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Uniform primeness of the Jordan algebra of hermitian quaternion matrices

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Abstract

In this note we improve the constant of ultraprimeness for the Euclidean algebra of hermitian operators on a quaternionic Hilbert space. More precisely we shall prove that a constant of ultraprimeness is $\sqrt{2} - 1$.

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1. Introduction

The topic of ultraprime algebras was started for the class of associative Banach algebras by Mathieu (see [9]). The original definition involved ultrafilters, hence the name ultraprimeness. It was also proved by Mathieu that ultraprimeness is equivalent to the existence of a certain norm estimate, which could be called uniform primeness.

To be more precise, let $\mathscr{A}$ be an associative Banach algebra and $a, b \in \mathscr{A}$. The multiplication operator $M_{a,b} : \mathscr{A} \to \mathscr{A}$ is defined by $M_{a,b}(x) = axb$. Then Mathieu proved that $\mathscr{A}$ is ultraprime if and only if there exists a constant $\kappa > 0$ such that the estimate $\|M_{a,b}\| \geqslant \kappa\|a\|\|b\|$ holds for all $a, b \in \mathscr{A}$. The best possible κ could be called the ultraprimeness constant of $\mathscr{A}$.

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It is obvious that every ultraprime associative Banach algebra is also a prime algebra but the converse is not true. It is well known that the algebra of Hilbert–Schmidt operators over an infinite dimensional Hilbert space is prime but not ultraprime.

The topic of ultraprimeness has been transferred to the nonassociative setting by Spanish school of Jordan algebras (see [2–7]). They proved that for the class of Jordan Banach algebras ultraprimeness is also equivalent to a certain uniform norm estimate $\|U_{a,b}\| \geqslant \kappa\|a\|\|b\|$. Here $U_{a,b}$ denotes the Jacobson–McCrimmon operator on a Jordan algebra $(\mathscr{J}, \circ)$ defined by $U_{a,b}(x) = a \circ (b \circ x) + b \circ (a \circ x) - (a \circ b) \circ x$.

Beside general theory there are also some explicit calculation for some important cases. Mathieu proved that for prime C^*-algebras we even have the equality $\|M_{a,b}\| = \|a\|\|b\|$. In Jordan theory mostly estimates are known. In [6] the authors proved that prime JB^*-algebras are in fact ultraprime. In [1] it has been shown that in complex prime JB^*-algebras and prime JB^*-triples the uniform estimate $\|U_{a,b}\| \geqslant \frac{1}{6}\|a\|\|b\|$ holds. Later in [7] it was shown that for real JB^*-algebras and real JB^*-triples the estimate $\left\|U_{a,b}\right\| \geqslant \frac{1}{12}\|a\|\|b\|$ holds. It is still an open question whether those two constants are sharp.

The aim of our paper is to continue the investigation of the ultraprimeness constant for another interesting class of real algebras which are called Euclidean algebras. They are intimately connected with the analysis on symmetric cones. The standard reference for this theory is [8]. The classical structure theorem for Euclidean algebras is the following: An Euclidean algebra is prime if and only if it is simple. In this case it is isomorphic to one of the algebras below

i. $\mathrm{Herm}(n, \mathbb{R})$ algebra of $n \times n$ hermitian matrices over $\mathbb{R}$;
ii. $\mathrm{Herm}(n, \mathbb{C})$ algebra of $n \times n$ hermitian matrices over $\mathbb{C}$;
iii. $\mathrm{Herm}(n, \mathbb{H})$ algebra of $n \times n$ hermitian matrices over quaternions $\mathbb{H}$;
iv. $\mathrm{Herm}(3, \mathbb{O})$ algebra of 3×3 hermitian matrices over octonions $\mathbb{O}$;
v. $\mathrm{Lor}(n)$ n-dimensional Lorentz algebra.

