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Ultraprimeness of the Lorentz algebra

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Abstract. We solve the problem of finding the best possible constant of ultraprimeness for the special class of Euclidean algebra called Lorentz algebra. More precisely, we prove that for Lorentz algebra, equipped with spectral norm, the best possible constant of ultraprimeness is $\frac{1}{2}$ regardless of dimension.

1. Introduction

In this paper we address the following problem: determine the best possible constant of ultraprimeness for the special class of Euclidean algebra called Lorentz algebra. The topic of ultraprime algebras was started for the class of associative Banach algebras by M. Mathieu (see [8]). The original definition involved ultrafilters, hence the name ultraprimeness. It was also proved by M. Mathieu that ultraprimeness is equivalent to the existence of a certain norm estimate. To be more precise, let $\mathcal{A}$ be an associative Banach algebra and $a, b \in \mathcal{A}$. The multiplication operator $M_{a,b} : \mathcal{A} \to \mathcal{A}$ is defined by $M_{a,b}(x) = axb$. Then Mathieu proved that $\mathcal{A}$ is ultraprime if and only if there exists a constant $\kappa > 0$ such that the estimate $\left\|M_{a,b}\right\| \geq \kappa \|a\| \|b\|$ holds for all $a, b \in \mathcal{A}$. The best possible κ could be called the ultraprimeness constant of $\mathcal{A}$.

It is obvious that every ultraprime associative Banach algebra is also a prime algebra but the converse is not true. It is well known that the algebra of Hilbert-Schmidt operators over an infinite dimensional Hilbert space is prime but not ultraprime.

The topic of ultraprimeness has been transferred to the nonassociative setting by numerous algebraists (see [2], [3], [4], [5] and [6]). They proved that for the class of Jordan Banach algebras ultraprimeness is also equivalent to a certain uniform norm estimate $\left\|U_{a,b}\right\| \geq \kappa \|a\| \|b\|$. Here $U_{a,b}$ denotes the Jacobson-McCrimmon operator on a Jordan algebra $(\mathcal{J}, \circ)$ defined by $U_{a,b}(x) = a\circ(b \circ x)+b\circ(a \circ x)-(a \circ b) \circ x$.

The aim of our paper is to continue the investigation of the ultraprimeness constant for class of real algebras which are called Euclidean algebras. These algebras belong to a class, which is related to the analysis on symmetric cones in R^n. The

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standard reference for this theory is [7]. The classical structure theorem for Euclidean algebras is the following: An Euclidean algebra is prime if and only if it is simple. In this case it is isomorphic to one of the algebras below

i. $Herm(n, R)$ algebra of $n \times n$ hermitian matrices over R;
ii. $Herm(n, C)$ algebra of $n \times n$ hermitian matrices over C;
iii. $Herm(n, H)$ algebra of $n \times n$ hermitian matrices over quaternions H;
iv. $Herm(3, O)$ algebra of 3×3 hermitian matrices over octonions O;
v. $Lor(n)$ n-dimensional Lorentz algebra.

The first paper to deal with the question of the ultraprimeness constant of Euclidean algebras is [9], where it was proved that for $Herm(n, R)$ and $Herm(n, C)$ there is an estimate $\left\|U_{a,b}\right\| \geq \frac{1}{2} \|a\| \|b\|$. Actually the results in [1] are proved not only for R^n and C^n but also in the setting of general real and complex Hilbert spaces. For $Herm(n, H)$ and Lorentz algebras we established the estimate $\left\|U_{a,b}\right\| \geq \left(\sqrt{2}-1\right) \|a\| \|b\|$ (see [10] and [11]). For $Herm(3, O)$ there is an estimate $\left\|U_{a,b}\right\| \geq \frac{1}{12} \|a\| \|b\|$, which is implicit in the results of C.H. Chu, A.M. Galindo and A.R. Palacios [6].

