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Surjective isometries between real JB*-triples

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1. *Introduction*

In [**19**], R. Kadison proved that every surjective linear isometry $\Phi: \mathcal{A} \to \mathcal{B}$ between two unital C*-algebras has the form

$$\Phi(x) = uT(x),\ x \in \mathcal{A},$$

where u is a unitary element in $\mathcal{B}$ and T is a Jordan $*$-isomorphism from $\mathcal{A}$ onto $\mathcal{B}$. This result extends the classical Banach–Stone theorem [**3**, **32**] obtained in the 1930s to non-abelian unital C*-algebras. A. L. Paterson and A. M. Sinclair extended Kadison's result to surjective isometries between C*-algebras by replacing the unitary element u by a unitary element in the multiplier C*-algebra of the range algebra [**28**]. Thus, every surjective linear isometry between C*-algebras preserves the triple products as

$$\{x, y, z\} = 2^{-1}(xy^*z + zy^*x).$$

In the non-associative case, J. Wright and M. Youngson [**35**, theorem 6], established that every unital surjective linear isometry between two unital JB*-algebras was a Jordan $*$-isomorphism. In 1995, J. M. Isidro and A. Rodríguez [**18**, theorem 1·9] showed that every surjective linear isometry Φ between two JB-algebras has the form

$$\Phi(x) = bT(x),$$

where b is a central symmetry in the multiplier algebra of the range JB-algebra and T is a surjective algebra isomorphism. It also follows, according to Isidro and Rodríguez [**18**, theorem 1·9], that a bijective linear map Φ between two JB-algebras is an isometry if and only if Φ is a triple-isomorphism with respect to the triple product

$$\{x, y, z\} = (x \circ y) \circ z + (z \circ y) \circ x - (x \circ z) \circ y.$$

Other extensions of the Banach–Stone theorem for non-associative Banach algebras can be found in [**30**] and [**20**].

C*-algebras and JB*-algebras belong to a more general class of algebraic-topological structures, known as (complex) JB*-triples (cf. Section 2 for the definition).

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W. Kaup's version of the Banach–Stone theorem for JB*-triples [**22**], establishes that a bijective linear map Φ between two JB*-triples is an isometry if and only if it is a triple isomorphism. An alternative proof of Kaup's theorem was obtained by T. Dang, Y. Friedman and B. Russo in [**8**].

Dang studied real linear surjective isometries between JB*-triples [**6**]. An examination of quantum mechanics reveals the existence of invertible affine maps on the unit sphere of the dual of JB*-triples and these maps coincide with the adjoints of real (not necessarily complex) linear surjective isometries (compare [**6**, first paragraph on page 972]). In general the Banach–Stone theorem does not hold good for real linear isometries between JB*-triples (cf. [**6**, remark 2·7]). Dang showed, however, [**6**, theorem 3·1] that if $\Phi: \mathcal{E} \rightarrow \mathcal{F}$ is a surjective real linear isometry between two JB*-triples, so that $\mathcal{E}^{**}$ does not contain non-trivial rank-1 Cartan factors, then Φ is a real linear triple isomorphism.

The structures of C*-algebras, JB*-algebras and JB*-triples have been generalised to real C*-algebras, J*B-algebras and real JB*-triples respectively (cf. [**5**, **13**], [**2**] and [**17**], cf. also Section 2 for completeness). The Banach–Stone theorem in these latter structures constitutes another line of generalisation. Several authors have obtained Banach–Stone theorems corresponding to some of these structures.

In 1990, M. Grzesiak proved an extension of the Banach–Stone theorem for abelian real C*-algebras [**24**, corollary 5·2·4]. For not-necessarily abelian real C*-algebras the Banach–Stone theorem was obtained by C.-H. H. Chu, T. Dang, B. Russo and B. Ventura (cf. [**5**, theorem 6·4]), who showed that a linear bijection between two real C*-algebras is an isometry if and only if it is a real triple isomorphism.

The study of surjective linear isometries between real JB*-triples begins in [**17**], where the authors prove that every triple isomorphism between real JB*-triples is an isometry, [**17**, theorem 4·8]. As we have seen before, however, not every surjective isometry is a triple isomorphism (cf. [**6**, remark 2·7]). Recently, W. Kaup [**23**] obtained a Banach–Stone theorem for some real Cartan factors (real forms of complex Cartan factors, see Section 2 for completeness). In [**23**, theorem 5·18] Kaup proved that every real linear mapping from a non-exceptional real or complex rank >1 Cartan factor onto a real JBW*-triple is a real triple isomorphism if and only if it is an isometry. He left open the cases of the two exceptional real Cartan factors $V^{\mathbb{O}_0}$ and $VI^{\mathbb{O}_0}$ [**23**, page 217]. This problem forms the starting point of our paper.

As our first goal, Proposition 2·14, we conclude that every surjective linear isometry between two real reduced Cartan factors is a triple isomorphism. The novelty of our technique resides in the concept of a real reduced JB*-triple, as already introduced by O. Loos [**26**, 11.9]. Most real Cartan factors are real reduced Cartan factors, especially the exceptional real Cartan factors $V^{\mathbb{O}_0}$ and $VI^{\mathbb{O}_0}$. This fact, together with our Proposition 2·14, provides a positive answer to the problem left open by Kaup. Moreover, our result for real reduced Cartan factors together with Kaup's result for non-reduced real and complex Cartan factors allow us to get rid of the hypothesis of non-exceptionality in [**23**, theorem 5·18] (cf. Corollary 2·16). Finally, in our main result, Theorem 3·2, we extend Dang's Banach–Stone theorem to real JB*-triples. As a consequence of our main result we also obtain a Banach–Stone theorem for J*B-algebras (Corollary 3·4).

Let X be either a real or complex Banach space and let $S \subset X$. Then X^* and S° denote the dual space of X and the polar of S in X^* respectively. If X is a dual

Banach space, X_* will denote a predual of X and $S_\circ$ will stand for the pre-polar of S in X_*. The canonical embedding of X into X^{**} is denoted by $j\colon X \to X^{**}$. The Banach space of all bounded linear operators between two Banach spaces, X and Y, is denoted by $\mathcal{L}(X, Y)$ and $\mathcal{L}(X)$ stands for $\mathcal{L}(X, X)$.

