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A REMARK ON LORENTZ ALGEBRAS

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Abstract. We consider the class of Euclidean algebras associated to Minkowski light cones and called Lorentz algebras. We prove that in Lorentz algebras the estimate $\| P(a,b)\|_\infty \geqq (\sqrt{2}-1)\,\|a\|_\infty \|b\|_\infty$ is valid for the spectral norm and is therefore independent of the dimension of the Lorentz algebra.

1. Introduction

Lorentz algebras are nonassociative structures, which arise from the Minkowski metric for the Einstein space-time of the general relativity theory. For $x, y \in \mathbf{R}^4$ the Minkowski form is defined by

$$[x,y] = x_1y_1 - x_2y_2 - x_3y_3 - x_4y_4.$$

The set

$$\Lambda = \left\{ x \in \mathbf{R}^4,\ [x,x] > 0 \text{ and } x_1 > 0 \right\}$$

is a light cone (see [2] for more details). This definition can be extended in an obvious way to any $\mathbf{R}^{n+1} = \mathbf{R} \times \mathbf{R}^n$ where for $(t,x),(s,y) \in \mathbf{R} \times \mathbf{R}^n$ we define

$$\left[(t,x),(s,y) \right] = ts - \langle x,y \rangle.$$

Here $\langle x,y \rangle$ denotes the classical inner product of $\mathbf{R}^n$. We thus obtain a family of Lorentz cones

$$\Lambda_{n+1} = \left\{ (t,x) \in \mathbf{R} \times \mathbf{R}^n, t > \sqrt{\langle x,x \rangle} \right\}.$$

If we define the binary product $\circ$ on $\mathbf{R}^{n+1} = \mathbf{R} \times \mathbf{R}^n$ by

$$(t,x) \circ (s,y) = ts + \langle x,y \rangle + ty + sx,$$

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we obtain a unital algebra (the element $(1,0)$ being the unit) which is not associative, but has a remarkable property. The set of its squares is the closure of the Lorentz cone. This example can be put in a more general framework, which is elaborated in [1]. Lorentz cones are symmetric and to every symmetric cone one can associate a Euclidean Jordan algebra, whose set of squares forms the closure of the original cone. Using this algebra as coordinates, it is possible to build analysis on symmetric cones (see [1] for details).

There are two natural norms in Euclidean Jordan algebras. Since they are modeled on the Euclidean space, they have the natural inner product norm $\|x\| = \sqrt{\langle x, x\rangle}$. Another norm can be defined with the aid of the spectral theorem [1], p. 43, as

$$\|x\|_\infty = \max\left\{ |\lambda|, \lambda \in \mathrm{Spectrum}\,(x)\right\}. \tag{1}$$

We note that for the Euclidean Jordan algebra $\mathrm{Sym}\,(m, \mathbf{R})$ of $m \times m$ real symmetric matrices, this spectral norm is the same as the operator norm.

A starting point for our investigation is [5], where the authors proved an interesting estimate for the quadratic operator in the algebra $\mathrm{Sym}\,(m, \mathbf{R})$, which can be rewritten in a form $\left\| P(a,b)\right\|_\infty \geqq \frac{1}{2}\|a\|_\infty \|b\|_\infty$, where operator $P(a,b)$ can be defined in all Jordan algebras as

$$P(a,b)(x) = a \circ (b \circ x) + b \circ (a \circ x) - (a \circ b) \circ x$$

(see [1], p. 32). We show that a similar estimate, independent of dimension, can be given for Lorentz algebras. It is interesting that the constant $\sqrt{2} - 1$ we obtain is the same as the one given by Stachó and Zalar in [4], where they considered standard operator algebras. For more details on the algebraic theory of nonassociative algebras and Jordan algebras in particular, we refer to [3].

2. Preliminaries

Let $\mathcal{H}$ be a real Hilbert space and let $\langle a, b\rangle$ be the inner product defined on $\mathcal{H}$. Defining on the vector space $\mathcal{L} = \mathbf{R} \oplus \mathcal{H}$ the product

$$(\lambda + a) \circ (\mu + b) = \lambda\mu + \langle a, b\rangle + \mu a + \lambda b,$$

$\mathcal{L}$ is a Euclidean Jordan algebra with the unit element $e = 1 + 0$. The algebra $\mathcal{L}$ belongs to the class of Euclidean Jordan algebras associated to symmetric Lorentz cones introduced in the first section and is therefore called Lorentz algebra. In the classical case, as in [1], only finite dimensional algebras are considered. In our work this restriction is not necessary, so we omit it.