Our aim in the sequel is to show that a better estimate is possible for the case of $Lor(n)$. More precisely we shall prove

Theorem. *Let $Lor(n)$ be a n-dimensional Lorentz algebra and $a, b \in Lor(n)$. Then the uniform estimate*

$$\left\|U_{a,b}\right\| \geq \frac{1}{2} \|a\| \|b\|$$

holds.

2. Preliminaries

Before our prove begins, let us settle on some definitions and known facts. Let $\mathcal{H}$ be a real Hilbert space and let $\langle a, b\rangle$ be the inner product defined on $\mathcal{H}$. Defining on the vector space $Lor(n) = R \oplus \mathcal{H}$ the product

$$(\lambda + a) \circ (\mu + b) = \lambda\mu + \langle a, b\rangle + \mu a + \lambda b,$$

$Lor(n)$ is a Euclidean algebra with the unit element $e = 1 + 0$. In the classical case, as in [7] only finite dimensional algebras are considered. In our work this restriction is not necessary, so we omit it. In the monograph of J. Faraut and A. Koranyi there are two norms on Euclidean algebras which are considered. The first one is the obvious Hilbert norm $\|x\| = \sqrt{\langle x, x\rangle}$ and the second one is the so called spectral norm which is defined as $\|x\|_\infty = \max\{|\lambda_i|\}$, where λ_i are eigenvalues of the spectral decomposition of $x \in Lor(n)$. We recall that by spectral decomposition, for each $x \in Lor(n)$ there exist unique real numbers $\lambda_1, \ldots, \lambda_k$ all distinct, and an unique complete system of orthogonal idempotents $\{c_1, \ldots, c_k\}$ such that $x = \lambda_1 c_1 + \cdots + \lambda_k c_k$. We also recall that $\{c_1, \ldots, c_k\}$ is a complete system of orthogonal idempotents if $c_i^2 = c_i$, $c_i \circ c_j = 0$ for $i \neq j$ and $c_1 + c_2 + \ldots + c_k = e$.

Since the rank of $Lor(n)$ is equal to 2 and the only possible nonzero idempotents in $Lor(n)$ are e and $\frac{1}{2} + \frac{1}{2}\frac{h}{\|h\|}$, $h \in \mathcal{H}$, the spectral decomposition of $x = \lambda + a$ is $x = \lambda_1 c_1 + \lambda_2 c_2 = \lambda_1 \left(\frac{1}{2} + \frac{1}{2}\frac{h_1}{||h_1||}\right) + \lambda_2 \left(\frac{1}{2} + \frac{1}{2}\frac{h_2}{||h_2||}\right)$. Since the idempotents c_1 and c_2 in the above decomposition are orthogonal, we have $h_1 = -h_2$ and so $2\|a\| = |\lambda_1 - \lambda_2|$. Obviously $2\lambda = \lambda_1 + \lambda_2$. Hence we have $\lambda_1 = \lambda + \|a\|$ and $\lambda_2 = \lambda - \|a\|$. Considering both identities, the spectral norm in the case of Lorentz algebra $Lor(n)$ can be rewritten as

$$\|u\|_\infty = |\lambda| + \|a\|,$$

which is also valid in the case of $Lor(n)$ being infinite dimensional. We also recall that in the case of Lorentz algebra $Lor(n)$ the Jacobson-McCrimmon operator $U_{a,b}(x)$ can be described as

$$\begin{aligned} U_{a,b}(x) &= \lambda\mu\rho + \lambda\langle v, w\rangle + \mu\langle u, w\rangle + \rho\langle u, v\rangle \\ &\quad + \lambda\mu w + \mu\rho u + \lambda\rho v + \langle v, w\rangle u + \langle u, w\rangle v - \langle u, v\rangle w, \end{aligned}$$

where $a = \lambda + u$, $b = \mu + v$ and $x = \rho + w \in Lor(n)$. In the sense of the spectral norm $\|U_{a,b}\|_\infty$ means $\sup_{\|x\|_\infty = 1} \|U_{a,b}(x)\|_\infty$. Since $\|U_{a,b}\|_\infty$ is the same as the operator norm in all other classes of Euclidean algebras, we will drop the subscript "∞" and just write $\|U_{a,b}\|$.