2. *Surjective isometries between real Cartan factors*

Recall that a *JB*-triple* is a complex Banach space, $\mathcal{E}$, together with a triple product $\{.,.,.\}\colon \mathcal{E} \times \mathcal{E} \times \mathcal{E} \to \mathcal{E}$, which is continuous, symmetric and linear in the outer variables and conjugate linear in the inner one, satisfying:

(a) *Jordan Identity*: for all $a, b, x, y, z \in \mathcal{E}$

$$L(a,b)\{x,y,z\} = \{L(a,b)x,y,z\} - \{x,L(b,a)y,z\} + \{x,y,L(a,b)z\},$$

where $L(a,b)x := \{a,b,x\}$;

(b) for each $a \in \mathcal{E}$ the operator $L(a,a)$ is hermitian with non-negative spectrum and $\|L(a,a)\| = \|a\|^2$.

Every C*-algebra is a complex JB*-triple with respect to the triple product $\{x,y,z\} = \frac{1}{2}(xy^*z + zy^*x)$, and in the same way every JB*-algebra with respect to $\{a,b,c\} = (a \circ b^*) \circ c + (c \circ b^*) \circ a - (a \circ c) \circ b^*$.

A real Banach space, E, together with a trilinear map $\{.,.,.\}\colon E \times E \times E \to E$ is called a *real JB*-triple* if there is a JB*-triple, $\mathcal{E}$, and an $\mathbb{R}$-linear isometry, λ, from E to $\mathcal{E}$ so that

$$\lambda\{x,y,z\} = \{\lambda x, \lambda y, \lambda z\}$$

for all x, y, z in E (cf. [**17**]).

Real JB*-triples are essentially the closed real subtriples of complex JB*-triples and, by [**17**, proposition 2·2], given a real JB*-triple, E, there exists a unique complex JB*-triple, $\mathcal{E}$, and a unique *conjugation* (conjugate linear and isometric mapping of period 2), τ, on $\mathcal{E}$ so that $E = \mathcal{E}^\tau := \{x \in \mathcal{E} : \tau(x) = x\}$. In fact, $\mathcal{E} = E \oplus iE$ is the complexification of the vector space, E, with a triple product, extending in a natural way the triple product of E, and a suitable norm. Throughout the paper the complexification of a real JB*-triple E, equipped with the structure of a complex JB*-triple, will be denoted by $\widehat{E}$. Given a complex Banach space X and a conjugation τ on X, the real Banach space $X^\tau = \{x \in X : \tau(x) = x\}$ is called a real form of X. Given a conjugation τ on a complex JB*-triple then, according to Kaup's Banach–Stone theorem for complex JB*-triples [**22**], we know that τ is a conjugate-linear triple isomorphism and hence that the real JB*-triples coincide with the real forms of complex JB*-triples.

Every complex JB*-triple is a real JB*-triple when considered as a real Banach space. The class of real JB*-triples also includes all JB-algebras [**14**], all real C*-algebras [**13**] and all J*B-algebras [**2**]. Other examples of real JB*-triples are the so-called real and complex Cartan factors, which are dealt with below.

Cartan Factors

Cartan factors can be classified into six different types (cf. [**21**]). The type-1 Cartan factor, denoted by $I_{n,m}$, is the complex Banach space, $L(H, K)$, of bounded linear operators between two complex Hilbert spaces, H and K, of dimensions n and m respectively, where the triple product is defined by $\{x,y,z\} = 2^{-1}(xy^*z + zy^*x)$.

It is worth remembering that, given a conjugation, q, on a complex Hilbert space, H, we can define the following linear involution $x \mapsto x^t := qx^*q$ on $L(H)$. The type-2 Cartan factor, denoted by II_n, is the subtriple of $L(H)$ formed by the skew-symmetric operators for the involution t; in the same way the type-3 Cartan factor, III_n, is formed by the t-symmetric operators. Moreover, II_n and III_n are, up to isomorphism, independent of the conjugation q on H.

The type-4 Cartan factor, IV_n (also called *complex spin factor*) is a complex Hilbert space of dimension n provided with a conjugation $x \mapsto \overline{x}$, triple product

$$\{x, y, z\} = \langle x/y\rangle\, z + \langle z/y\rangle\, x - \langle x/\bar{z}\rangle\, \bar{y},$$

and norm given by $\|x\|^2 = \langle x/x\rangle + \sqrt{\langle x/x\rangle^2 - |\langle x/\overline{x}\rangle|^2}$.

The type-6 Cartan factor is the 27-dimensional exceptional JB*-algebra $VI = H_3(\mathbb{O}^{\mathbb{C}})$ of all hermitian three by three matrices with entries in the complex octonions $\mathbb{O}^{\mathbb{C}}$ [**37**]. The type-5 Cartan factor is the subtriple, $V = M_{1,2}(\mathbb{O}^{\mathbb{C}})$, of the type-6 Cartan factor formed by all one by two matrices with entries in $\mathbb{O}^{\mathbb{C}}$. We also refer to [**1**, **14**, **26**, **31**, **34**] as basic reading concerning the exceptional Cartan factors.

In accordance with [**23**], real Cartan factors are real forms of complex Cartan factors. They are completely described in [**23**, corollary 4·4] and [**26**, pages 11·5–11·7]. Real Cartan factors can be described, up to isomorphism, as follows:

Let X and Y be two real Hilbert spaces of dimensions n and m respectively. Let P and Q be two Hilbert spaces of dimensions p and q respectively, over the quaternion field $\mathbb{H}$, and finally, let H be a complex Hilbert space of dimension n.

(i) $I^{\mathbb{R}}_{n,m} := \mathcal{L}(X, Y)$

(ii) $I^{\mathbb{H}}_{2p,2q} := \mathcal{L}(P, Q)$

(iii) $I^{\mathbb{C}}_{n,n} := \{z \in \mathcal{L}(H) : z^* = z\}$

(iv) $II^{\mathbb{R}}_{n} := \{x \in \mathcal{L}(X) : x^* = -x\}$

(v) $II^{\mathbb{H}}_{2p} := \{w \in \mathcal{L}(P) : w^* = w\}$

(vi) $III^{\mathbb{R}}_{n} := \{x \in \mathcal{L}(X) : x^* = x\}$

(vii) $III^{\mathbb{H}}_{2p} := \{w \in \mathcal{L}(P) : w^* = -w\}$

(viii) $IV^{r,s}_n := E$, where $E = X_1 \oplus^{\ell_1} X_2$ and X_1,X_2 are closed linear subspaces, of dimensions r and s, of a real Hilbert space, X, of dimension greater or equal to three, so that $X_2 = X_1{}^{\perp}$, with triple product

$$\{x, y, z\} = \langle x/y\rangle\, z + \langle z/y\rangle\, x - \langle x/\bar{z}\rangle\, \bar{y},$$

where $\langle ./.\rangle$ is the inner product in X and the involution $x \to \bar{x}$ on E is defined by $\bar{x} = (x_1, -x_2)$ for every $x = (x_1, x_2)$. This factor is known as a *real spin factor*.