Let $u = \lambda + a$ be an arbitrary element in the Lorentz algebra $\mathcal{L}$. Since the rank of $\mathcal{L}$ is equal to 2 and the only possible nonzero idempotents in $\mathcal{L}$ are e and $\frac{1}{2} + \frac{1}{2}\frac{h}{\|h\|}$, $h \in \mathcal{H}$, the spectral decomposition of u is

$$u = \lambda_1 c_1 + \lambda_2 c_2 = \lambda_1 \left(\frac{1}{2} + \frac{1}{2}\frac{h_1}{\|h_1\|}\right) + \lambda_2 \left(\frac{1}{2} + \frac{1}{2}\frac{h_2}{\|h_2\|}\right).$$

We recall that by spectral decomposition, for each $u \in \mathcal{L}$ there exist unique real numbers $\lambda_1, \dots, \lambda_k$, all distinct, and a unique complete system of orthogonal idempotents $\{c_1, \dots, c_k\}$ such that $u = \lambda_1 c_1 + \ldots + \lambda_k c_k$. We also recall that $\{c_1, \dots, c_k\}$ is a complete system of orthogonal idempotents if $c_i^2 = c_i$, $c_i \circ c_j = 0$ for $i \neq j$ and $c_1 + c_2 + \ldots + c_k = e$. Since the idempotents c_1 and c_2 in the above decomposition are orthogonal, we have $h_1 = -h_2$ and so $2\|a\| = |\lambda_1 - \lambda_2|$. Obviously $2\lambda = \lambda_1 + \lambda_2$. Hence we have $\lambda_1 = \lambda + \|a\|$ and $\lambda_2 = \lambda - \|a\|$. Considering both identities, the norm (1) in the case of Lorentz algebra $\mathcal{L}$ can be rewritten as $\|u\|_\infty = |\lambda| + \|a\|$, which is also valid in the case of $\mathcal{L}$ being infinite dimensional.

The purpose of this note is to prove the following

THEOREM. *Let $\mathcal{L}$ be a Lorentz algebra with* $\dim \mathcal{L} \geqq 3$. *Then we have*

$$\big\| P(a,b) \big\|_\infty \geqq \big(\sqrt{2} - 1\big) \|a\|_\infty \|b\|_\infty,$$

for all $a, b \in \mathcal{L}$.

Note that $\big\| P(a,b) \big\|_\infty$ means $\sup_{\|x\|_\infty \leqq 1} \big\| P(a,b)(x) \big\|_\infty$.

3. Proof of the Theorem

In the case of Lorentz algebra $\mathcal{L}$ the operator $P(a,b)$ can be represented as

$$P(a,b)(x) = \lambda\mu\rho + \lambda\langle v, w\rangle + \mu\langle u, w\rangle + \rho\langle u, v\rangle$$
$$+ \lambda\mu w + \mu\rho u + \lambda\rho v + \langle v, w\rangle u + \langle u, w\rangle v - \langle u, v\rangle w,$$

where $a = \lambda + u$, $b = \mu + v$ and $x = \rho + w \in \mathcal{L}$.

LEMMA 1. *Let $\mathcal{L}$ be a Lorentz algebra and let $a = \lambda + u$, $b = \mu + v \in \mathcal{L}$ be ∞-norm one elements. Then we have*

$$\big\| P(a,b) \big\|_\infty \geqq \max\big\{ \big| 2\|u\| - 1 \big|, \big| 2\|v\| - 1 \big| \big\}.$$

PROOF. From the definitions of norm and $P(a,b)$ we can easily calculate that

$$\big\| P(a,b)(x)\big\|_\infty = \big|\lambda\mu\rho + \lambda\langle v,w\rangle + \mu\langle u,w\rangle + \rho\langle u,v\rangle\big|$$

$$+ \big\| \lambda\mu w + \mu\rho u + \lambda\rho v + \langle v,w\rangle u + \langle u,w\rangle v - \langle u,v\rangle w\big\| .$$

Choose $x = \rho + w \in \mathcal{L}$ such that $\rho = -\lambda$ and $w = u$ and compute the expression $\big\| P(a,b)(x)\big\|$. We have

$$\big\| P(a,b)\big\|_\infty \geqq \big\| P(a,b)(x)\big\|_\infty \geqq \big| -\lambda^2\mu + \lambda\langle u,v\rangle + \mu\langle u,u\rangle - \lambda\langle u,v\rangle\big|$$

$$+ \big\| \lambda\mu u - \lambda\mu u - \lambda^2 v + \langle u,v\rangle u + \langle u,u\rangle v - \langle u,v\rangle u\big\|$$

$$= \big| -\lambda^2\mu + \|u\|^2\mu\big| + \big\| -\lambda^2 v + \|u\|^2 v\big\| \geqq \big| -\lambda^2 + \|u\|^2\big| \big(|\mu| + \|v\|\big) .$$

Since $|\lambda| + \|u\| = |\mu| + \|v\| = 1$ we have

$$\big\| P(a,b)\big\|_\infty \geqq \big| 2\|u\| - 1\big| .$$

If we replace ρ by $-\mu$ and w by v in the same way as above, we get

$$\big\| P(a,b)\big\|_\infty \geqq \big| 2\|v\| - 1\big| .$$

Considering both estimates, we conclude the proof. □

LEMMA 2. *Let $\mathcal{L}$ be a Lorentz algebra and let $a = \lambda + u$, $b = \mu + v \in \mathcal{L}$ be ∞-norm one elements. Then we have*

$$\big\| P(a,b)\big\|_\infty \geqq \big(1 - \|u\|\big)\big(1 - \|v\|\big) + \|u\|\|v\|.$$

