Available online at www.sciencedirect.com
SCIENCE DIRECT®
Linear Algebra and its Applications 416 (2006) 580–587
LINEAR ALGEBRA AND ITS APPLICATIONS
www.elsevier.com/locate/laa

On Euclidean algebra of hermitian operators on a quaternionic Hilbert space

Rok Strašek

Department of Basic Sciences, Faculty of Civil Engineering, University of Maribor, Smetanova 17, 2000 Maribor, Slovenia

Received 19 October 2005; accepted 3 December 2005
Available online 2 February 2006
Submitted by R.A. Brualdi

Abstract

We solve the problem of finding the best possible constant of ultraprimeness for the special class of Euclidean algebra called algebra of hermitian operators on a quaternionic Hilbert space. More precisely, we prove that for algebra of hermitian operators, equipped with spectral norm, the best possible constant of ultraprimeness is $\frac{1}{2}$.

AMS classification: 17C65; 46L70

Keywords: Euclidean algebras; Quaternionic Hilbert space; Ultraprime algebras; Jacobson–McCrimmon operator; Spectral norm

1. Introduction

In this paper we address the following problem: determine the best possible constant of ultraprimeness for the special class of Euclidean algebra called algebra of hermitian operators on a quaternionic Hilbert space. The topic of ultraprimeness was started for the class of associative Banach algebras by Mathieu (see [9]). The original definition involved ultrafilters, hence the name ultraprimeness. To be more precise, let $\mathscr{A}$ be an associative Banach algebra and $a, b \in \mathscr{A}$. The multiplication operator $M_{a,b} : \mathscr{A} \to \mathscr{A}$ is defined by $M_{a,b}(x) = axb$. Then Mathieu proved that $\mathscr{A}$ is ultraprime if and only if there exists a constant $\kappa > 0$ such that the estimate

E-mail address: rok.strasek@uni-mb.si

0024-3795/$ - see front matter
doi:10.1016/j.laa.2005.12.005

$\|M_{a,b}\| \geqslant \kappa\|a\|\|b\|$ holds for all $a, b \in \mathscr{A}$. The best possible κ could be called the ultraprimeness constant of $\mathscr{A}$. It is obvious that every ultraprime associative Banach algebra is also a prime algebra but the converse is not true. It is well known that the algebra of Hilbert–Schmidt operators over an infinite dimensional Hilbert space is prime but not ultraprime.

The topic of ultraprimeness has been transferred to the nonassociative setting by numerous algebraists (see [2–7]). They proved that for the class of Jordan Banach algebras ultraprimeness is also equivalent to a certain uniform norm estimate $\|U_{a,b}\| \geqslant \kappa\|a\|\|b\|$. Here $U_{a,b}$ denotes the operator on a Jordan algebra $(\mathscr{J}, \circ)$ defined by $U_{a,b}(x) = a \circ (b \circ x) + b \circ (a \circ x) - (a \circ b) \circ x$ called Jacobson–McCrimmon operator. Similar work has also been done in the context of ternary compositions (see [1,7]).

The purpose of our paper is to continue the investigation of the ultraprimeness constant for a particular class of Euclidean algebras which is called algebra of hermitian operators on a quaternionic Hilbert space. Throughout the rest of the article it will be denoted by $\mathrm{Herm}(\mathscr{H})$. This algebra belongs to a class, which is related to the analysis on symmetric cones in $\mathbb{R}^n$. The standard reference for this theory is [8].

The first paper to deal with the question of the ultraprimeness constant of $\mathrm{Herm}(\mathscr{H})$ is [10], where it was proved that there is an estimate $\|U_{a,b}\| \geqslant \left(\sqrt{2}-1\right)\|a\|\|b\|$. Our aim in the sequel is to show that a better estimate is possible for the case of $\mathrm{Herm}(\mathscr{H})$. More precisely we shall prove that our estimate $\frac{1}{2}$ is also the best possible.