(ix) $V^{\mathbb{O}_0} := M_{1,2}(\mathbb{O}_0)$

(x) $V^{\mathbb{O}} := M_{1,2}(\mathbb{O})$

(xi) $VI^{\mathbb{O}_0} := H_3(\mathbb{O}_0)$

(xii) $VI^{\mathbb{O}} := H_3(\mathbb{O})$

where $\mathbb{O}_0$ is the real split Cayley algebra over the field of the real numbers and $\mathbb{O}$ is the real division Cayley algebra (known also as the algebra of real division octonions). The real Cartan factors (ix)–(xii) are called *exceptional real Cartan factors*.

Given a real or complex JB*-triple U and a tripotent $e \in U$ (i.e. $\{e,e,e\} = e$), then e induces the following two decompositions of U

$$U = U_0(e) \oplus U_1(e) \oplus U_2(e) = U^1(e) \oplus U^{-1}(e) \oplus U^0(e)$$

where $U_k(e) := \{x \in U : L(e,e)x = \frac{k}{2}x\}$ is a subtriple of U and $U^k(e) := \{x \in U : Q(e)(x) := \{e,x,e\} = kx\}$ is a real Banach subspace of U (compare [**26**, theorem 3·13]). The natural projections of U onto $U_k(e)$ and $U^k(e)$ will be denoted by $P_k(e)$ and $P^k(e)$, respectively. The first decomposition is called the Peirce decomposition with respect to the tripotent, e. The following *Peirce rules* are satisfied for the Peirce decomposition

$$\{U_i(e), U_j(e), U_k(e)\} \subseteq U_{i-j+k}(e), \text{ where } i,j,k = 0,1,2 \text{ and}$$

$$U_l(e) = 0 \text{ for } l \neq 0,1,2.$$

$$\{U_0(e), U_2(e), U\} = \{U_2(e), U_0(e), U\} = 0.$$

The following identities and rules are also satisfied

$$U_2(e) = U^1(e) \oplus U^{-1}(e), \qquad U_1(e) \oplus U_0(e) = U^0(e)$$

$$\{U^i(e), U^j(e), U^k(e)\} \subseteq U^{ijk}(e), \text{ whenever } ijk \neq 0.$$

It is known that for every tripotent e in a real or complex JB*-triple U, the mapping $Q(e)$ is a period-2 conjugate linear automorphism of $U_2(e)$.

Two non-zero elements, x and y, in a real or complex JB*-triple, U, are said to be *orthogonal*, and we write $x \perp y$, if $L(x,y) = 0$ (equivalently $L(y,x) = 0$). Thus if e and f are tripotents in U, we have $e \perp f$ if and only if $e \in U_0(f)$. The tripotents e and f are said to be *colinear*, $e \top f$, if $e \in U_1(f)$ and $f \in U_1(e)$. We say that e *governs* f, $e \vdash f$, whenever $f \in U_2(e)$ and $e \in U_1(f)$. A tripotent e is called *unitary* if $U = U_2(e)$, whilst a non-zero tripotent e is called *minimal* if $U^1(e) = \mathbb{R}e$ (since in the complex case $U^{-1}(e) = i\, U^1(e)$, this definition is equivalent to $U_2(e) = \mathbb{C}e$).

A real or complex JBW*-triple is a real or complex JB*-triple that is a dual Banach space. Every real or complex JBW*-triple has a unique predual and its triple product is separately weak*-continuous (compare [**4**] and [**27**]).

According to [**26**, 11.9], we say that a real JB*-triple E is *reduced* whenever $E_2(e) = \mathbb{R}e$ (equivalently, $E^{-1}(e) = 0$) for every minimal tripotent $e \in E$. The reduced real Cartan factors have been studied and classified in [**26**, 11.9] for the finite dimensional case and in [**23**, table 1] (in the last case they correspond to those factors with the parameter $z = 1$). The non-reduced real Cartan factors are the following: $IV_n^{n,0}$, $V^{\mathbb{O}}$, $I_{2p,2q}^{\mathbb{H}}$ and $II_{2p}^{\mathbb{H}}$.

Remark 2·1. Let E be a real Cartan factor of type $IV_n^{n,0}$. It is easy to check that every norm-one element, e, in E is a minimal tripotent which is also unitary (i.e. $E_2(e) = E$), and thus $E_1(e) = 0$.

Let E now denote the real Cartan factor $V^{\mathbb{O}}$ and let $e = (1,0)$ in E. In this case we can easily see that

$$E_1(e) = \{(0,z) : z \in \mathbb{O}\}, \quad E^1(e) = \mathbb{R}e \text{ and}$$

$$E^{-1}(e) = \{(y,0) : y \in \mathrm{Span}_{\mathbb{R}}\{e_1, \dots, e_7\}\},$$

where $\{1, e_1, \ldots, e_7\}$ denotes the canonical basis of $\mathbb{O}$. Every tripotent element in $E_1(e)$ must have the form $f = (0, z)$, with $zz^* = 1$. The tripotent e lies in $E_1(f)$.

In the two remaining non-reduced real Cartan factors, $I^{\mathbb{H}}_{2p,2q}$ and $III^{\mathbb{H}}_{2p}$, it is easy to see that, given a minimal tripotent, e, and a tripotent, $f \in E_1(e)$, we have $e \in E_1(f) \cup E_2(f)$.

The next lemma shows that the situation in Remark 2·1 for non-reduced real Cartan factors is the same for every real JB*-triple.

Lemma 2·2. *Let E be a real or a complex JB*-triple, v a minimal tripotent in E and e a tripotent in $E_1(v)$. Then $v \in E_2(e) \cup E_1(e)$.*

Proof. If E is a complex JB*-triple, the proof follows from [**9**, lemma 2·1].

Suppose now that E is a real JB*-triple. By [**17**, lemma 4·2 and theorem 4·4] the bidual, E^{**}, of E is a real JBW*-triple with a separate weak*-continuous triple product, extending the product of E. Therefore, given a tripotent $e \in E$ we can ascertain via Banach–Alaoglu's theorem that

$$(E^{**})^j(e) = \overline{E^j(e)}^{w^*} \quad \text{and} \quad (E^{**})_k(e) = \overline{E_k(e)}^{w^*},$$

for every $j \in \{0, 1, -1\}$, $k \in \{0, 1, 2\}$ and consequently every minimal tripotent in E is also a minimal tripotent in E^{**}. Thus, we can assume from now on that E is a real JBW*-triple.