PROOF. We may, upon replacing b by $-b$, assume that λ has the same sign as μ. If $u \neq -v$ we choose $x = \rho + w \in \mathcal{L}$ such that $\rho = 0$ and $w = \frac{\|v\|u + \|u\|v}{\|\|v\|u + \|u\|v\|}$. Then $\|w\| = 1$ and so

$$\big\| P(a,b)\big\|_\infty \geqq \big\| P(a,b)(x)\big\|_\infty$$

$$\geqq \left|\lambda\left\langle v, \frac{\|v\|u + \|u\|v}{\big\|\,\|v\|u + \|u\|v\big\|}\right\rangle + \mu\left\langle u, \frac{\|v\|u + \|u\|v}{\big\|\,\|v\|u + \|u\|v\big\|}\right\rangle\right|$$

$$+ \left\|\lambda\mu\frac{\|v\|u + \|u\|v}{\big\|\,\|v\|u + \|u\|v\big\|} + \left\langle v, \frac{\|v\|u + \|u\|v}{\big\|\,\|v\|u + \|u\|v\big\|}\right\rangle u\right.$$

$$+ \left\langle u, \frac{\|v\|u + \|u\|v}{\big\| \|v\|u + \|u\|v \big\|} \right\rangle v - \langle u, v\rangle \frac{\|v\|u + \|u\|v}{\big\| \|v\|u + \|u\|v \big\|} \Bigg\|$$

$$\geqq \frac{1}{\big\| \|v\|u + \|u\|v \big\|} \Big\| \lambda\mu\big(\|v\|u + \|u\|v\big) + \|v\|\langle v, u\rangle u$$

$$+ \|u\|\|v\|^2 u + \|v\|\|u\|^2 v + \|u\|\langle v, u\rangle v - \|v\|\langle u, v\rangle u - \|u\|\langle u, v\rangle v \Big\|$$

$$= \frac{1}{\big\| \|v\|u + \|u\|v \big\|} \Big\| \lambda\mu\big(\|v\|u + \|u\|v\big) + \|u\|\|v\|^2 u + \|v\|\|u\|^2 v \Big\|$$

$$= \frac{1}{\big\| \|v\|u + \|u\|v \big\|} \Big\| \big(\lambda\mu + \|u\|\|v\|\big)\big(\|v\|u + \|u\|v\big) \Big\|$$

$$= \big| \lambda\mu + \|u\|\|v\| \big| = \big(1 - \|u\|\big)\big(1 - \|v\|\big) + \|u\|\|v\|.$$

If $u = -v$, we must proceed in a different way. Since

$$\|\lambda + u\|_\infty = |\lambda| + \|u\| = 1 = |\mu| + \| - u\| = \|\mu - u\|_\infty$$

we have $\lambda = \mu$, so we must show that

$$\big\| P(\lambda + u, \lambda - u)\big\|_\infty \geqq \lambda^2 + \|u\|^2.$$

As $\dim \mathcal{L} \geqq 3$, there exist $w \in \mathcal{H}$ such that $\|w\| = 1$ and $w \perp u$. If we take $x = 0 + w$, then $\|x\|_\infty = 1$ while

$$P(\lambda + u, \lambda - u)(x) = \big(\lambda^2 + \|u\|^2\big)\, w$$

which clearly implies

$$\big\| P(\lambda + u, \lambda - u)\big\|_\infty \geqq \big|\lambda^2 + \|u\|^2\big| \cdot \|w\| = \lambda^2 + \|u\|^2. \qquad \square$$

Proof of Theorem. Without loss of generality we may assume that $\|a\|_\infty = \|b\|_\infty = 1$. Let $a = \lambda + u$ and $b = \mu + v$. Denote

$$\kappa = \max\big\{ \|u\|, \|v\|\big\} \quad \text{and} \quad \omega = \min\big\{ \|u\|, \|v\|\big\}.$$

If $\kappa \geqq \frac{1}{2}\sqrt{2}$ or $\omega \leqq 1 - \frac{1}{2}\sqrt{2}$, by Lemma 1, we have $\big\| P(a,b)\big\|_\infty \geqq \sqrt{2} - 1$. If $1 - \frac{1}{2}\sqrt{2} \leqq \omega, \kappa \leqq \frac{1}{2}\sqrt{2}$, then the second lemma yields

$$\big\| P(a,b)\big\|_\infty \geqq \sqrt{2} - 1.$$

Considering both cases, we have

$$\left\| P(a,b)\right\|_{\infty} \geqq \left(\sqrt{2}-1\right)\|a\|_{\infty}\|b\|_{\infty}. \quad \square$$

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