[5] Pham, Joseph. *An Operator Theoretic Approach to Wavelet Theory with Applications.* Ph.D. dissertation, University of California, Los Angeles, 1999.

[6] Rosenblum, M. and Rovnyak, J. *Hardy Classes and Operator Theory.* Oxford University Press, New York, 1985.

[7] Sz.-Nagy, B. and Foias, C. *Harmonic Analysis of Operators on Hilbert Space.* American Elsevier Publishing Company, Inc., New York, 1970.

[8] Wickerhauser, M.V. *Adapted Wavelet Analysis from Theory to Software.* A.K. Peters, Wellesley, Massachusetts, 1994.

Joseph N.Q. Pham, Department of Electrical Engineering, University of California, Los Angeles, 66-124 Engineering 4, Los Angeles, California 90095-1594, U.S.A.

E-mail address: pham@ee.ucla.edu

Progress in Nonlinear Differential Equations
and Their Applications, Vol. 42

ASYMPTOTIC BEHAVIOR OF SEMIGROUPS OF HOLOMORPHIC MAPPINGS

MARK ELIN
BRAUDE COLLEGE

SIMEON REICH
TECHNION-ISRAEL INSTITUTE OF TECHNOLOGY

DAVID SHOIKHET
BRAUDE COLLEGE
TECHNION-ISRAEL INSTITUTE OF TECHNOLOGY

Abstract. We present several new results on the asymptotic behavior of nonlinear semigroups of holomorphic mappings on the open unit balls of complex Banach and Hilbert spaces.

Let X be a complex Banach space and let $D \subset X$ be a domain (that is, an open connected subset of X). Recall that a mapping $f : D \to X$ is called holomorphic if it is Fréchet differentiable at each point of D [9]. The set of all holomorphic mappings from D into X will be denoted by $Hol(D, X)$.

Definition 1. *Let D be a domain in X and let $g \in Hol(D, X)$. The mapping g is said to be a semi-complete vector field on D if the Cauchy problem*

$$\begin{cases} \frac{dv}{dt} + g(v) = 0 \\ v(0) = x \end{cases} \tag{1}$$

has a solution $v(\cdot, x) : \mathbb{R}^+ \to D$ which is well-defined on all of $\mathbb{R}^+$ for each initial datum $x \in D$.

Note that since any $g \in Hol(D, X)$ is locally bounded (hence locally Lipschitzian), this solution is unique and the family $\{S(t)\}_{t\geq 0}$, defined by $S(t) := v(t, \cdot)$, is a one-parameter semigroup (flow) of holomorphic self-mappings of D, i.e.,

$$\begin{cases} S(t+s) = S(t) \circ S(s) \quad \text{for all} \quad t, s, \geq 0 \\ S(0) = I, \end{cases} \tag{2}$$

where $\circ$ denotes the composition operation and I is the restriction of the identity operator on X to D. In the case where this flow consists of automorphisms of D it can be extended to a one parameter group and the Cauchy problem (1) has a unique solution $u(\cdot, x) : \mathbb{R} \to D$ defined on all of $\mathbb{R} = (-\infty, \infty)$ for each $x \in D$. The converse is also true. In such a situation the mapping g is said to be a complete vector field (see [4, 15]).

We observe that if the solution $S(t) = v(t, \cdot)$ of (1) is known, then $g \in Hol(D, X)$ can be recovered as the strong limit

$$g(x) = \lim_{t \to 0^+} \frac{x - S(t)x}{t} \tag{3}$$

$(S(t)x = v(t, x))$, i.e., $-g$ is the right derivative of the semigroup $S(t)$ at zero. Therefore the mapping g is seen to be the infinitesimal generator of the semigroup.

As a matter of fact, for hyperbolic domains the converse is also true: If for a given semigroup $\{S(t) : t \geq 0\}$ of holomorphic self-mappings of D, which is continuous in $t \geq 0$, the strong limit $g(x)$ in (3) exists for all $x \in D$, then $v(t, x) = S(t)x$ is the solution of the Cauchy problem (1) for all $t \geq 0$ and $x \in D$. In other words, $g \in Hol(D, X)$ is a generator of a flow on D if and only if it is a semi-complete vector field (see, for example, [12]).