By [**29**, theorem 3·6] there are two weak*-closed ideals, A and N of E, so that

$$E = A \oplus^\infty N,$$

where A is the weak*-closed real linear span of all minimal tripotents of E, N contains no minimal tripotents and $A \perp N$. Moreover it follows from the proof of [**29**, theorem 3·6] that A can be decomposed in the following ℓ_∞-sum

$$A = \oplus^\infty C_\alpha,$$

where each C_α is a real Cartan factor or a complex Cartan factor when considered as a real JB*-triple and every minimal tripotent of E belongs to a unique C_α. Thus we can suppose that $v \in C_\gamma$ for a unique γ. Since for $\beta \neq \alpha$ we have $C_\alpha \perp C_\beta$, then $E_1(v) = (C_\gamma)_1(v)$. Therefore we can assume that $E = C_\gamma$ is either a real Cartan factor or a complex Cartan factor when considered as real.

If E is a complex Cartan factor when considered as real the statement follows from [**9**, lemma 2·1].

Suppose now that E is a reduced real Cartan factor. Then v is a minimal tripotent in $\widehat{E}$, the complexification of E. Therefore $v \in \widehat{E}_2(e) \cup \widehat{E}_1(e)$. But, since $\tau(v) = v$ and $\tau(e) = e$, we have $v \in (\widehat{E}_2(e) \cup \widehat{E}_1(e))^\tau = E_2(e) \cup E_1(e)$.

Finally, we assume that E is a non-reduced real Cartan factor. By [**23**, table 1, page 210] (see also [**26**, 11·9]), E is one of the following $I^{\mathbb{H}}_{2p,2q}$, $III^{\mathbb{H}}_{2p}$, $IV^{n,0}_n$, $V^{\mathbb{O}}$. By [**23**, proposition 5·8], given two minimal tripotents, u and $v \in E$, there is an automorphism of E interchanging u and v. This implies that to finish the proof it is enough to check the statement of the lemma for one particular minimal tripotent in each one of the previous four factors . Therefore, the statement follows from Remark 2·1 above.

Let U be a real or complex JB*-triple. Recall (cf. [**7**]) that an ordered triplet $(v, u, \tilde{v})$ of tripotents in U, is called a *triangle* if $v \perp \tilde{v}$, $u \vdash v$, $u \vdash \tilde{v}$ and $v = Q(u)\tilde{v}$. If $u \vdash v$, we say that (v, u) forms a *pre-triangle*. It is easy to see that $(v, u, \tilde{v})$ forms a triangle with $\tilde{v} = Q(u)v$. An ordered quadruple (u_1, u_2, u_3, u_4) of tripotents is called a *quadrangle* if $u_1 \perp u_3$, $u_2 \perp u_4$, $u_1 \top u_2 \top u_3 \top u_4 \top u_1$ and $u_4 = 2\{u_1, u_2, u_3\}$ (the Jordan Identity maintains the above equality to still be true if the indices are permutated cyclically, e.g. $u_2 = 2\{u_3, u_4, u_1\}$). If u_1, u_2, u_3 are tripotents in such a way that $u_1 \perp u_3$, $u_1 \top u_2 \top u_3$, we say that (u_1, u_2, u_3) forms a *pre-quadrangle*. In this case $u_4 = 2\{u_1, u_2, u_3\}$ is a tripotent and (u_1, u_2, u_3, u_4) forms a quadrangle. The following lemma can be obtained by applying Peirce rules and the definition of quadrangle.

LEMMA 2·3. *Let (u_1, u_2, u_3, u_4) be a quadrangle in a real or complex JB*-triple U. Then $\varepsilon(u_1 + u_2 + u_3 + u_4)$ is a tripotent if and only if $|\varepsilon| = 2^{-1}$, and in the same way $\varepsilon(u_1 + u_2 + u_3 - u_4)$ is a tripotent if and only if $|\varepsilon| = 2^{-\frac{1}{2}}$.*

Lemma 2·2 allows us to translate the result known as "Triple System Analyzer" (cf. [**7**, proposition 2·1]) to the setting of real JB*-triples, replacing [**9**, lemma 2·1] in the proof of [**7**, proposition 2·1] with Lemma 2·2.

PROPOSITION 2·4. *Let U be a real or complex JBW*-triple containing a minimal tripotent v. Let u be a tripotent in $U_1(v)$. Then exactly one of the following 3 cases will pertain*:

(i) *u is minimal in U; this happens if and only if u and v are colinear*;

(ii) *u is not minimal in U but is minimal in $U_1(v)$; in this case (v, u) forms a pre-triangle and $\tilde{v} = \{u, v, u\}$ is a minimal tripotent in U*;

(iii) *if u is not minimal in $U_1(v)$, then two orthogonal minimal tripotents of U, u_1, $\tilde{u}_1$ exist, both contained in $U_1(v)$, so that $u = u_1 + \tilde{u}_1$. Moreover $\tilde{v} = \{u, v, u\}$ is a minimal tripotent of U and $(u_1, v, \tilde{u}_1, \tilde{v})$ forms a quadrangle.*

Let U be a real or complex JB*-triple. We recall that the rank of U, $r(U)$, is the minimal cardinal number r satisfying $card(S) \leqslant r$ whenever S is an orthogonal subset of U, i.e. $0 \notin S$ and $x \perp y$ for every $x \neq y$ in S. The rank of a real or complex JB*-triple is preserved by surjective isometries (cf. Proposition 2·9).

COROLLARY 2·5. *Let v be a minimal tripotent in a real or complex JB*-triple U. Then $r(U_1(v)) \leqslant 2$, i.e., $U_1(v)$ does not contain more than two mutually orthogonal tripotents.*

Remark 2·6. Let $E = X_1 \oplus^{\ell_1} X_2$ be a real spin factor of dimension $\geqslant 3$.

If X_1 and X_2 are both non-zero, then it is easy to check that the set of minimal tripotents of E is

$$\mathrm{MinTrip}(E) = \left\{ \frac{1}{2}(x_1 + x_2) : x_1 \in X_1, x_2 \in X_2 \text{ and } \|x_1\| = \|x_2\| = 1 \right\}.$$

Let $u = \frac{1}{2}(x_1 + x_2)$ be a minimal tripotent in E. It can be seen that $E_0(u) = \mathbb{R}\bar{u}$, $E_2(u) = \mathbb{R}u$, and $E_1(u) = (\{x_1\}^\perp \cap X_1) \oplus (\{x_2\}^\perp \cap X_2)$.