2. Preliminaries

We summarize here only the essential properties of the n-dimensional quaternionic Hilbert space $\mathscr{H}$. Let $\mathbb{H}$ be the division ring of real quaternions. For any $\mathbf{h} \in \mathbb{H}$, $\mathbf{h}^*$ will denote the conjugate and $|\mathbf{h}| = \sqrt{\mathbf{h}^*\mathbf{h}}$ the absolute value of $\mathbf{h}$. Let $\mathscr{H} = \mathbb{H}^n$ be a right n-dimensional quaternionic Hilbert space with inner product $\langle\,,\rangle$. We recall that an inner product on $\mathscr{H}$ is a mapping $\langle\,,\rangle : \mathscr{H} \times \mathscr{H} \longrightarrow \mathbb{H}$ with the following properties:

(i) $\langle \mathbf{h}, \mathbf{k}\rangle = \langle \mathbf{k}, \mathbf{h}\rangle^*$,
(ii) $\langle \mathbf{ha} + \mathbf{kb}, \mathbf{g}\rangle = \langle \mathbf{h}, \mathbf{g}\rangle\mathbf{a} + \langle \mathbf{k}, \mathbf{g}\rangle\mathbf{b}$,
(iii) $\langle \mathbf{h}, \mathbf{ka} + \mathbf{gb}\rangle = \mathbf{a}^*\langle \mathbf{h}, \mathbf{k}\rangle + \mathbf{b}^*\langle \mathbf{h}, \mathbf{g}\rangle$,
(iv) $\langle \mathbf{h}, \mathbf{h}\rangle \geqslant 0$ and $\langle \mathbf{h}, \mathbf{h}\rangle = 0$ if and only if $\mathbf{h} = 0$,

for all $\mathbf{h}, \mathbf{k}, \mathbf{g} \in \mathscr{H}$ and $\mathbf{a}, \mathbf{b} \in \mathbb{H}$.

Let $B(\mathscr{H})$ be the algebra of bounded linear transformations on $\mathscr{H}$. With respect to an inner product $\langle\,,\rangle$, we define the adjoint of an operator $A \in B(\mathscr{H})$, denoted by A^*, by $\langle A\mathbf{h}, \mathbf{k}\rangle = \langle \mathbf{h}, A^*\mathbf{k}\rangle$, for all $\mathbf{h}, \mathbf{k} \in \mathscr{H}$. Hence the definitions of hermitian, unitary and normal operators follow in the usual way.

There are two natural norms on Euclidean algebras. Since they are modeled on the Hilbert space, they have the natural Hilbert space norm $\|A\|_2 = \sqrt{\mathrm{Trace}(AA^*)}$. Another norm can be defined with the aid of the spectral theorem [8, p. 43], as

$$\|A\|_\infty = \max\{|\lambda_i|\},$$

where λ_i are eigenvalues of the spectral decomposition of $A \in \mathrm{Herm}(\mathscr{H})$. We note that for the Euclidean algebra $\mathrm{Herm}(\mathscr{H})$ this spectral norm is the same as the operator norm $\|A\|_{\mathrm{op}} = \sup_{\|\mathbf{h}\|=1}\|A\mathbf{h}\|$. So, it is natural to use it in our work as well. In the sequel, we will drop the subscript “op” and just write $\|A\|$.

In order to prove the ultraprimeness of $\mathrm{Herm}(\mathscr{H})$, we use some results proved in [10] and the method introduced in the sequel. First we consider the two-dimensional case, and in the proof of the main theorem, we will apply it to the n-dimensional case. By $\mathrm{Herm}(\mathbb{H}^2)$ we denote the algebra of hermitian 2×2 quaternionic matrices equipped with the usual operator norm. We recall that the operator norm of hermitian matrix

$$A = \begin{bmatrix} \alpha & \mathbf{u} \\ \mathbf{u}^* & \beta \end{bmatrix} \in \mathrm{Herm}(\mathbb{H}^2)$$

is

$$\begin{aligned} \|A\| &= \frac{1}{2}\left(|\alpha+\beta| + \sqrt{(\alpha-\beta)^2 + 4|\mathbf{u}|^2}\right) \\ &= \frac{1}{2}\left(|\mathrm{Trace}(A)| + \sqrt{(\mathrm{Trace}(A))^2 - 4\det(\mathbf{A})}\right), \end{aligned}$$

which is a result that follows immediately from the determination of the eigenvalues of A:

$$\lambda_{\pm} = \frac{1}{2}\left(\alpha+\beta \pm \sqrt{(\alpha-\beta)^2 + 4|\mathbf{u}|^2}\right).$$