3. The result

The main point in the proof of the theorem is to find an element $x = \rho + w \in Lor(n)$ for which $U_{a,b}(x)$ has an expression which is simple enough to allow universal estimates independent of ρ,w, and at the same time large enough for those estimates to be useful. Once the correct idea for x is found, the remaining proof is quite easy. In order to get the correct idea, we made excessive experiments with the aid of symbolic computer algebra.

First we remind the reader of the next lemma follows from results in [10]. We include a direct proof for the sake of completeness.

Lemma 1. *Let $Lor(n)$ be a Lorentz algebra and let $a = \lambda + u$, $b = \mu + v$ be norm one elements. Then we have*

a) $\|U_{a,b}\| \geq 1 - \|u\| - \|v\| + 2\|u\|\|v\|$,

b) $\|U_{a,b}\| \geq \max\{|1 - 2\|u\||, |1 - 2\|v\||\}$.

Proof. From the definitions of norm and $U_{a,b}$ we can easily calculate that

$$\begin{aligned} \|U_{a,b}(x)\| &= |\lambda\mu\rho + \lambda\langle v, w\rangle + \mu\langle u, w\rangle + \rho\langle u, v\rangle| \\ &\quad + \|\lambda\mu w + \mu\rho u + \lambda\rho v + \langle v, w\rangle u + \langle u, w\rangle v - \langle u, v\rangle w\|. \end{aligned}$$

Assume first that λ has the same sign as μ. If $u \neq -v$ we choose $x = \rho + w \in Lor(n)$ such that $\rho = 0$, we have

$$\begin{aligned} \|U_{a,b}\| \geq \|U_{a,b}(x)\| &= |\langle\lambda v + \mu u, w\rangle| \\ &\quad + \|\lambda\mu w + \langle v, w\rangle u + \langle u, w\rangle v - \langle u, v\rangle w\|, \end{aligned}$$

where $\|w\| = 1$. Now, choosing $w = \frac{\|v\|u+\|u\|v}{\|\|v\|u+\|u\|v\|}$, a tedious straightforward computation shows that

$$\left\|U_{a,b}\right\| \geq \frac{|\lambda\|v\| + \mu\|u\|| \cdot |\langle u, v\rangle + \|u\|\,\|v\||}{\|\|v\|\,u + \|u\|\,v\|} + |\lambda\mu + \|u\|\,\|v\||$$
$$\geq |\lambda\mu + \|u\|\,\|v\|| = 1 - \|u\| - \|v\| + 2\,\|u\|\,\|v\|\,.$$

If $u = -v$, we must proceed in a different way. Since

$$\|\lambda + u\| = |\lambda| + \|u\| = 1 = |\mu| + \|-u\| = \|\mu - u\|$$

we have $\lambda = \mu$, so we must show that $\left\|U_{\lambda+u,\lambda-u}\right\| \geq \lambda^2 + \|u\|^2$. As $\dim Lor(n) \geq 3$, there exist $w \in \mathcal{H}$ such that $\|w\| = 1$ and w is orthogonal to u. If we take $x = 0 + w$, then $\|x\| = 1$ while $U_{\lambda+u,\lambda-u}(x) = \left(\lambda^2 + \|u\|^2\right) w$ which clearly implies

$$\left\|U_{\lambda+u,\lambda-u}\right\| \geq \left|\lambda^2 + \|u\|^2\right| \cdot \|w\| = \lambda^2 + \|u\|^2\,.$$

If λ has the opposite sign as μ, we choose $x = \rho + w \in Lor(n)$ such that $\rho = 0$ and $w = \frac{\|v\|u-\|u\|v}{\|\|v\|u-\|u\|v\|}$. In the same way as above, we complete a).