The first paper to deal with the question of the ultraprimeness constant of Euclidean algebras is [10], where it was proved that for $\mathrm{Herm}(n, \mathbb{R})$ and $\mathrm{Herm}(n, \mathbb{C})$ there is an estimate $\|U_{a,b}\| \geqslant \frac{1}{2}\|a\|\|b\|$. Actually the results in [1] are proved not only for $\mathbb{R}^n$ and $\mathbb{C}^n$ but also in the setting of general real and complex Hilbert spaces. For Lorentz algebras we established the estimate $\|U_{a,b}\| \geqslant \frac{1}{3}\|a\|\|b\|$ in [11]. For $\mathrm{Herm}(n, \mathbb{H})$ and $\mathrm{Herm}(3, \mathbb{O})$ there is an estimate $\|U_{a,b}\| \geqslant \frac{1}{12}\|a\|\|b\|$, which is implicit in the results of Chu et al. [7]. Our aim below is to show that a better estimate is possible for the case of $\mathrm{Herm}(n, \mathbb{H})$. More precisely we shall prove

Theorem 1.1. *Let $\mathscr{H}$ be a n-dimensional quaternionic Hilbert space and $a, b \in$* $\mathrm{Herm}(\mathscr{H})$. *Then the uniform estimate*

$$\|U_{a,b}\| \geqslant (\sqrt{2} - 1)\|a\|\|b\|$$

holds.

2. Preliminaries

Let $\mathbb{H} = \{\mathbf{h} = h_0 + h_1\mathrm{i} + h_2\mathrm{j} + h_3\mathrm{k}\}$ be the division ring of real quaternions. For any $\mathbf{h} \in \mathbb{H}$, $\mathbf{h}^* = h_0 - h_1\mathrm{i} - h_2\mathrm{j} - h_3\mathrm{k}$ will denote the conjugate of $\mathbf{h}$, $|\mathbf{h}| = \sqrt{\mathbf{h}^*\mathbf{h}} = \sqrt{h_0^2 + h_1^2 + h_2^2 + h_3^2}$ the absolute value of $\mathbf{h}$, $\mathrm{Re}(\mathbf{h}) = h_0$ the real part, $\mathrm{Co}(\mathbf{h}) = h_0 + h_1\mathrm{i}$ the complex part and $\mathrm{Im}(\mathbf{h}) = h_1\mathrm{i} + h_2\mathrm{j} + h_3\mathrm{k}$ the imaginary part of $\mathbf{h}$. Let $\mathscr{H} = \mathbb{H}^n$ be a right n-dimensional quaternionic Hilbert space with inner product $\langle\cdot,\cdot\rangle$ defined by

$$\langle \mathbf{h}, \mathbf{k}\rangle = \sum_{i=1}^{n} \mathbf{k}_i^* \mathbf{h}_i.$$

We recall that an inner product on $\mathscr{H}$ is a mapping $\langle\cdot,\cdot\rangle : \mathscr{H} \times \mathscr{H} \to \mathbb{H}$ with the following properties:

i. $\langle \mathbf{h}, \mathbf{k}\rangle = \langle \mathbf{k}, \mathbf{h}\rangle^*$,
ii. $\langle \mathbf{h}\mathbf{a} + \mathbf{k}\mathbf{b}, \mathbf{g}\rangle = \langle \mathbf{h}, \mathbf{g}\rangle\, \mathbf{a} + \langle \mathbf{k}, \mathbf{g}\rangle\, \mathbf{b}$,
iii. $\langle \mathbf{h}, \mathbf{k}\mathbf{a} + \mathbf{g}\mathbf{b}\rangle = \mathbf{a}^* \langle \mathbf{h}, \mathbf{k}\rangle + \mathbf{b}^* \langle \mathbf{h}, \mathbf{g}\rangle$,
iv. $\langle \mathbf{h}, \mathbf{h}\rangle \geqslant 0$ and $\langle \mathbf{h}, \mathbf{h}\rangle = 0$ if and only if $\mathbf{h} = 0$,

for all $\mathbf{h}, \mathbf{k}, \mathbf{g} \in \mathscr{H}$ and $\mathbf{a}, \mathbf{b} \in \mathbb{H}$.