We recall again that $U_{A,B} : \mathrm{Herm}(\mathbb{H}^2) \to \mathrm{Herm}(\mathbb{H}^2)$ is defined by

$$U_{A,B}(X) = \frac{1}{2}(AXB + BXA).$$

Without loss of generality we may suppose that A and B have norm one and therefore both have 1 or -1 in their spectrum. The norm of $U_{A,B}$ does not change if we replace A with $-A$ or if we switch the roles of A and B. Therefore we may assume that $\sigma(A) = \{1, a\}$ and $\sigma(B) = \{1, b\}$, where $|a|, |b| \leqslant 1$, $a, b \in \mathbb{R}$, and that there exist $\varphi, \vartheta \in [0, 2\pi]$ and $\mathbf{h}, \mathbf{k} \in \mathbb{H}$ of norm one, such that

$$U_\varphi = \begin{bmatrix} \cos\varphi & \mathbf{h}\sin\varphi \\ \mathbf{h}^*\sin\varphi & -\cos\varphi \end{bmatrix}, \quad U_\vartheta = \begin{bmatrix} \cos\vartheta & \mathbf{k}\sin\vartheta \\ \mathbf{k}^*\sin\vartheta & -\cos\vartheta \end{bmatrix},$$

and

$$A = U_\varphi \begin{bmatrix} 1 & 0 \\ 0 & a \end{bmatrix} U_\varphi \quad \text{and} \quad B = U_\vartheta \begin{bmatrix} 1 & 0 \\ 0 & b \end{bmatrix} U_\vartheta.$$

Note that in the decomposition of norm-one hermitian matrix $A = \begin{bmatrix} \alpha & \mathbf{u} \\ \mathbf{u}^* & \beta \end{bmatrix}$, with $\sigma(A) = \{1, a\}$, we have first $a = -1 + \alpha + \beta$. In the case of $\mathbf{u} = \mathbf{0}$, we have two possibilities. When $\alpha = 1$, we can take $\mathbf{h} = \mathbf{1}$ and $\varphi = 0$. When $\alpha \neq 1$, we can take $\mathbf{h} = \mathbf{1}$ and $\varphi = \frac{\pi}{2}$. In the case when $\mathbf{u} \neq \mathbf{0}$, it follows $a \neq 1$, so we can define $\varphi \in [0, 2\pi]$ by the conditions

$$\sin 2\varphi = \frac{2|\mathbf{u}|}{1-a} \quad \text{and} \quad (\alpha-\beta)\cos 2\varphi \geqslant 0.$$

Finally we define $\mathbf{h}$ by

$$\mathbf{h} = \frac{2\mathbf{u}}{(1-a)\sin 2\varphi}.$$

The above decomposition is indeed a direct consequence of Schur decomposition theorem for quaternionic matrices. We refer the reader to [11].

Let $\mathscr{M}_2$ denote the algebra of 2×2 quaternionic matrices. If we define $\phi : \mathscr{M}_2 \to \mathscr{M}_2$ by $\phi(X) = U_\varphi X U_\varphi$, then ϕ is an algebra isomorphism which is isometric. Hence $\phi U_{A,B} = U_{\phi A, \phi B}$, and therefore

$$\|U_{A,B}\| = \|U_{\phi A,\phi B}\| = \left\| U_{\begin{bmatrix}1 & 0\\ 0 & a\end{bmatrix}, U_\varphi U_\vartheta \begin{bmatrix}1 & 0\\ 0 & b\end{bmatrix} U_\vartheta U_\varphi} \right\|.$$

Thus we may assume in the sequel that $A = \begin{bmatrix}1 & 0\\ 0 & a\end{bmatrix}$, where $|a| \leqslant 1$. In this case B is some hermitian matrix of norm one and it may be written as $B = U_\delta \begin{bmatrix}1 & 0\\ 0 & b\end{bmatrix} U_\delta$, where $|b| \leqslant 1$ and $\delta \in [0, 2\pi]$.