Let D be a domain in X and let g be a semi-complete vector field on D. Suppose that $S(t)$ is the semigroup of holomorphic mappings generated by g. The uniqueness of the solution of the Cauchy problem (1) implies that the null point set of g in D coincides with the common fixed point set of $S(t)$, i.e.,

$$Null_D g = \bigcap_{t \geq 0} Fix_D S(t). \tag{4}$$

In the theory of evolution equations this set is usually called the stationary point set of the semigroup.

Definition 2. *Let g be a semi-complete vector field on a domain D in X with $Null_D g \neq \phi$. A point $a \in Null_D g$ is said to be locally uniformly attractive if the semigroup $S(t)$ generated by g converges to a in the topology of local uniform convergence over D (see [5], [10]).*

Definition 3. *Let D be a domain in a Banach space X and let $\mathcal{G}(D)$ be the family of all semi-complete vector fields on D. A mapping $g \in \mathcal{G}(D)$ is said to be a strongly semi-complete vector field if it has a unique null point in D which is a locally uniformly attractive fixed point for the semigroup generated by g.*

Let $\sigma(A)$ denote the spectrum of a bounded linear operator $A : X \to X$. It is known [12] that if D is a bounded domain, then $g \in \mathcal{G}(D)$ with $g(\tau) = 0$, $\tau \in D$, is strongly semi-complete if and only if there is $\varepsilon > 0$ such that $Re\lambda \geq \varepsilon > 0$ for all $\lambda \in \sigma(g'(\tau))$. Such a point τ is sometimes said to be strictly regular.

In this paper we will give several sufficient conditions for $f \in Hol(D, X)$ to be strongly semi-complete on the open unit ball D of X and obtain rates of convergence for the semigroups generated by such mappings.

Let X^* be the dual of X. By $\langle x, x^* \rangle$ we denote the action of a linear functional x^* in X^* on an element x of X.

The mapping $J : X \to 2^{X^*}$ defined by

$$J(x) = \left\{x^* \in X^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\right\}, \quad x \in X,$$

is called the (normalized) duality mapping.

Theorem 4. *Let D be the open unit ball in X and let $g \in Hol(D, X)$ satisfy*

$$Re\,\langle g(x), x^* \rangle \geq \alpha(\|x\|)\|x\|, \quad x \in D, \quad x^* \in J(x), \tag{5}$$

where α is a real continuous function on $[0, 1]$ such that

$$\alpha(1) = \omega > 0. \tag{6}$$

Then

(i) g is strongly semi-complete;

(ii) if $\{S(t)\}_{t \geq 0}$ is the semigroup generated by g, then for each pair of points x and y in D the following estimate holds:

$$\rho(S(t)x, S(t)y) \leq e^{-\frac{\omega}{2}t} \rho(x, y), \tag{7}$$

where ρ is the hyperbolic metric on D. In particular, if τ is the null point of g, then

$$\rho(S(t)x, \tau) \leq e^{-\frac{\omega}{2}t} \rho(x, \tau), \tag{8}$$

for all $x \in D$.

Proof. Consider for each $n = 1, 2, \ldots$ the mappings $g_n \in Hol(D, X)$ defined by

$$g_n(x) = x + \frac{t}{n} g(x) - y, \quad x \in D, \tag{9}$$

where $t \geq 0$ and $y \in D$. Let D_r be the open ball of radius $r \in [0, 1)$ centered at the origin. For all $x \in \partial D_r = \{x \in X : \|x\| = r\}$ and for all $x^* \in J(x)$ we have by (5),

$$\begin{aligned} Re\ \langle g_n(x), x^* \rangle &= \|x\|^2 + \tfrac{t}{n} Re\,\langle g(x), x^* \rangle - Re\,\langle y, x^* \rangle \geq r^2 + \tfrac{t}{n} r\alpha(r) - r\|y\| \\ &= r\left(r + \tfrac{t}{n}\alpha(r) - \|y\|\right). \end{aligned} \tag{10}$$

Since $\alpha(1) > 0$, it is easy to see that for sufficiently large n the equation

$$\varphi_n(r) := r + \frac{t}{n} \alpha(r) = 1 \tag{11}$$

has a solution $r_n \in [0,1)$.