If $X_i = 0$ for some $i = 1, 2$, then

$$\mathrm{MinTrip}(E) = \{x : x \in E, \|x\| = 1\}.$$

In the latter case, given a tripotent $e \in E$ we have $E_0(e) = E_1(e) = 0$, $E_2(e) = E$, $E^1(e) = \mathbb{R}e$, and $E^{-1}(e) = \{e\}^\perp$.

When X_1 and X_2 are non-zero then $r(E)=2$, while $r(E)=1$ whenever X_1 or X_2 is zero.

Let $(u, v, \tilde{u})$ be a triangle in E with $u, \tilde{u}$ minimal. We note that in this case $r(E)=2$ and hence $X_1, X_2 \neq 0$. One can check that $u, v, \tilde{u}$ must have the following form $u=\frac{1}{2}(x_1+x_2)$, $\tilde{u}= \pm \frac{1}{2}(x_1-x_2)$ and $v=y$, where $\|x_1\|=\|x_2\|=1$, y is in the disjoint union of $\left(\{x_1\}^{\perp} \cap X_1\right)$ and $\left(\{x_2\}^{\perp} \cap X_2\right)$ and $\|y\|=1$. Moreover, since $\{u, v, \tilde{u}\} = \frac{1}{2}v$, it may be concluded that $\varepsilon(u+v+\tilde{u})$ is a minimal tripotent if and only if $\varepsilon = \pm \frac{1}{2}$, whilst $\varepsilon(u+v-\tilde{u})$ is a tripotent if and only if $\varepsilon = \pm \frac{1}{\sqrt{2}}$.

The following lemma is well known and immediately follows from the classification of JB*-triples of finite rank (c.f. [**21**, theorem 4·10]).

Lemma 2·7. *Let $\mathcal{U}$ be a JBW*-triple and let u and v be two orthogonal minimal tripotents in $\mathcal{U}$. Then $\mathcal{U}_2(u+v)$ is either $\mathbb{C} \oplus^{\infty} \mathbb{C}$ or a spin factor.*

In [**23**, page 215], Kaup affirms that $E_2(u+v)$ is a spin factor whenever u and v are minimal tripotents in the real JBW*-triple $E=I^{\mathbb{R}}_{n,m}$. Since the latter is a real reduced JBW*-triple, our next result includes the above affirmation.

Corollary 2·8. *Let E be a real reduced JBW*-triple and let u, v be orthogonal minimal tripotents in E. Then $E_2(u+v)$ is either $\mathbb{R} \oplus^{\infty} \mathbb{R}$ or a real spin factor.*

Proof. Clearly $E_2(u+v)=\mathbb{R}u \oplus \mathbb{R}v \oplus [E_1(u) \cap E_1(v)]$. If $E_1(u) \cap E_1(v)=\{0\}$, then $E_2(u+v)$ can be identified as a real JBW*-triple with $\mathbb{R} \oplus^{\infty} \mathbb{R}$. Otherwise, since u and v are two orthogonal minimal tripotents in $\widehat{E}$, then $\widehat{E}_2(u+v)$ is a JBW*-triple with $dim \widehat{E}_2(u+v) \geqslant 3$. Now, by the above lemma, we may conclude that $\widehat{E}_2(u+v)$ is a spin factor, and thus $E_2(u+v)$ is a real spin factor.

The next proposition summarizes some known facts about surjective isometries between real JB*-triples from [**17**].

Proposition 2·9. *Let $\Phi: E \to F$ be a surjective linear isometry between two real JB*-triples. The following assertions hold*:

(i) $\Phi(x) \perp \Phi(y)$ *iff* $x \perp y$;

(ii) *for every tripotent $e \in E$, Φ maps the spaces $E^1(e)$, $E_0(e)$, and $E^{-1}(e) \oplus E_1(e)$ into the corresponding spaces with respect to $\Phi(e)$*;

(iii) Φ *preserves the symmetrized triple product*

$$\langle x, y, z\rangle = \frac{1}{3}(\{x, y, z\} + \{z, x, y\} + \{y, z, x\}).$$

Proof. By passing to the bi-transpose of Φ we can suppose that we have a surjective weak*-continuous linear isometry between two real JBW*-triples [**17**, lemma 4·2]. Since in a real JBW*-triple the algebraic elements are dense (compare (i) $\Rightarrow$ (ii) in the proof of [**17**, theorem 4·8]), the statement follows from [**17**, proposition 3·8, theorem 4·8].

When we have a surjective real linear isometry from a complex Cartan factor with a unitary element to another complex Cartan factor we can deduce from the last statement of the above proposition that the isometry is in fact a triple isomorphism.

Corollary 2·10. *Let $\Phi: \mathcal{E} \to \mathcal{F}$ be a surjective real linear isometry between two complex Cartan factors. Suppose that $\mathcal{E}$ contains a unitary element u. Then Φ is a real triple* *isomorphism. Here the factor $\mathcal{E}$ can be any of the following: $I_{n,n}$, II_{2k}, III_n, IV_n, and VI.*

Proof. We may assume $r(\mathcal{E}) > 1$. By [**23**, proposition 5·7], we have $\mathcal{F} = \Phi(\mathcal{E}) = \Phi(\mathcal{E}_2(u)) = \mathcal{F}_2(\Phi(u))$, which proves that $v = \Phi(u)$ is a unitary element in $\mathcal{F}$. It is well known that $\mathcal{E}$ and $\mathcal{F}$ are JBW*-algebras with products and involutions given by

$$x \circ_1 y = \{x, u, y\}\,, \quad x \circ_2 y = \{x, v, y\}\,,$$

$$x^{\sharp_1} = \{u, x, u\}\,, \quad x^{\sharp_2} = \{v, x, v\}\,,$$

respectively [**33**, propositions 19·13 and 19·7]. Moreover, in this case the triple product is determined by the algebraic structure via the identity

$$\{x, y, z\} = (x \circ y^*) \circ z + (z \circ y^*) \circ x - (x \circ z) \circ y^*,$$

where $\circ$ denotes $\circ_1$ and $\circ_2$ and $*$ denotes $\sharp_1$ and $\sharp_2$, respectively. Therefore, Φ is a unital surjective real linear isometry between two JBW*-algebras. By Proposition 2·9 (iii), Φ preserves the symmetrized triple product and is unital. Then it is easy to see that $\Phi\sharp_1 = \sharp_2\Phi$, $\Phi(x \circ_1 y) = \Phi(x) \circ_2 \Phi(y)$, $(x, y \in \mathcal{E})$. Thus Φ is a Jordan *-isomorphism and hence a triple isomorphism. The last statement in the corollary follows by [**15**, proposition 2] and the fact that every complex spin factor has a unitary element [**7**, corollary, page 313].

