuev, N.M. (1986): On the rate of convergence in the central limit theorem for $m(d)$-dependent random fields. Vestsi Akad. Nauk BSSR, Ser. Fiz.-Mat. Navuk **4**, 28–32 (Russian). Zbl. 614.60019

Zuev, N.M. (1989): Estimation of the rate of convergence in the central limit theorem for $m(d)$-dependent random fields. Vestsi Akad. Nauk BSSR, Ser. Fiz.-Mat. Navuk **3**, 17–22 (Russian). Zbl. 687.60053

Zuparov, T.M. (1981): Moment inequalities and estimation of the remainder in the central limit theorem for a sequence of weakly dependent random variables. Limit Theorems for Random Processes and Statistical Inferences. Fan, Tashkent, 69–87 (Russian). Zbl. 543.60029

Zuparov, T.M. (1983): Estimates of the rate of convergence in the central limit theorem for absolutely regular Banach-valued random variable. Dokl. Akad. Nauk. SSSR **272**, No. 5, 1042–1045. English transl.: Soviet Math. Dokl. **28**, No. 2, 475–478. Zbl. 553.41027

Zuparov, T.M. (1984): Estimates of the rate of convergence in the central limit theorem for absolutely regular Banach-valued random variables. Asymptotic Problems for Probability Distributions, Fan, Tashkent, 78–87 (Russian).

Zuparov, T.M. (1985): On estimates of the rate of convergence in the invariance principle for weakly dependent random variables. Limit Theorems for Probability Distributions. Fan, Tashkent, 32–52 (Russian). Zbl. 616.60036