Indeed, $\varphi_n(0) = \frac{t}{n}\alpha(0) \leq 1$ for $n \geq t|\alpha(0)|$ and $\varphi_n(1) = 1 + \frac{t}{n}\omega > 1$. The inequality (5) implies in turn that for such n and r_n, and for all x with $\|x\| = r_n$ and $x^* \in J(x)$, the following inequality holds:

$$Re\,\langle g_n(x), x^*\rangle \geq r_n(1 - \|y\|). \tag{12}$$

Since g_n is bounded on $\bar{D}_{r_n}$ [8], it follows by [3] that the equation

$$g_n(x) = x + \frac{t}{n}g(x) - y = 0$$

has a unique solution $x = J_{\frac{t}{n}}(y) := \left(I + \frac{t}{n}g\right)^{-1}(y) \in D_{r_n}$ for each $y \in D$. In other words, the resolvent mapping $J_{\frac{t}{n}}$ maps D into D_{r_n}.

It now follows by the Earle-Hamilton fixed point theorem [6] that $J_{\frac{t}{n}}$ has a unique fixed point τ in D. This point is also a null point of g. In addition, repeating the proof of this fixed point theorem we obtain the estimate

$$\rho\left(J_{\frac{t}{n}}(x), J_{\frac{t}{n}}(y)\right) \leq \frac{1}{1 + \left(\frac{t}{n}\right)\left(\frac{\alpha(r_n)}{2}\right)}\rho(x,y) \tag{13}$$

for each pair of points x and y in D.

Since $\alpha(r)$ is continuous on the interval $[0,1]$, it follows by (11) that $r_n \to 1$ and $\alpha(r_n) \to \omega$ as $n \to \infty$. Therefore, by using the exponential formula [13, 14]

$$S(t)x = \lim_{n\to\infty} J^n_{\frac{t}{n}}(x)$$

and (13), we get by induction the estimates (7) and (8). Theorem 4 is proved.

Example. Let $D = \Delta$ be the open unit disk in the complex plane $\mathbb{C}$ and let $g \in Hol(\Delta, \mathbb{C})$ be defined by

$$g(z) = a - \bar{a}z^2 + bz\frac{1 - cz}{1 + cz}$$

where $a \in \mathbb{C}$, $Re\,b > 0$ and $0 \leq c < 1$. If we take

$$\alpha(s) = -|a|(1 - s^2) + (Re\,b)s\frac{1 - cs}{1 + cs},$$

then we get

$$Re\,g(z)\bar{z} \geq \alpha(|z|)|z|$$

and $\alpha(1) = Re\,b\,\frac{1-c}{1+c} > 0$. Hence $g(z)$ is a strongly semi-complete vector field on Δ.

Remark. Note that if $g \in Hol(D, X)$ is known to be a semi-complete vector field on D, then condition (6) can be replaced by a slightly more general condition, namely,

$$\alpha(l) > 0 \quad \text{for some} \quad l \in (0,1], \tag{6'}$$

which will still ensure the validity of assertion (i) of Theorem 4. This implies the following very simple and interesting sufficient condition.

Recall that a bounded linear operator $A : X \to X$ is called strongly accretive if

$$Re\,\langle Ax, x^*\rangle \geq k\|x\|^2 \tag{14}$$

for some $k > 0$ and all $x \in X$, $x^* \in J(x)$.

Corollary 5. *Let $g \in \mathcal{G}(D)$ and suppose that the bounded linear operator $A = g'(0)$ is strongly accretive with constant $k > 0$. If*

$$k > 4\|g(0)\|, \tag{15}$$

then g is a strongly semi-complete vector field.