Let $B(\mathscr{H})$ be the algebra of bounded linear transformations on $\mathscr{H}$. Since the geometry of quaternionic Hilbert spaces is entirely similar to that of complex Hilbert spaces, the Riesz representation theorem for quaternionic Hilbert space can be used to show the existence of adjoints (see [12]). With respect to an inner product $\langle\cdot,\cdot\rangle$, we define the adjoint of an operator $A \in B(\mathscr{H})$, denoted by A^*, by $\langle A\mathbf{h}, \mathbf{k}\rangle = \langle \mathbf{h}, A^*\mathbf{k}\rangle$, for all $\mathbf{h}, \mathbf{k} \in \mathscr{H}$. Hence the definitions of hermitian, unitary and normal operators follow in the usual way. Throughout the rest of the article $\mathrm{Herm}(\mathscr{H})$ will denote the Euclidean algebra of hermitian operators on $\mathscr{H}$.

In the monograph [8] of Faraut and Koranyi there are two important norms on Euclidean algebras which are considered. The first one is the Hilbert space norm $\|A\|_2 = \sqrt{\mathrm{Trace}(AA^*)}$, for all $A \in \mathrm{Herm}(\mathscr{H})$. The second one, which is the framework of most existing results on ultraprimeness is the so called spectral norm and is defined by

$$\|A\|_\infty = \max\{|\lambda_i|\},$$

where λ_i are eigenvalues of the spectral decomposition of $A \in \mathrm{Herm}(\mathscr{H})$. We recall that by spectral decomposition, for each $A \in \mathrm{Herm}(\mathscr{H})$ there exist unique real numbers $\lambda_1, \dots, \lambda_m$ all distinct, and an unique complete system of orthogonal idempotents $\{P_1, \dots, P_m\}$ such that $A = \lambda_1 P_1 + \cdots + \lambda_m P_m$. We also recall that $\{P_1, \dots, P_m\}$ is a complete system of orthogonal idempotents if $P_i^2 = P_i$, $P_i P_j = 0$, for $i \neq j$ and $P_1 + \cdots + P_m = I$. Since $\|A\|_{\mathrm{op}} = \sup_{\|\mathbf{h}\|=1} \|A\mathbf{h}\|$ for the case of $\mathrm{Herm}(\mathscr{H})$ the spectral norm is the operator norm, it is natural to use it in our work as well. In the sequel, we will drop the subscript "op" and just write $\|A\|$.

3. The two-dimensional case

In order to prove the uniform primeness of Herm($\mathscr{H}$), we first consider the two-dimensional case. In the last section, we will apply it to the n-dimensional case. By Herm($\mathbb{H}^2$) we denote the algebra of hermitian 2×2 quaternionic matrices equipped with the usual operator norm. We recall that the operator norm of a matrix

$$A = \begin{bmatrix} \alpha & \mathbf{u} \\ \mathbf{u}^* & \beta \end{bmatrix} \in \mathrm{Herm}(\mathbb{H}^2)$$

is

$$\|A\| = \frac{1}{2}\Big(|\alpha + \beta| + \sqrt{(\alpha - \beta)^2 + 4|\mathbf{u}|^2}\Big),$$

which is a result that follows immediately from the determination of the eigenvalues of A:

$$\lambda_\pm = \frac{1}{2}\Big(\alpha + \beta \pm \sqrt{(\alpha - \beta)^2 + 4|\mathbf{u}|^2}\Big).$$

We recall again that $U_{A,B} : \mathrm{Herm}(\mathbb{H}^2) \to \mathrm{Herm}(\mathbb{H}^2)$ is defined by

$$2\,U_{A,B}(X) = AXB + BXA.$$

Without loss of generality we may suppose that A and B have norm one and therefore both have 1 or -1 in their spectrum. The norm of $U_{A,B}$ does not change if we replace A with $-A$ or if we switch the roles of A and B. Therefore we may assume that $\sigma(A) = \{1, a\}$ and $\sigma(B) = \{1, b\}$, where $|a|, |b| \leqslant 1$, $a, b \in \mathbb{R}$, and that there exist $\varphi, \vartheta \in [0, 2\pi]$ and $\mathbf{h}, \mathbf{k} \in \mathbb{H}$ of norm one, such that