3. The result

We first give some preliminary results which play an important role in the proof of the main theorem. Following the proofs of Lemmas 3.1 and 3.2 in [10], we can prove the following lemma.

Lemma 1. *Let $A, B \in \mathrm{Herm}(\mathbb{H}^2)$. Then the estimates*

(a) $\|U_{A,B}\| \geqslant \max\{|a|, |b|\}$,
(b) $\|U_{A,B}\| \geqslant \frac{1}{2}(1 + ab)$

hold.

It is interesting to note that the method introduced in [10] and the estimates of the above lemma allow us to achieve the ultraprimeness constant of $\sqrt{2} - 1$. In order to estimate a better constant of ultraprimeness, a more elaborate method is required.

Our method consists of choosing appropriate matrix X for which $AXB + BXA$ has an expression which is simple enough to allow universal estimates independent of U_δ, and at the same time large enough for those estimates to be useful. Once the correct idea for X is at hand, the remaining proofs are quite easy. In order to get the correct idea, we made excessive experiments with the aid of numerical computer algebra.

Lemma 2. *Let $A, B \in \mathrm{Herm}(\mathbb{H}^2)$, $0 \leqslant |a|, |b| < \frac{1}{2}$ and $ab < 0$. Then*

$$\|U_{A,B}\| \geqslant \frac{1}{2}[(1 + |ab|)|\cos\delta| + (1 - |ab|)].$$

Proof. Choosing

$$X = U_{\delta-\vartheta} = \begin{bmatrix}\cos(\delta-\vartheta) & \mathbf{h}\sin(\delta-\vartheta)\\ \mathbf{h}^*\sin(\delta-\vartheta) & -\cos(\delta-\vartheta)\end{bmatrix},$$

a tedious but straightforward computation shows that $\|X\| = 1$ and

$$U_{A,B}(X) = \frac{1}{2}\left(\begin{bmatrix}2\cos\delta & \mathbf{h}(1+ab)\sin\delta\\ \mathbf{h}^*(1+ab)\sin\delta & -2ab\cos\delta\end{bmatrix}\cos\vartheta + \begin{bmatrix}2b\sin\delta & -\mathbf{h}(a+b)\cos\delta\\ -\mathbf{h}^*(a+b)\cos\delta & -2a\sin\delta\end{bmatrix}\sin\vartheta\right).$$

Denoting

$$L=\begin{bmatrix} 2\cos\delta & \mathbf{h}(1+ab)\sin\delta \\ \mathbf{h}^*(1+ab)\sin\delta & -2ab\cos\delta \end{bmatrix},$$

and

$$R=\begin{bmatrix} 2b\sin\delta & -\mathbf{h}(a+b)\cos\delta \\ -\mathbf{h}^*(a+b)\cos\delta & -2a\sin\delta \end{bmatrix}$$

we have

$$\|U_{A,B}\| \geqslant \|U_{A,B}(U_{\delta-\vartheta})\| \geqslant \frac{1}{2}\max_{\vartheta}\|L\cos\vartheta + R\sin\vartheta\|.$$