Proof. Consider the function $\alpha(s) = -\|g(0)\|(1 - s^2) + ks\,\frac{1-s}{1+s}$. Using (15) we see that $\alpha(1) = 0$ and $\alpha'(1) < 0$. Hence there is $l \in (0,1)$ such that $\alpha(l) > 0$. By [2] we know that

$$Re\,\langle g(x), x^*\rangle \geq \alpha(\|x\|)\|x\|, \quad x \in D.$$

Therefore the result follows from the above Remark.

Note that if $A = g'(0)$ is strongly accretive and $g(0) = 0$, then condition (15) is fulfilled automatically. Hence the origin is an attractive fixed point of the semigroup generated by g. Actually, this fact follows from more general considerations and in this case one can obtain an exponential rate of convergence.

To see this, we shall need the following lemma, the proof of which is omitted because it is similar to part of the proof of Theorem 4.

Lemma 6. *Let D be the open unit ball in a complex Banach space X and let $g \in Hol(D, X)$ satisfy the following condition:*

$$Re\,\langle g(x), x^*\rangle \geq \alpha(\|x\|)\|x\| \tag{16}$$

for all $x \in D$ and some $x^* \in J(x)$, where α is a real continuous function on $[0,1]$ such that for all $\mu \in [0,1]$ and for all sufficiently small $r > 0$ the equation

$$s + r\alpha(s) = \mu \tag{17}$$

has a unique solution $s(\mu)$ in $[0,1]$. Then

(i) g is a semi-complete vector-field on D;
(ii) if $\beta(t,s)$ is the solution of the Cauchy problem

$$(18)\qquad \begin{cases} \frac{\partial\beta(t,s)}{\partial t}+\alpha(\beta(t,s))=0 \\ \beta(0,s)=s\in[0,1) \end{cases}$$

and $v(t,x)$ is the solution of (1), then the following estimate holds:

$$(19)\qquad \|v(t,x)\|\le\beta(t,\|x\|),\quad x\in D.$$

Proposition 7. *Let $g\in\mathcal{G}(D)$ be such that $g(0)=0$ and $A=g'(0)$ is strongly accretive with $Re\,\langle Ax,x^*\rangle\ge k\|x\|^2$. Suppose that $\{S(t)\}_{t\ge0}$ is the semigroup generated by g. Then the following estimates hold:*

(i)

$$\|S(t)x\|\le\|x\|e^{-k\frac{1-\|x\|}{1+\|x\|}t},\quad x\in D,\ t\ge0;$$

(ii)

$$\frac{\|S(t)x\|}{(1-\|S(t)x\|)^2}\le e^{-kt}\frac{\|x\|}{(1-\|x\|)^2}.$$

Proof. Both estimates follow directly from Lemma 6 if we set

$$\alpha(s)=ks\frac{1-s}{1+s}.$$

In this case

$$\beta(t,s)\le se^{-k\frac{1-s}{1+s}t},$$

where $\{\beta(t,\cdot)\}$ is the semigroup generated by α.

Note that the estimate (i) is due to Gurganus [7, Proposition 2.5.4] while (ii) was obtained by Poreda [11]. Note also that the condition $g(0)=0$ is essential in their considerations as well as in the above approach.

For the case where X is a Hilbert space we would like to obtain more general estimates when g has an arbitrary null point which is strictly regular.

For a step in this direction we shall need the following general lemma. We omit its simple proof.