The following result is the main tool for the study of surjective isometries between real reduced JB*-triples.

Theorem 2·11. *Let $\Phi: E \to F$ be a surjective linear isometry between two real reduced JBW*-triples. Then Φ preserves quadrangles consisting of minimal tripotents. Moreover, if $(u, v, \widetilde{u})$ is a triangle in E with $u, \tilde{u}$ minimal, then $(\Phi(u), \Phi(v), \Phi(\widetilde{u}))$ is a triangle in F.*

Proof. By Proposition 2·9, Φ preserves tripotents and the relations of minimality and orthogonality between them. Φ also preserves colinearity since E and F are reduced. Hence, if (u_1, u_2, u_3, u_4) is a quadrangle of minimal tripotents in E, then $(\Phi(u_1), \Phi(u_2), \Phi(u_3), \Phi(u_4))$ forms a quadrangle, except possibly for the property $\Phi(u_4) = 2\,\{\Phi(u_1), \Phi(u_2), \Phi(u_3)\}$. The rest of the proof is devoted to establishing the last equality.

Denote by $v_i = \Phi(u_i)$ for $i = 1, 2, 3, 4$. By Corollary 2·8, since v_2 and v_4 belong to $F_2(v_1 + v_3)$, $F_2(v_1 + v_3)$ is a real spin factor of dimension $\geqslant 4$ and rank 2. Furthermore, since $Q(e)$ is an automorphism, we get that $2\,\{v_1, v_2, v_3\} = Q(v_1 + v_3)(v_2)$ is a minimal tripotent in F orthogonal to v_2. Therefore, since in a real spin factor of rank 2 the orthogonal space relative to a minimal tripotent has dimension 1 (see Remark 2·6), then $2\,\{v_1, v_2, v_3\} = \pm\, v_4$. Suppose that $2\,\{v_1, v_2, v_3\} = -\, v_4$. By Lemma 2·3, $\frac{1}{\sqrt{2}}(u_1 + u_2 + u_3 - u_4)$ is a tripotent while its image by Φ, $\frac{1}{\sqrt{2}}(v_1 + v_2 + v_3 - v_4)$, is not a tripotent, which contradicts the fact that Φ preserves tripotents.

To see the last statement of the theorem, let $(u, v, \widetilde{u})$ be a triangle in E with $u, \tilde{u}$ minimal. As we have seen in the first part of the proof,

$$\Phi(E_1(u)) = F_1(\Phi(u)), \quad \Phi(E_1(\tilde{u})) = F_1(\Phi(\tilde{u}))$$

and hence, by [**7**, lemma 2·4], it follows that

$$\Phi|_{E_2(u+\tilde u)}: E_2(u+\tilde u) \longrightarrow F_2(\Phi(u)+\Phi(\tilde u))$$

is a surjective isometry between two real spin factors of rank 2 (compare Remark 2·6). Since v is a tripotent in $E_1(u)\bigcap E_1(\tilde u)$, and E is reduced, then $\Phi(v)\in F_1(\Phi(u))\bigcap F_1(\Phi(\tilde u))$. By Lemma 2·2, Proposition 2·4 and the fact that $\Phi(v)$ is not minimal, it follows that $\Phi(v)\vdash\Phi(u)$ and $\Phi(v)\vdash\Phi(\tilde u)$. Therefore we only have to show that

$$Q(\Phi(v))(\Phi(u)) = \Phi(\tilde u)$$

to get the statement. By Peirce rules and since $Q(\Phi(v))$ is an automorphism on $F_2(\Phi(v)$, we have $Q(\Phi(v))(\Phi(u))\in F_0(\Phi(u)) = \mathbb{R}\Phi(\tilde u)$, which implies that

$$Q(\Phi(v))(\Phi(u)) = \pm\Phi(\tilde u).$$

Suppose that

$$Q(\Phi(v))(\Phi(u)) = -\Phi(\tilde u),$$

in which case $(\Phi(u),\Phi(v),-\Phi(\tilde u))$ is a triangle. By Remark 2·6, $\frac{1}{2}(u+v+\tilde u)$ is a tripotent in E, whilst

$$\Phi\left(\frac{1}{2}(u+v+\tilde u)\right) = \frac{1}{2}(\Phi(u)+\Phi(v)+\Phi(\tilde u))$$

is not a tripotent, which is a contradiction.

Remark 2·12. Let C be a rank >1 complex Cartan factor. By [**7**], there exist a rectangular grid, a symplectic grid, a hermitian grid, a spin grid, or a first- or second-type exceptional grid, built up of triangles and quadrangles in C. Moreover, if C is not a type III_n Cartan factor, then C is the weak*-closed linear span of the elements of the corresponding grid and all the non-vanishing triple products among the elements of the grid are those associated to quadrangles of minimal tripotents or to triangles $(u,v,\tilde u)$ with $u,\tilde u$ minimal.

If C is a type $I^{\mathbb{R}}_{n,m}$, $I^{\mathbb{H}}_{2p,2q}$, $II^{\mathbb{R}}_{n}$ or $III^{\mathbb{H}}_{2p}$ real Cartan factor, then, as in the complex case, we can define a grid built up of quadrangles of minimal tripotents (compare [**23**, proofs of propositions 5·14, 5·16 and 5·17]).

Let $C = IV^{r,s}_n = X_1\oplus^{\ell_1} X_2$ be a real spin factor. We may assume that $r\geqslant s$. Let $\{e_i\}_{i\in I}$ and $\{f_j\}_{j\in J}$ be orthonormal bases of X_1 and X_2 respectively. By hypothesis, there is a set, J_1, so that $I = J\cup J_1$. We define $u_i = 2^{-1}(e_i+f_i)$, $\tilde u_i = 2^{-1}(e_i-f_i) = \overline{u}_i$, whenever $i\in J$ whilst $u_i = e_i$ for all $i\in I\backslash J = J_1$. It thus follows that:

(i) u_i is a minimal tripotent for all $i\in J$ and u_i is a tripotent for all $i\in J_1$;

(ii) $u_j\vdash u_i$, $u_j\vdash\tilde u_i$, $Q(u_j)(u_k) = -\overline{u}_k$, for all $j\in J_1$, $i\in J$ and $k\in I$;

(iii) $(u_i,u_j,\tilde u_i,\tilde u_j)$ are odd quadrangles for $i\neq j$, $i,j\in J$;

(iv) $C = C_2(u_i+\tilde u_i) = C_2(u_j)$ for all $i\in J$ and $j\in J_1$;

(v) the non-vanishing triple products among elements of the set correspond to those described in (2) and (3).