By specializing $\vartheta = 0$ we obtain

$$\|U_{A,B}\| \geqslant \frac{1}{2}\left\|\begin{bmatrix} 2\cos\delta & \mathbf{h}(1+ab)\sin\delta \\ \mathbf{h}^*(1+ab)\sin\delta & -2ab\cos\delta \end{bmatrix}\right\|,$$

and therefore

$$\begin{aligned}
&\|U_{A,B}\| \\
&\quad\geqslant \frac{1}{4}\left[|2(1-ab)\cos\delta| + \sqrt{4(1-ab)^2\cos^2\delta - 4(-4ab\cos^2\delta - \mathbf{h}^*\mathbf{h}(1+ab)^2\sin^2\delta}\right] \\
&\quad= \frac{1}{4}\left[|2(1-ab)\cos\delta| + 2\sqrt{(1-ab)^2\cos^2\delta + 4ab\cos^2\delta + |\mathbf{h}|^2(1+ab)^2\sin^2\delta}\right] \\
&\quad= \frac{1}{4}\left[|2(1-ab)\cos\delta| + 2\sqrt{(1+ab)^2\cos^2\delta + (1+ab)^2\sin^2\delta}\right] \\
&\quad= \frac{1}{2}[(1+|ab|)|\cos\delta| + (1-|ab|)],
\end{aligned}$$

which completes the proof. □

Lemma 3. *Let* $A, B \in \mathrm{Herm}(\mathbb{H}^2)$, $0 \leqslant |a|, |b| < \frac{1}{2}$ *and* $ab < 0$. *Then*

$$\|U_{A,B}\| \geqslant \frac{1}{2}[(1+|ab|) - (1-|ab|)|\cos\delta|].$$

Proof. We consider the unit vector $\mathbf{u} = \begin{bmatrix} \cos\frac{\delta}{2} \\ \mathbf{h}^*\sin\frac{\delta}{2} \end{bmatrix} \in \mathbb{H}^2$, which is an eigenvector of U_δ. By specializing $\mathbf{u}$ in

$$\|U_{A,B}\|^2 \geqslant |\langle U_{A,B}(X)\mathbf{u}, \mathbf{u}\rangle|^2$$

we obtain

$$\begin{aligned}
\|U_{A,B}\|^2 &\geqslant \left|\left\langle \frac{1}{2}(AXU_\delta BU_\delta + U_\delta BU_\delta XA)\mathbf{u}, \mathbf{u}\right\rangle\right|^2 \\
&= \left|\frac{1}{2}\langle (AXU_\delta BU_\delta)\mathbf{u}, \mathbf{u}\rangle + \frac{1}{2}\langle (U_\delta BU_\delta XA)\mathbf{u}, \mathbf{u}\rangle\right|^2 \\
&= \left|\frac{1}{2}\langle (AXU_\delta BU_\delta)\mathbf{u}, \mathbf{u}\rangle + \frac{1}{2}\langle \mathbf{u}, (AXU_\delta BU_\delta)\mathbf{u}\rangle\right|^2 \\
&= |\mathrm{Re}\langle (AXU_\delta BU_\delta)\mathbf{u}, \mathbf{u}\rangle|^2.
\end{aligned}$$

Now, choosing

$$X = \begin{bmatrix} \cos(\delta - \varphi) & \mathbf{h}\sin(\delta - \varphi) \\ \mathbf{h}^* \sin(\delta - \varphi) & -\cos(\delta - \varphi) \end{bmatrix},$$

a straightforward computation shows that

$$\|U_{A,B}\| \geqslant \frac{1}{2} \max_{\varphi} |((1 + ab) + (1 - ab)\cos\delta)\cos\varphi + (b - a)\sin\delta\sin\varphi|.$$

Taking into account that for any real numbers u, v the estimate

$$\max_{\varphi} |u \sin\varphi + v\cos\varphi| = \sqrt{u^2 + v^2}$$

holds, we have

$$\|U_{A,B}\|^2 \geqslant \frac{1}{4}[((1 + ab) + (1 - ab)\cos\delta)^2 + (b - a)^2 \sin^2\delta].$$