$$U_\varphi = \begin{bmatrix} \cos\varphi & \mathbf{h}\sin\varphi \\ \mathbf{h}^*\sin\varphi & -\cos\varphi \end{bmatrix}, \quad U_\vartheta = \begin{bmatrix} \cos\vartheta & \mathbf{k}\sin\vartheta \\ \mathbf{k}^*\sin\vartheta & -\cos\vartheta \end{bmatrix},$$

and

$$A = U_\varphi \begin{bmatrix} 1 & 0 \\ 0 & a \end{bmatrix} U_\varphi \quad \text{and} \quad B = U_\vartheta \begin{bmatrix} 1 & 0 \\ 0 & b \end{bmatrix} U_\vartheta.$$

Note that in the decomposition of norm-one hermitian matrix

$$A = \begin{bmatrix} \alpha & \mathbf{u} \\ \mathbf{u}^* & \beta \end{bmatrix},$$

with $\sigma(A) = \{1, a\}$, we have first $a = -1 + \alpha + \beta$. In the case of $\mathbf{u} = \mathbf{0}$, we have two possibilities. When $\alpha = 1$, we can take $\mathbf{h} = \mathbf{1}$ and $\varphi = 0$. When $\alpha \neq 1$, we can take $\mathbf{h} = \mathbf{1}$ and $\varphi = \pi/2$. In the case when $\mathbf{u} \neq \mathbf{0}$, it follows $a \neq 1$, so we can define $\varphi \in [0, 2\pi]$ by the conditions

$$\sin 2\varphi = \frac{2|\mathbf{u}|}{1 - a} \quad \text{and} \quad (\alpha - \beta)\cos 2\varphi \geqslant 0.$$

Finally we define $\mathbf{h}$ by

$$\mathbf{h} = \frac{2\mathbf{u}}{(1 - a)\sin 2\varphi}.$$

The above decomposition is indeed a direct consequence of Schur decomposition theorem for quaternionic matrices. We refer the reader to Zhang [12].

Let $\mathscr{M}_2$ denote the algebra of 2×2 quaternionic matrices. If we define $\phi : \mathscr{M}_2 \to \mathscr{M}_2$ by $\phi(X) = U_\varphi X U_\varphi$, then ϕ is an algebra isomorphism which is isometric. Hence $\phi U_{A,B} = U_{\phi A,\phi B}$, and therefore

$$\|U_{A,B}\| = \|U_{\phi A,\phi B}\| = \left\| U_{\left[\begin{smallmatrix} 1 & 0 \\ 0 & a \end{smallmatrix}\right], U_\varphi U_\vartheta \left[\begin{smallmatrix} 1 & 0 \\ 0 & b \end{smallmatrix}\right] U_\vartheta U_\varphi} \right\|.$$

Thus we may assume in the sequel that

$$A = \begin{bmatrix} 1 & 0 \\ 0 & a \end{bmatrix},$$

where $|a| \leqslant 1$. In this case B is some hermitian matrix of norm one and it may be written as

$$B = U_\delta \begin{bmatrix} 1 & 0 \\ 0 & b \end{bmatrix} U_\delta,$$

where $|b| \leqslant 1$ and $\delta \in [0, 2\pi]$.

Lemma 3.1. *Let $A, B \in \mathrm{Herm}(\mathbb{H}^2)$. Then the estimate*

$$\|U_{A,B}\| \geqslant \tfrac{1}{2}(1 + ab)$$

holds.

Proof. Let $\mathbf{h} \in \mathbb{H}$ be the unit quaternion in the representation of U_δ, i.e.

$$U_\delta = \begin{bmatrix} \cos\delta & \mathbf{h}\sin\delta \\ \mathbf{h}^*\sin\delta & -\cos\delta \end{bmatrix}.$$