Lemma 8. *Let D and Ω be two domains in a complex Banach space X such that $\Omega=f(D)$ for some biholomorphic mapping $f:D\to\Omega$. Then there is a linear invertible operator T from the space $Hol(\Omega,X)$ onto the space $Hol(D,X)$ which takes the set $\mathcal{G}(\Omega)$ onto the set $\mathcal{G}(D)$ (i.e. $\mathcal{G}(D)=T(\mathcal{G}(\Omega))$. In other words, the classes of semi-complete vector fields on Ω and D are linearly isomorphic. Moreover, such an isomorphism $T:\mathcal{G}(\Omega)\to\mathcal{G}(D)$ can be given by the formulae*

$$(20)\qquad T(g)(\cdot)=[f'(\cdot)]^{-1}g(f(\cdot))$$

and

$$(21)\qquad T^{-1}(\varphi)(\cdot)=[f'(f^{-1}(\cdot))]\varphi(f^{-1}(\cdot)),$$

where $g\in\mathcal{G}(\Omega)$ and $\varphi\in\mathcal{G}(D)$.

Now let $\alpha:[0,1]\to\mathbb{R}^+$ be a continuous function on the interval $[0,1]$ such that for some $\delta>0$ and for each $r\in[0,\delta)$ the function $s+r\alpha(s)$ is increasing on $[0,1]$. We also assume that α satisfies the following range condition: For each $r\in[0,\delta)$ and for all $p\in[0,1]$, the equation

$$(22)\qquad s+r\alpha(s)=p$$

has a (unique) solution $s=s(r,p)\in[0,1]$. This solution is an increasing function of $p\in[0,1]$ for each fixed $r\in[0,\delta)$. Also, for each $t\ge0$ there exists the limit

$$(23)\qquad \beta(t,p)=\lim_{n\to\infty}s^{(n)}\left(\frac{1}{n}t,p\right),$$

where by $s^{(n)}(t,p)$ we denote the n-th fold iterate of $s(r,p)$, i.e., $s^{(0)}(r,p)=p$, $s^{(n)}(r,p)=s^{(n-1)}(r,s(r,p))$, $n=1,2,\dots$ This limit is the solution of the Cauchy problem

$$(24)\qquad \begin{cases} \frac{\partial\beta(t,p)}{\partial t}+\alpha(\beta(t,p))=0 \\ \beta(0,p)=p. \end{cases}$$

Assume now that X is a Hilbert space H and that g is a holomorphic mapping on $\mathbb{B}$, the open unit ball of H. As we have already mentioned, since g is locally bounded, hence locally Lipschitzian, the Cauchy problem

$$(25)\qquad \begin{cases} \frac{\partial u(t,x)}{\partial t}+g(u(t,x))=0 \\ u(0,x)=x \end{cases}$$

has a unique local solution $u=u(t,x)$ which is real-analytic in t in some neighborhood of zero, and holomorphic in a neighborhood U_x of x. We would like to compare this solution with the function $\beta(t,p)$ defined by (24) and to find out when $u=u(t,x)$ can be extended to a global solution of (25) defined on $\mathbb{R}^+\times\mathbb{B}$. To this end, we define, for a given $\tau\in\mathbb{B}$, a function $m_\tau:[0,\delta)\times U_x\to\mathbb{R}^+$ by

$$(26)\qquad m_\tau(t,y)=\|M_{-\tau}(u(t,y))\|,$$

where $M_{-\tau}$ is the Möbius transformation which takes τ to zero [6]. It is natural to compare this function with the function $\beta(t,\|M_{-\tau}(y)\|)$. As a matter of fact, we will see below that both local and global relations between these two functions are completely determined by their derivatives at merely one point.

For each $\tau\in\mathbb{B}$, let $N_\tau=\{g\in\mathcal{G}(\mathbb{B}):g(\tau)=0\}$.

Theorem 9. *Let $g \in Hol(\mathbb{B}, H)$ and let α, β and m be the functions defined above. Then the following assertions are equivalent:*
(i) for some $\tau \in \mathbb{B}$,

$$m_\tau(t,x) \le \beta(t, \|M_{-\tau}(x)\|)$$

whenever $u(t,x)$ is defined;
(ii) for some $\tau \in \mathbb{B}$,

$$\left.\frac{\partial m_\tau}{\partial t}\right|_{t=0^+} \le \left.\frac{\partial \beta}{\partial t}\right|_{t=0^+} = -\alpha(\|M_{-\tau}(x)\|);$$

(iii) for some $\tau \in \mathbb{B}$, the mapping g belongs to N_τ and satisfies

$$Re\,\langle g(x), x^*\rangle \ge \frac{\alpha(\|M_{-\tau}(x)\|)\|M_{-\tau}(x)\|}{\sigma(\tau,x)}$$

where $x \in \mathbb{B}$, $\sigma(x,y) = 1 - \|M_{-y}(x)\|^2$ and $z^* = \frac{z}{1-\|z\|^2} - \frac{\tau}{1-\langle\tau,z\rangle}$.