Since $(b - a)^2 = b^2 - 2ab + a^2 = b^2 + a^2 + 2|ab| \geqslant 4|ab|$, we have

$$\begin{aligned} &\|U_{A,B}\|^2 \\ &\geqslant \frac{1}{4}[(1 - |ab|)^2 + 2(1 - |ab|)(1 + |ab|)\cos\delta + (1 + |ab|)^2\cos^2\delta + 4|ab|(1 - \cos^2\delta)] \\ &\geqslant \frac{1}{4}[(1 - |ab|)^2 - 2(1 - |ab|)(1+|ab|)|\cos\delta| + (1 + |ab|)^2\cos^2\delta + 4|ab| - 4|ab|\cos^2\delta] \\ &= \frac{1}{4}[(1 + |ab|)^2 - 2(1 - |ab|)(1 + |ab|)|\cos\delta| + (1 - |ab|)^2\cos^2\delta] \\ &= \frac{1}{4}[(1 + |ab|) - (1 - |ab|)|\cos\delta|]^2, \end{aligned}$$

which completes the proof. □

Lemma 4. *Let* $A, B \in \mathrm{Herm}(\mathbb{H}^2)$, $0 \leqslant |a|, |b| < \frac{1}{2}$ *and* $ab < 0$. *Then the estimate*

$$\|U_{A,B}\| \geqslant \frac{1}{2}$$

holds.

Proof. The estimate is a direct consequence of the statements of Lemmas 2 and 3. Adding both estimations, we get

$$\|U_{A,B}\| \geqslant \frac{1}{2} + \frac{1}{2}|ab||\cos\delta| \geqslant \frac{1}{2},$$

which completes the proof. □

We are now in a position to establish a lower bound of the constant of ultraprimeness for the algebra of hermitian operators on a quaternionic Hilbert space.

Theorem 5. *Let* $\mathscr{H}$ *be a* n*-dimensional quaternionic Hilbert space and* $A, B \in \mathrm{Herm}(\mathscr{H})$. *Then the uniform estimate*

$$\|U_{A,B}\| \geqslant \frac{1}{2}\|A\|\|B\|$$

holds.

Proof. Without loss of generality we may suppose that $\|A\| = \|B\| = 1$. Denote by $\kappa = \max\{|a|, |b|\}$. If $\kappa \geqslant \frac{1}{2}$, by Lemma 1(a), we have $\|U_{A,B}\| \geqslant \frac{1}{2}$. In the case of $\kappa < \frac{1}{2}$, we have two possibilities. When $ab \geqslant 0$, the statement is true by Lemma 1(b). In the case when $ab < 0$, we have $\|U_{A,B}\| \geqslant \frac{1}{2}$ by Lemma 4. This completes the proof in the case of two-dimensional quaternionic Hilbert space. In order to complete the proof of ultraprimeness for the n-dimensional case, we made it by reduction to the two-dimensional case.

Note first that one-dimensional case is trivial, since $\mathrm{Herm}(\mathbb{H}) \simeq \mathbb{R}$ with the absolute value $|\;|$ as a norm. In this case, given any $a, b \in \mathbb{R}$, we have $U_{a,b}(x) = abx$ for all $x \in \mathbb{R}$ and so $\|U_{a,b}\| = |a||b|$.

Suppose next that the quaternionic Hilbert space $\mathscr{H}$ is n-dimensional and $A, B \in \mathrm{Herm}(\mathscr{H})$. Then there exist unit vectors $\mu, \eta \in \mathscr{H}$ such that $A\mu = \mu$ and $B\eta = \eta$. Let $\mathscr{K}$ be the span of $\{\mu, \eta\}$. If $\mathscr{K}$ is one-dimensional, then the inequality $\|U_{A,B}\| \geqslant 1$ is obvious. If $\mathscr{K}$ is two-dimensional, then let $P : \mathscr{H} \to \mathscr{H}$ be the orthogonal projection onto $\mathscr{K}$. Define $\phi : \mathrm{Herm}(\mathscr{H}) \to \mathrm{Herm}(\mathscr{H})$ by $\phi(X) = PXP$. Then it is obvious that $\|\phi\| \leqslant 1$ and that $\mathrm{ran}(\phi)$ is isometrically isomorphic as an Euclidean algebra to $\mathrm{Herm}(\mathbb{H}^2)$. Since $P\mu = \mu$ and $P\eta = \eta$, we have $\|\phi(A)\| = \|\phi(B)\| = 1$. Therefore, considering Lemma 4, we get