The family $\{u_i,\tilde u_i,u_j : i\in J, j\in J_1\}$ is called a *real spin grid*.

If C is a type $I^{\mathbb{C}}_{n,n}$ real Cartan factor, the real "hermitian grid" in this case is $\{v^l_{\alpha,\beta}\}$, where $v^l_{\alpha,\beta} = (le_\alpha\otimes e_\beta + e_\beta\otimes le_\alpha)$ if $\alpha\neq\beta$, $v^l_{\alpha,\alpha} = e_\alpha\otimes e_\alpha$, and $\{e_\alpha\}$ is an orthonormal basis of the complex Hilbert space H, $l = 1, i$ and $(h\otimes k)(x) = (x|k)h$. The same ideas hold good for $III^{\mathbb{R}}_n$ and $II^{\mathbb{H}}_{2p}$ taking real and quaternionic Hilbert spaces, X and P, and $l=1$ and $l=1,i,j,k$, where $1,i,j,k$ is the quaternion's canonical basis, repectively. The same reasoning can be applied to get a hermitian grid for the exceptional JB-algebra, $VI^{\mathbb{O}}=H_3(\mathbb{O})$.

Finally, we study the grids in the exceptional real Cartan factors. Let C be a type $V^{\mathbb{O}_0}$ real Cartan factor. It is easy to check that C contains two minimal orthogonal tripotents, v and $\tilde{v}$, so that dim $(C_2(v+\tilde{v}))=8$ and $C_1(v+\tilde{v}) \neq 0$. Therefore, bearing in mind Proposition 2·4 and the fact that C is reduced, the proof of [**7**, proposition on page 322] can be literally adapted to get an *exceptional grid of the first type* in C. By adapting the proof of [**7**, proposition on page 323], the above arguments can be applied to get an *exceptional grid of the second type* in $VI^{\mathbb{O}_0}$.

It is proved in [**6**, lemma 2·5] that every surjective real linear isometry between two complex Cartan factors of greater rank than one is w^*-continuous. Our next result shows that the same conclusion continues to be true for surjective linear isometries between real JBW*-triples.

Let X be a real or complex Banach space. Following [**11**], we define $\mathcal{B}(X)$ as the set of all functionals $\varphi \in X^{***}$ so that for every non-empty closed convex subset, $C \subset X$, the mapping

$$\varphi|_{\overline{C}^{\sigma(X^{**},X^*)}} : \left(\overline{C}^{\sigma(X^{**},X^*)}, \sigma(X^{**},X^*)\right) \longrightarrow \mathbb{F}$$

has at least one point of continuity, where $\mathbb{F}$ denotes the base field. The *universal frame* of X, $\gamma(X)$, is defined as

$$\gamma(X) = (\mathcal{B}(X) \cap j(X)^{\circ})_{\circ}.$$

The space X is called *well-framed* if and only if $\gamma(X) = j(X)$. In [**11**, théorème 16] it is shown that the well-framed property is inherited by subspaces. The duals (and hence the preduals) of von Neumann algebras and real and complex JBW*-triples are examples of well-framed Banach spaces (compare [**11**, théorème 18], [**16**], [**27**, lemma 2·2]).

LEMMA 2·13. *Every linear surjective isometry between real JBW*-triples is w^*-continuous.*

Proof. By [**27**, lemma 2·2], the predual of every real JBW*-triple is well-framed. This fact establishes that the predual of every JBW*-triple satisfies the condition (*) of [**12**, theorem V·1]. Finally, the statement follows by [**12**, theorem VII·8].

Our next goal is to prove that the surjective real linear isometries between two real reduced Cartan factors are triple isomorphisms.

PROPOSITION 2·14. *Let $\Phi: E \to F$ be a surjective linear isometry between two real reduced Cartan factors. Then Φ is a triple isomorphism.*

Proof. We assume first that both factors are of rank greater than one. Since, as we have seen in Remark 2·12 above, each reduced real Cartan factor of rank greater than one, except types $III_n^{\mathbb{R}}$, $I_{n,n}^{\mathbb{C}}$, $II_{2p}^{\mathbb{H}}$ and $VI^{\mathbb{O}}$, is the w^*-closed real linear span of a grid built up of quadrangles of minimal tripotents and triangles $(u,v,\widetilde{u})$ with $u,\tilde{u}$ minimal, the result follows from Theorem 2·11 and Lemma 2·13. When both factors are rank-one reduced, they coincide with a type $I_{1,n}^{\mathbb{R}}$ real Cartan factor (compare

[**23**, proposition 5·4] and [**26**, 11·9]). By [**23**, lemma 5·12] every surjective isometry between type $I_{1,n}^{\mathbb{R}}$ real Cartan factors is a triple isomorphism.

Factors $VI^{\mathbb{O}}$, $I_{n,n}^{\mathbb{C}}$, $II_{2p}^{\mathbb{H}}$, and $III_n^{\mathbb{R}}$ are JB-algebras and hence every surjective isometry between them is a triple isomorphism [**18**].

Following [**23**, page 214] we denote by $\mathcal{IS}$ the class of all real JB*-triples, E, where the surjective (real)-linear isometries $\Phi: E \to E$ coincide with the triple automorphisms. Proposition 2·14 establishes that every real reduced Cartan factor is in the class $\mathcal{IS}$. The exceptional real Cartan factors $V^{\mathbb{O}_0}$ and $VI^{\mathbb{O}_0}$ are real reduced Cartan factors (compare [**23**, table 1, page 210]), and hence, they are in the class $\mathcal{IS}$. This gives a positive answer to the question posed by Kaup in [**23**, page 217].