Thus, if one of the above assertions holds, then $u(t,x)$ has a unique extension to all of $\mathbb{R}^+ \times \mathbb{B}$ and the estimate (i) holds globally. The points τ in (i)-(iii) are one and the same.

Proof. Since $m_\tau(0,x) = \beta(0, \|M_{-\tau}(x)\|)$, the implication (i)⇒(ii) is clear. Further, by direct calculations we obtain that

$$\left.\frac{\partial m_\tau}{\partial t}\right|_{t=0^+} = -\frac{\sigma(\tau,x)}{\|M_{-\tau}(x)\|} Re\,\langle g(x), x^*\rangle .$$

Since the inequality in (iii) and [1] imply that $g \in N_\tau$, the implication (ii)⇒(iii) follows because $\left.\frac{\partial\beta(t,\|M_{-\tau}(x)\|)}{\partial t}\right|_{t=0^+} = -\alpha(\|M_{-\tau}(x)\|)$. Thus it remains to be shown that (iii)⇒(i). To this end, we use Lemma 8. Since $g \in N_\tau$, $u(t,x)$ is well-defined and belongs to $\mathbb{B}$ for all $(t,x) \in \mathbb{R}^+ \times \mathbb{B}$. Hence the operator $T : Hol(\mathbb{B}, H) \to Hol(\mathbb{B}, H)$ defined by

(27) $$T(g)(\cdot) = [(M_\tau)'(\cdot)]^{-1} g(M_\tau(\cdot)) = (M_{-\tau})'(M_\tau(\cdot))g(M_\tau(\cdot))$$

takes $g \in N_\tau$ to $\varphi \in N_0$. Now we observe that since the explicit expression for the linear operator A $(= A(x)) := (M_{-\tau})'(M_\tau(x))$ is

(28) $$A = \frac{1 + \langle x,\tau\rangle}{1 - \|\tau\|^2}\left(P_\tau + \sqrt{1-\|\tau\|^2}\,(I - P_\tau) + \langle\cdot,\tau\rangle\, x\right),$$

where P_τ is the orthogonal projection of the Hilbert space H onto the one-dimensional subspace spanned by τ,

we get

$$\langle\varphi(x), x\rangle = \langle Ag(M_\tau(x)), x\rangle = \langle g(M_\tau(x)), A^*x\rangle =$$

(29) $$= \frac{|1 + \langle x,\tau\rangle|^2}{1 - \|\tau\|^2}\left\langle g(M_\tau(x)),\ M_\tau(x) - \tau\frac{1-\|x\|^2}{1+\langle x,\tau\rangle}\right\rangle.$$

Also, if $z = M_\tau(x)$, then

$$1 + \langle x,\tau\rangle = 1 + \langle M_{-\tau}(z),\tau\rangle = \frac{1-\|\tau\|^2}{1-\langle z,\tau\rangle}$$

and

$$1 - \|x\|^2 = 1 - \|M_{-\tau}(z)\|^2 = \sigma(\tau, z).$$

Hence by (29) we obtain

$$\langle\varphi(x), x\rangle = \frac{1-\|\tau\|^2}{|1-\langle z,\tau\rangle|^2}\left\langle g(z), z - \tau\frac{1-\|z\|^2}{1-\langle\tau,z\rangle}\right\rangle =$$

$$= \sigma(\tau,z)\left\langle g(z), \frac{z}{1-\|z\|^2} - \frac{\tau}{1-\langle\tau,z\rangle}\right\rangle.$$