$$\begin{aligned}
\|U_{A,B}\| &= \sup_{\|X\|=1, X=X^*} \frac{1}{2}\|AXB + BXA\| \\
&\geqslant \sup_{\|X\|=1, X=X^*} \frac{1}{2}\|PAXBP + PBXAP\| \\
&\geqslant \sup_{\|PYP\|=1, Y=Y^*} \frac{1}{2}\|PAPYPBP + PBPYPAP\| \\
&= \sup_{Z\in \mathrm{Im}\,\phi, \|Z\|=1} \|U_{\phi(A),\phi(B)}(Z)\| \geqslant \|\widetilde{U}_{\phi(A),\phi(B)}\| \geqslant \frac{1}{2}\|\phi(A)\|\|\phi(B)\| = \frac{1}{2},
\end{aligned}$$

where $\widetilde{U} : \mathrm{Im}\,\phi \to \mathrm{Im}\,\phi$. □

Remark 6. We wish to mention that the proved constant of ultraprimeness is the best possible, regardless of the dimension of $\mathscr{H}$. Indeed, if $\mathscr{H}$ is at least two-dimensional, we can take orthogonal unit vectors $h, k \in \mathscr{H}$ and form $P, Q \in \mathrm{Herm}(\mathscr{H})$ by $Pu = h\langle h, u\rangle$ and $Qu = k\langle k, u\rangle$. Then, it is easy to check that $\|U_{P,Q}\| = \frac{1}{2}$.

4. Conclusion

As our final remark we wish to note that it is still unclear how to compute the precise value of

$$\max_{\|A\|,\|B\|=1} \|U_{A,B}\|,$$

or even how to give some useful estimate in the form $\|U_{A,B}\| = \frac{1}{2} + m(A, B)$. In many cases $\|U_{A,B}\| = \frac{1}{2}(\|A\|\|B\| + \|AB\|)$ but we do not know whether this holds in general.

References

[1] L.J. Bunce, C.H. Chu, L.L. Stacho, B. Zalar, On prime JB^*-triples, Quart. J. Math. Oxford (2) 49 (1998) 279–290.
[2] M. Cabrera, J. Martinez, Inner derivations on ultraprime normed real algebras, Arch. Math. (Basel) 68 (3) (1997) 221–227.
[3] M. Cabrera, J. Martinez, Inner derivations on ultraprime normed algebras, Proc. Amer. Math. Soc. 125 (7) (1997) 2033–2039.
[4] M. Cabrera, A. Rodriguez, Nonassociative ultraprime normed algebras, Extracta Math. 5 (3) (1990) 127–129.
[5] M. Cabrera, A. Rodriguez, Nonassociative ultraprime normed algebras, Quart. J. Math. Oxford Ser. (2) 43 (169) (1992) 1–7.
[6] M. Cabrera, A. Rodriguez, Non-degenerately ultraprime Jordan–Banach algebras: a zel'manovian treatment, Proc. London Math. Soc. (3) 69 (1994) 576–604.
[7] C.H. Chu, A.M. Galindo, A.R. Palacios, On prime real JB^*-triples, Contemp. Math. 232 (1999) 105–109.
[8] J. Faraut, A. Koranyi, Analysis on Symmetric Cones, Clarendon Press, Oxford, 1994.
[9] M. Mathieu, Rings of quotients of ultraprime Banach algebras, with application to elementary operators, in: Conference on Automatic Continuity and Banach Algebras, Proc. Centre Math. Anal. Austral. Nat. Univ., vol. 21, Canberra, 1989, pp. 297–317.
[10] R. Strašek, Uniform primeness of the Jordan algebra of Hermitian Quaternion Matrices, Linear Algebra Appl. 367 (2003) 235–242.
[11] F. Zhang, Quaternions and matrices of Quaternions, Linear Algebra Appl. 251 (1997) 21–57.