Note that an elementary calculation shows that $\|U_\delta\| = 1$. Let

$$\xi = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \in \mathbb{H}^2.$$

Considering

$$\|U_{A,B}\| = \|U_\delta\| \cdot \|U_{A,B}\| \geqslant \|U_{A,B}(U_\delta)\| \geqslant \|U_{A,B}(U_\delta)(\boldsymbol{\xi})\|,$$

we obtain

$$\begin{aligned} \|U_{A,B}(U_\delta)(\boldsymbol{\xi})\|^2 &= \left\| \begin{bmatrix} \cos\delta & \mathbf{h}\frac{1}{2}(1+ab)\sin\delta \\ \mathbf{h}^*\frac{1}{2}(1+ab)\sin\delta & -ab\cos\delta \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\|^2 \\ &\geqslant \left\| \begin{bmatrix} \cos\delta \\ \mathbf{h}^*\frac{1}{2}(1+ab)\sin\delta \end{bmatrix} \right\|^2 \\ &= \cos^2\delta + \tfrac{1}{4}(1+ab)^2\sin^2\delta. \end{aligned}$$

Since $1 + ab \leqslant 2$, we have $\cos^2\delta \geqslant \frac{1}{4}(1+ab)^2\cos^2\delta$ and finally

$$\|U_{A,B}\|^2 \geqslant \tfrac{1}{4}(1+ab)^2\cos^2\delta + \tfrac{1}{4}(1+ab)^2\sin^2\delta = \tfrac{1}{4}(1+ab)^2,$$

which completes the proof. □

Remark 3.1. The main point in the proof of the above lemma, as well as the forthcoming one, is to find a matrix for which $AXB + BXA$ has an expression which is simple enough to allow universal estimates independent of U_δ, and at the same time large enough for those estimates to be useful. Once the correct idea for X is at hand, the remaining proofs are quite easy. In order to get the correct idea, we made excessive experiments with the aid of symbolic computer algebra.

Lemma 3.2. *Let* $A, B \in \mathrm{Herm}(\mathbb{H}^2)$. *Then the estimate*

$$\|U_{A,B}\| \geqslant \max\{|a|, |b|\}$$

holds.

Proof. First we prove that $\|U_{A,B}\| \geqslant |b|$. This is clear whenever $b = 0$. Otherwise, $\cos^2\delta + b^2\sin^2\delta \neq 0$, and we can consider $\rho = 1/\sqrt{\cos^2\delta + b^2\sin^2\delta}$. Now, choosing

$$X = \rho\begin{bmatrix} \cos^2\delta + b\sin^2\delta & \mathbf{h}(1-b)\sin\delta\cos\delta \\ \mathbf{h}^*(1-b)\sin\delta\cos\delta & -\cos^2\delta - b\sin^2\delta \end{bmatrix},$$

a tedious but straightforward computation shows that $\|X\| = 1$ and

$$U_{A,B}(X) = \rho\begin{bmatrix} \cos^2\delta + b^2\sin^2\delta & \mathbf{h}\frac{1}{4}(1-b^2)\sin 2\delta \\ \mathbf{h}^*\frac{1}{4}(1-b^2)\sin 2\delta & -ab \end{bmatrix}.$$

By specializing

$$\mathbf{u} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \in \mathbb{H}^2$$

in

$$\|U_{A,B}\| \geqslant \left|\left\langle U_{A,B}(X)\mathbf{u}, \mathbf{u}\right\rangle\right|$$

we obtain

$$\begin{aligned} \|U_{A,B}\| &\geqslant \left|\left\langle \rho\begin{bmatrix} \cos^2\delta + b^2\sin^2\delta & \mathbf{h}\frac{1}{4}(1-b^2)\sin 2\delta \\ \mathbf{h}^*\frac{1}{4}(1-b^2)\sin 2\delta & -ab \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\rangle\right| \\ &= \frac{1}{\sqrt{\cos^2\delta + b^2\sin^2\delta}}\left|\left\langle \begin{bmatrix} \cos^2\delta + b^2\sin^2\delta \\ \mathbf{h}^*\frac{1}{4}(1-b^2)\sin 2\delta \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\rangle\right| \\ &= \frac{\cos^2\delta + b^2\sin^2\delta}{\sqrt{\cos^2\delta + b^2\sin^2\delta}} = \sqrt{\cos^2\delta + b^2\sin^2\delta}. \end{aligned}$$

Since $b^2 \leqslant 1$, we have

$$\cos^2 \delta + b^2 \sin^2 \delta \geqslant b^2 \cos^2 \delta + b^2 \sin^2 \delta = b^2$$

and so

$$\|U_{A,B}\| \geqslant |b|.$$

If we now take into account the fact that the norm of $U_{A,B}$ does not change if we switch the roles of A and B, we obtain

$$\|U_{A,B}\| \geqslant |a|.$$

Considering both estimates, we conclude the proof. □

Preposition 3.1. *The statement of Theorem* 1.1 *is true for* $n = 2$.