Thus (iii) implies that

(30) $$Re\,\langle\varphi(z), z\rangle \ge \alpha(\|z\|)\|z\|, \quad z \in \mathbb{B}.$$

Now, if $v(t,z)$ is the solution of the Cauchy problem

$$\begin{cases} \frac{\partial v}{\partial t} + \varphi(v) = 0 \\ v(0,z) = z, \end{cases}$$

then it follows by Lemma 6 that

$$\|v(t,z)\| \le \beta(t, \|z\|), \quad z \in \mathbb{B}.$$

But $v(t, M_{-\tau}(x)) = M_{-\tau}(u(t,x))$ and this concludes the proof.

Remark. If $\beta(t,s) \to 0$ as $t \to \infty$ for a fixed $s \in [0,1)$, then g is seen to be strongly semi-complete and assertion (i) establishes a rate of convergence of the semigroup $S(t) = u(t,\cdot)$ to its stationary point $\tau \in \mathbb{B}$. It would be nice, of course, to find a universal class of functions α such that this convergence will be of exponential type. We intend to study this problem elsewhere.

References

[1] D. Aharonov, M. Elin, S. Reich and D. Shoikhet, Parametric representations of semi-complete vector fields on the unit balls in $\mathbb{C}^n$ and in Hilbert space, *Atti. Accad. Naz. Lincei*, in press.

[2] D. Aharonov, S. Reich and D. Shoikhet, Flow invariance conditions for holomorphic mappings in Banach spaces, *Math. Proc. Royal Irish Academy*, in press.

[3] L. Aizenberg, S. Reich and D. Shoikhet, One-sided estimates for the existence of null points of holomorphic mappings in Banach spaces, *J. Math. Anal. Appl.* **203** (1996), 38-54.

[4] S. Dineen, *The Schwartz Lemma*, Clarendon Press, Oxford, 1989.

[5] T. Franzoni and E. Vesentini, *Holomorphic Maps and Invariant Distances*, North-Holland, Amsterdam, 1980.

[6] K. Goebel and S. Reich, *Uniform Convexity, Hyperbolic Geometry and Nonexpansive Mappings*, Marcel Dekker, New York and Basel, 1984.

[7] K.R. Gurganus, Φ-like holomorphic functions in $\mathbb{C}^n$ and Banach space, *Trans. Amer. Math. Soc.* **205** (1975), 389-406.

[8] L.A. Harris, S. Reich and D. Shoikhet, Dissipative holomorphic functions, Bloch radii, and the Schwarz lemma, *Technion Preprint Series* **1050** (1999), 1-14.

[9] E. Hille and R. S. Phillips, *Functional Analysis and Semigroups*, Amer. Math. Soc., Providence, RI, 1957.

[10] J.M. Isidro and L.L. Stacho, *Holomorphic Automorphism Groups in Banach Spaces: An Elementary Introduction*, North Holland, Amsterdam, 1984.

[11] T. Poreda, On generalized differential equations in Banach space, *Dissert. Math.* **310** (1991), 1-50.

[12] S. Reich and D. Shoikhet, Generation theory for semigroups of holomorphic mappings in Banach spaces, *Abstr. Appl. Anal.* **1** (1996), 1-44.

[13] S. Reich and D. Shoikhet, Semigroups and generators on convex domains with the hyperbolic metric, *Atti. Accad. Naz. Lincei* **8** (1997), 231-250.

[14] S. Reich and D. Shoikhet, Metric domains, holomorphic mappings and nonlinear semigroups, *Abstr. Appl. Anal.* **3** (1998), 203-228.

[15] H. Upmeier, *Jordan Algebras in Analysis, Operator Theory and Quantum Mechanics*, CBMS, Reg. Conf. Ser. in Math., Vol. **67**, Amer. Math. Soc., Providence, RI, 1986.