For b), we consider the norm one $x = \rho + w \in Lor(n)$ such that $\rho = -\lambda$ and $w = u$. Then we have

$$\left\|U_{a,b}\right\| \geq \left|-\lambda^2\mu + \|u\|\,\mu\right| + \left\|-\lambda^2 v + \|u\|^2\,v\right\| \geq \left|-\lambda^2 + \|u\|\right|(|\mu| + \|v\|)\,.$$

Since $|\lambda| + \|u\| = |\mu| + \|v\| = 1$ we have

$$\left\|U_{a,b}\right\| \geq |1 - 2\,\|u\||\,.$$

If we replace ρ by $-\mu$ and w by v in the same way as above, we get

$$\left\|U_{a,b}\right\| \geq |1 - 2\,\|v\||\,.$$

Considering both estimates, we conclude the proof. □

Lemma 2. *Let $Lor(n)$ be a Lorentz algebra and let $a = \lambda + u$, $b = \mu + v$ be norm one elements. Then we have*

$$\left\|U_{a,b}\right\|^2 \geq \left(1 - 2\,\|u\| + 2\,\|u\|^2\right)\left(1 - 2\,\|v\| + 2\,\|v\|^2\right).$$

Proof. Consider first the norm one $x = \rho + w \in Lor(n)$ such that $\rho = 0$. Then we have

$$\left\|U_{a,b}\right\| \geq \left\|U_{a,b}(x)\right\| = |\langle\lambda v + \mu u, w\rangle| + \|(\lambda\mu - \langle u, v\rangle)\,w + \langle v, w\rangle\,u + \langle u, w\rangle\,v\|\,.$$

If we take into account the fact that

$$\|(\lambda\mu - \langle u, v\rangle)\,w + \langle v, w\rangle\,u + \langle u, w\rangle\,v\|$$
$$\geq \max_{\|k\|=1} |\langle(\lambda\mu - \langle u, v\rangle)\,w + \langle v, w\rangle\,u + \langle u, w\rangle\,v, k\rangle|\,,$$

we obtain

$$\left\|U_{a,b}\right\| \geq \max_{\|k\|=1} \left|\langle(\mu + \langle v, k\rangle)\, u + (\lambda + \langle u, k\rangle)\, v + (\lambda\mu - \langle u, v\rangle)\, k, w\rangle\right|.$$

Now, by specializing

$$w = \frac{(\mu + \langle v, k\rangle)\, u + (\lambda + \langle u, k\rangle)\, v + (\lambda\mu - \langle u, v\rangle)\, k}{\|(\mu + \langle v, k\rangle)\, u + (\lambda + \langle u, k\rangle)\, v + (\lambda\mu - \langle u, v\rangle)\, k\|},$$

we obtain

$$\left\|U_{a,b}\right\|^2 \geq \max_{\|k\|=1} \|(\mu + \langle v, k\rangle)\, u + (\lambda + \langle u, k\rangle)\, v + (\lambda\mu - \langle u, v\rangle)\, k\|^2$$

and after elementary calculation

$$\begin{aligned} \left\|U_{a,b}\right\|^2 \geq \max_{\|k\|=1} \Big|&\lambda^2\mu^2 + \langle u, v\rangle^2 + \lambda^2 \|v\|^2 + \mu^2 \|u\|^2 + \langle u, k\rangle^2 \|v\|^2 \\ &+ \langle v, k\rangle^2 \|u\|^2 + 2\lambda \left(\|v\|^2 + \mu^2\right) \langle u, k\rangle + 2\mu \left(\|u\|^2 + \lambda^2\right) \langle v, k\rangle \\ &+4\lambda\mu \langle u, k\rangle \langle v, k\rangle - 2 \langle u, k\rangle \langle v, k\rangle \langle u, v\rangle\Big|. \end{aligned}$$

