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ZEROS OF SCHROEDINGER EIGENFUNCTIONS AT POTENTIAL SINGULARITIES

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Let Ω be a domain in $\mathbf{R}^N$ $(N \geqq 3)$, $0 \in \Omega$ and let $q \in L^1_{\text{loc}}(\Omega)$ denote a positive valued function. Recently Alimov and Joó [1], [2] achieved interesting results concerning the summability of eigenfunction expansions of the Schroedinger operator L defined as a self-adjoint extension of L_0: $C_0^\infty(\Omega) \ni f \mapsto -\Delta f + qf$ under the hypothesis $q = q_0 + q_1$ where $q_0 \in L^2(\Omega)$ is a radially symmetric positive function, $q(x) \leqq \text{const} \cdot |x|^{-2}$ and $q_1 \in L^\infty(\Omega)$. Their considerations are based upon Titchmarsh's mean value formula whose proof requires, as far we know (cf. [3]), the Gauss—Green theorem and hence the smoothness of the eigenfunctions of L. Nevertheless, the above conditions on q seem not too restrictive from the view points of applications e.g. in quantum chemistry, except for the assumption on the singularity of q_0. The aim of this short note is to point out that a bit stronger singularity of the potential q at 0 entails the vanishing of all the Schroedinger eigenfunctions at 0 even in a much more general setting. This fact may have interest in itself from the view points of quantum chemistry [4].

PROPOSITION. *Suppose $u \in C(\Omega)$ is a function satisfying the equation $-\Delta u + qu = \lambda u$ in distribution sense for some λ. Then $u(0) = 0$ unless $\varliminf_{r \downarrow 0} r^{-N+2} \int_{|x|<r} q(x)\,dx < \infty$.*

PROOF. Let $u(0) \neq 0$. Then we may assume without loss of generality $u(0) = 1$ and we may choose $r_0 > 0$ such that $\{x\colon |x| < r_0\} \subset \Omega$ and $\sup_{|x|<r_0} |u(x) - 1| \leqq 1/2$. Let us fix $r \in (0, r_0)$ arbitrarily and consider the function $\psi_0(r) \equiv \operatorname{sgn}\left(\tau - \frac{3}{4} r\right)\chi(\tau)$ where χ denotes the characteristic function of the interval $\left(\frac{r}{2}, r\right)$. Observe that for a suitable sequence $\psi_n \in C_0^\infty\left(\frac{r}{2}, r\right)$, $|\psi_n| \leqq 1$ $(n = 1, 2, \ldots)$ we have $\psi_n(\tau) \to \psi_0(\tau)$ $(n \to \infty)$ whenever $\tau \in [0, \infty)$. Now define the functions $\varphi_n\colon [0, \infty) \to \mathbf{R}$ and $f_n\colon \Omega \to \mathbf{R}$ by

$$\varphi_n(\varrho) \equiv \int_\varrho^\infty \int_\xi^\infty \psi_n(\tau)\,d\tau\,d\xi \quad \text{and} \quad f_n(x) \equiv \varphi_n(|x|) \quad (n = 0, 1, 2, \ldots).$$

Clearly, $f_n \in C_0^\infty(\Omega)$ $(n = 1, 2, \ldots)$ and

$$\Delta f_n(x) = \varphi_n''(|x|) + \frac{N-1}{|x|}\varphi_n'(|x|) \to \Delta f_0(x) \quad (n \to \infty)$$

while

$$|\Delta f_n(x)| \leqq 1+\frac{N-1}{|x|}\left||x|-\frac{3}{4}r\right| \leqq \frac{N+1}{2} \quad \text{if} \quad \frac{r}{2} \leqq |x| \leqq r$$

and $\Delta f_n(x)=0$ elsewhere. Thus

$$\int_\Omega u[-\Delta f_n-\lambda f_n] \to \int_\Omega u[-\Delta f_0-\lambda f_0]$$

and

$$\int_\Omega quf_n \to \int_\Omega quf_0 \quad (n\to\infty).$$

By hypothesis,

$$\int_\Omega u(-\Delta f+qf) = \lambda \int_\Omega uf$$

for all $f\in C_0^\infty(\Omega)$, in particular for $f=f_1, f_2, \dots$. By passing to the limit, hence we obtain

$$\int_\Omega u(\Delta f_0+\lambda f_0) = \int_\Omega quf_0. \tag{1}$$

However,

$$\left|\int_\Omega u(\Delta f_0+\lambda f_0)\right| \leqq \left(\frac{N+1}{2}+|\lambda|\right)\int_{r/2<|x|<r}|u(x)|\,dx \leqq \left(\frac{N+1}{2}+|\lambda|\right)\frac{3}{2}\omega_N r^N$$

where ω_N denotes the volume of the unit ball in $\mathbf{R}^N$. On the other hand,

$$\operatorname{Re}\int_\Omega quf_0 = \int_\Omega q(\operatorname{Re} u)f_0 \geqq \int_{|x|<r/2} q(x)\,\frac{1}{2}\,\frac{r^2}{16}\,dx$$

since $q, f_0 \geqq 0$ and $f_0(x)=r^2/16$ whenever $|x|<r/2$. Therefore, by (1), $r^2\cdot \int_{|x|<r/2} q \leqq \leqq cr^N$ where the constant c does not depend on $r(<r_0)$. This completes the proof.

COROLLARY. *If* $q(x)\geqq c|x|^{-(2+\varepsilon)}$ $(x\in\Omega)$ *for some* $c, \varepsilon>0$ *then every continuous solution u of the Schroedinger equation* $-\Delta u+qu=\lambda u$ *vanishes at* 0.

References

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