MARK ELIN, DEPARTMENT OF APPLIED MATHEMATICS, BRAUDE COLLEGE, 21982 KARMIEL, ISRAEL

SIMEON REICH, DEPARTMENT OF MATHEMATICS, TECHNION-ISRAEL INSTITUTE OF TECHNOLOGY, 32000 HAIFA, ISRAEL
E-mail address: sreich@techunix.technion.ac.il

DAVID SHOIKHET, DEPARTMENT OF APPLIED MATHEMATICS, BRAUDE COLLEGE, 21982 KARMIEL; DEPARTMENT OF MATHEMATICS, TECHNION-ISRAEL INSTITUTE OF TECHNOLOGY, 32000 HAIFA, ISRAEL

Progress in Nonlinear Differential Equations
and Their Applications, Vol. 42

FRACTIONAL POWERS OF OPERATORS VIA HYPERSINGULAR INTEGRALS

STEFAN SAMKO
UNIVERSIDADE DO ALGARVE

Introduction

The well known Balakrishnan formula represents the fractional power $(-A)^\alpha$ in case of the generator A of a semigroup $T_t, t > 0$, in terms of a (hyper)-singular integral with respect to the variable $t \in R^1_+$, that is,

$$(-A)^\alpha f = \frac{1}{\Gamma(-\alpha)} \int_0^\infty t^{-\alpha-1} (T_t - I) f dt ,$$

where $0 < \alpha < 1, \varphi \in D(A)$, and I is the identity operator. In the case $\alpha > 1$, this formula can be written with the usage of "finite differences" $(T_t - I)^\ell, \ell = 1, 2, 3, ..., \ell > \alpha$:

$$(1) \qquad (-A)^\alpha f = \frac{1}{\varkappa(\alpha, \ell)} \int_0^\infty t^{-\alpha-1} (I - T_t)^\ell f dt , \ell > \alpha,$$

with $\varkappa(\alpha, \ell) = -\Gamma(-\alpha) A_\alpha(\ell)$, where $A_\alpha(\ell) = \sum_{k=0}^{\ell} (-1)^{k-1} \binom{\ell}{k}$. In particular, the fractional power of the Laplace operator is given by (1) with $T_t = P_t$ where P_t is the Poisson semigroup of operators:

$$P_t f = c_n \int_{R^n} \frac{t f(x-y)}{(|x|^2 + t^2)^{(n+1)/2}} dy , \quad t > 0.$$

On the other hand, positive fractional powers of the Laplace operator can be given also in the form

$$(2) \qquad (-\Delta)^{\frac{\alpha}{2}} f = \frac{1}{d_{n,\ell}(\alpha)} \lim_{\varepsilon \to 0} \int_{|y| > \varepsilon} \frac{(\Delta_y^\ell f)(x)}{|y|^{n+\alpha}} dy ,$$

see [23], p.56, which is also known as the Riesz fractional derivative and denoted as $\mathbb{D}^\alpha f = (-\Delta)^{\frac{\alpha}{2}} f$. Here $d_{n,\ell}(\alpha)$ is the known normalizing constant and the finite difference $(\Delta_y^\ell f)(x)$, generated by the standard shift $\tau_y f = f(x-y)$, may be centered one and then $0 < \alpha < \ell$, see [15], or a non-centered and then $0 < \alpha < 2\left[\frac{\ell}{2}\right]$, where $\left[\frac{\alpha}{2}\right]$ stands for the entire part of $\frac{\alpha}{2}$, see [23], Ch.3, Section 1.

Hypersingular constructions of the type (2) can be used for an effective realization of fractional powers of some differential operators of mathematical physics, such as fractional powers $(I - \Delta)^{\frac{\alpha}{2}}, \Delta$ being the Laplace operator; fractional powers $(-\Delta_x + \frac{\partial}{\partial t})^{\frac{\alpha}{2}}$ of *parabolic (heat) operator* or $(I - \Delta_x + \frac{\partial}{\partial t})^{\frac{\alpha}{2}}$, the Laplace operator being applied in the spatial variable $x = (x_1, \cdots, x_n)$; fractional powers of the *wave operator*, of *Schrödinger operator* and others.