If we replace k by $-k$ and add up both estimates, we obtain

$$\begin{aligned} \left\|U_{a,b}\right\|^2 \geq \max_{\|k\|=1} \Big|&\lambda^2\mu^2 + \langle u, v\rangle^2 + \lambda^2 \|v\|^2 + \mu^2 \|u\|^2 + \langle u, k\rangle^2 \|v\|^2 + \langle v, k\rangle^2 \|u\|^2 \\ &+ 4\lambda\mu \langle u, k\rangle \langle v, k\rangle - 2 \langle u, k\rangle \langle v, k\rangle \langle u, v\rangle\Big|. \end{aligned}$$

Denote the angle between u and v with δ. As $\dim Lor(n) \geq 3$, there exist $k \in \mathcal{H}$ such that $\|k\| = 1$, k is orthogonal to u and the angle between k and v is $\left|\delta - \frac{\pi}{2}\right|$. Then we have

$$\begin{aligned} \left\|U_{a,b}\right\|^2 &\geq \left|\lambda^2\mu^2 + \lambda^2 \|v\|^2 + \mu^2 \|u\|^2 + \|u\|^2 \|v\|^2 \left(\cos^2 \delta + \cos^2 \left|\delta - \frac{\pi}{2}\right|\right)\right| \\ &= \left|\lambda^2\mu^2 + \lambda^2 \|v\|^2 + \mu^2 \|u\|^2 + \|u\|^2 \|v\|^2\right|, \end{aligned}$$

which completes the proof. □

Proof of the theorem. Without loss of generality we may assume that $\|a\| = \|b\| = 1$. Let $a = \lambda + u$, $b = \mu + v$. Denote

$$\kappa = \max\{\|u\|, \|v\|\} \quad \textit{and} \quad \omega = \min\{\|u\|, \|v\|\}.$$

If $\kappa \geq \frac{3}{4}$ or $\omega \leq \frac{1}{4}$, by Lemma 1 b), we have $\left\|U_{a,b}\right\| \geq \frac{1}{2}$. If $\frac{1}{4} < \kappa, \omega < \frac{1}{2}$ or $\frac{1}{2} < \kappa, \omega < \frac{3}{4}$, then Lemma 1 a) yields $\left\|U_{a,b}\right\| \geq \frac{1}{2}$. Finally, if $\frac{1}{4} < \omega < \frac{1}{2}$ and $\frac{1}{2} < \kappa < \frac{3}{4}$, the same estimate is a consequence of Lemma 2. Considering all estimations, we get $\left\|U_{a,b}\right\| \geq \frac{1}{2}$, which completes the proof. □

4. Conclusion

As remark we wish to mention that the proved constant of ultraprimeness is the best possible, regardless of dimension of $Lor(n)$. This can be seen by taking any $\mathcal{H}$ of dimension at least 2 and such $a = \lambda + u$ and $b = \mu + v$ that $\lambda = \mu = \|u\| = \|v\| = \frac{1}{2}$ and $u = -v$. Then, a straightforward computation shows that $\|U_{a,b}\| \geq \|U_{a,b}(x)\| = \left\|\frac{1}{2}w - 2\langle v, w\rangle v\right\|$, where $x = \rho + w \in Lor(n)$. It is easy to check that the maximum value of above estimate is $\frac{1}{2}$ which is attained by taking w of norm one and orthogonal to v.

As our final remark we wish to note that it is still unclear how to compute the precise value of $\max_{\|a\|,\|b\|=1} \|U_{a,b}\|$, or even how to give some useful estimate in the form $\|U_{a,b}\| = \frac{1}{2} + m(a, b)$. In many cases $\|U_{a,b}\| = \frac{1}{2}(\|a\|\,\|b\| + \|ab\|)$ but we do not know whether this holds in general.

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