Proof. Without loss of generality we may suppose that $\|A\| = \|B\| = 1$. Denote by

$$\kappa = \max\{|a|, |b|\}.$$

If $\kappa < \sqrt{2} - 1$, by Lemma 3.1, we have

$$\|U_{A,B}\| \geqslant \tfrac{1}{2}(1 - |a||b|) \geqslant \tfrac{1}{2}(1 - \kappa^2) > \sqrt{2} - 1.$$

On the other hand, if $\kappa \geqslant \sqrt{2} - 1$, the same estimate is direct consequence of Lemma 3.2. Considering both estimations, we get

$$\|U_{A,B}\| \geqslant \sqrt{2} - 1,$$

which completes the proof in the case $\dim \mathscr{H} = 2$. □

4. The conclusion of the proof of Theorem 1.1

In this section we complete the proof of uniform primeness of $\mathrm{Herm}(\mathscr{H})$ by reduction to the two-dimensional case. Note first that one-dimensional case is trivial, since $\mathrm{Herm}(\mathbb{H}) \simeq \mathbb{R}$ with the absolute value $|\cdot|$ as a norm. In this case, given any $a, b \in \mathbb{R}$, we have $U_{a,b}(x) = abx$ for all $x \in \mathbb{R}$ and so $\|U_{a,b}\| = |a||b|$.

Suppose first that the quaternionic Hilbert space $\mathscr{H}$ is n-dimensional and $A, B \in \mathrm{Herm}(\mathscr{H})$. Without loss of generality we may assume that $\|A\| = \|B\| = 1$ and both have 1 in their spectrum. Thus there exist unit vectors $\mu, \eta \in \mathscr{H}$ such that $A\mu = \mu$ and $B\eta = \eta$. Let $\mathscr{K}$ be the span of $\{\mu, \eta\}$. If $\mathscr{K}$ is one-dimensional, then the inequality $\|U_{A,B}\| \geqslant 1$ is obvious. If $\mathscr{K}$ is two-dimensional, then let $P : \mathscr{H} \to \mathscr{H}$ be the orthogonal projection onto $\mathscr{K}$. Define $\phi : \mathrm{Herm}(\mathscr{H}) \to \mathrm{Herm}(\mathscr{H})$ by $\phi(X) = PXP$. Then it is obvious that $\|\phi\| \leqslant 1$ and that $\mathrm{ran}(\phi)$ is isometrically isomorphic as an Euclidean algebra to $\mathrm{Herm}(\mathbb{H}^2)$. Since $P\mu = \mu$ and $P\eta = \eta$, we have $\|\phi(A)\| = \|\phi(B)\| = 1$. Therefore, considering Proposition 3.1, we get

$$
\begin{aligned}
\|U_{A,B}\| &= \sup_{\|X\|=1, X=X^*} \frac{1}{2}\|AXB + BXA\| \\
&\geqslant \sup_{\|X\|=1, X=X^*} \frac{1}{2}\|PAXBP + PBXAP\| \\
&\geqslant \sup_{\|PYP\|=1, Y=Y^*} \frac{1}{2}\|PAPYPBP + PBPYPAP\| \\
&= \sup_{Z\in \mathrm{Im}\,\phi, \|Z\|=1} \|U_{\phi(A),\phi(B)}(Z)\| \\
&\geqslant \|U'_{\phi(A),\phi(B)}\| \geqslant (\sqrt{2}-1)\|\phi(A)\|\|\phi(B)\| = \sqrt{2}-1,
\end{aligned}
$$

where $U' : \mathrm{Im}\,\phi \to \mathrm{Im}\,\phi$.

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