Our techniques (Corollary 2·10 and Proposition 2·14) cannot be applied to the non-reduced real Cartan factors $I_{2p,2q}^{\mathbb{H}}$ and $III_{2n}^{\mathbb{H}}$ (compare [**23**, table 1]) nor to the type $I_{n,m}$ complex Cartan factor with $n \neq m$, $n, m \geqslant 2$, II_{2k-1} with $k \geqslant 3$, and V. The remaining rank > 1 complex Cartan factors are in the class $\mathcal{IS}$ by [**6**] and the non-reduced real Cartan factors not covered by our result are also in the class $\mathcal{IS}$ according to [**23**, theorem 5·18]. In fact, our results overlap those of Dang [**6**] and Kaup [**23**]. Actually Corollary 2·10 is an alternative proof of [**6**, proposition 2·6] for those complex Cartan factors with a unitary element and Proposition 2·14 overlaps Kaup's results, showing that every non-exceptional rank > 1 real Cartan factor is in the class $\mathcal{IS}$. The following corollary holds good.

Corollary 2·15. *Every real or complex Cartan factor of rank greater than one is in the class $\mathcal{IS}$.*

The previous corollary allows us to extend [**23**, theorem 5·18] to exceptional rank > 1 real Cartan factors by the same arguments given in [**23**].

Corollary 2·16. *Let C be a real or complex Cartan factor of rank greater than one and F a real JBW*-triple. Then a bijective $\mathbb{R}$-linear map $\Phi: C \to F$ is an isometry if and only if it is a real triple isomorphism.*

3. *Real JB*-triples*

We begin with the following Gelfand–Naimark type theorem for real JB*-triples, the proof of which, as in the complex case (cf. [**10**]), is based upon the atomic decomposition of a real JBW*-triple.

Proposition 3·1. *Let E be a real JB*-triple. Then E can be isometrically embedded as a real subtriple of an ℓ_∞-sum of real Cartan factors and complex Cartan factors regarded as real. More specifically, if A denotes the atomic part of E^{**} and $\pi: E^{**} \to A$ is the canonical projection, then A is an ℓ_∞-sum of real or complex Cartan factors and the mapping $\pi \circ j: E \to A$ is an isometric triple embedding.*

Proof. It is known that E^{**} is a real JBW*-triple, the triple product of which extends the product of E [**17**]. In particular, $j: E \to E^{**}$ is a triple homomorphism. It should be remembered [**29**, theorem 3·6] that E^{**} decomposes in the form

$$E^{**} = A \oplus^{\infty} N,$$

where A and N are weak*-closed ideals, A being the weak*-closed real linear span of all the minimal tripotents of E^{**}, N containing no minimal tripotents and $A \perp N$. It follows by the proof of [**29**, theorem 3·6] that A is an ℓ_∞-sum of real and complex Cartan factors. It is clear that $\pi\colon E^{**} \to A$ is a triple homomorphism, and hence, $\pi \circ j$ is a triple homomorphism with norm less or equal to one. Therefore, we only have to show that $\pi \circ j$ is an isometry to get the statement.

Let $x \in E$ with $\|x\| = 1$. By the Krein–Milman, Hahn–Banach and Banach–Alaoglu theorems there exists an extreme point of the unit ball of E^*, φ, such that $\varphi(x) = 1$. By [**29**, corollary 2·1 and lemma 2·7], there is a minimal tripotent $u \in E^{**}$ such that $\varphi = \varphi \circ P^1(u)$. Thus $\varphi(N) = 0$ and hence $\varphi = \varphi \circ \pi$. Finally

$$1 = \|x\| = \|j(x)\| \geqslant \|\pi(j(x))\| \geqslant \varphi(\pi(j(x))) = \varphi(x) = 1,$$

which proves that $\pi \circ j$ is an isometry.

The following theorem extends [**6**, theorem 3·1] to the real setting.

THEOREM 3·2. *Let $\Phi\colon E \to F$ be a surjective linear isometry between two real JB*-triples. Suppose that E^{**} does not contain (real or complex) rank-one Cartan factors. Then Φ is a triple isomorphism.*

Proof. The mapping $\Phi^{**}\colon E^{**} \to F^{**}$ is a surjective weak*- continuous real-linear isometry between JBW*-triples. By Proposition 2·9, Φ^{**} preserves tripotents and the relations of minimality and orthogonality between them. Therefore, Φ^{**} maps the atomic part of E^{**}, $A_{E^{**}} = \oplus^{\ell_\infty} C_\alpha$, into the atomic part of F^{**}, $A_{F^{**}} = \oplus^{\ell_\infty} C_\beta$. Thus

$$\Psi = \Phi^{**}|_{A_{E^{**}}}\colon \oplus^{\ell_\infty} C_\alpha \longrightarrow \oplus^{\ell_\infty} C_\beta$$

is a surjective real-linear isometry from an ℓ_∞-sum of a family of real or complex rank > 1 Cartan factors to another ℓ_∞-sum of the same type.

We claim that for every C_α, there is a unique C_β such that $\Psi(C_\alpha) \subseteq C_\beta$. Indeed, since every real reduced or complex Cartan factor of rank greater than one is spanned by a grid built up of quadrangles of minimal tripotents or triangles, $(u, v, \tilde{u})$, with $u, \tilde{u}$ minimal (compare Remark 2·12 and [**7**]), we only have to show that each of the above quadrangles or triangles is mapped by Ψ into a unique C_β. Let (u_1, u_2, u_3, u_4) be a quadrangle of minimal tripotents in a fixed C_α. Since Ψ maps minimal tripotents into minimal tripotents and every minimal tripotent belongs to a unique C_β, it follows that each $\Psi(u_i)$ belongs to a unique C_β. Now, if $\Psi(u_1)$ and $\Psi(u_2)$ lie in different factors then they, and hence u_1 and u_2, must be orthogonal, which is impossible. The same reasoning holds for $(\Psi(u_2), \Psi(u_3))$ and $(\Psi(u_3), \Psi(u_4))$ and (u_2, u_3) and (u_3, u_4), respectively.

Let now $(u, v, \tilde{u})$ be a triangle in C_α. Since the subtriple $E_2(u+\tilde{u}) = (C_\alpha)_2(u+\tilde{u})$ is a spin factor, we can assume that C_α is a spin factor. By Remark 2·6, $m = 2^{-1}(u+v+\tilde{u})$ is a minimal tripotent in C_α, which is not orthogonal to u nor $\tilde{u}$. As in the case of quadrangle, this implies that the triangle is contained in a unique C_β.

Therefore, $\Psi|_{C_\alpha}\colon C_\alpha \to C_\beta$ is a (real) linear surjective isometry between two real or complex Cartan factors of rank greater than one and then a triple isomorphism by Corollary 2·16. This implies that Ψ is a real triple isomorphism. Finally, let $\pi_E\colon E^{**} \to A_{E^{**}}$, $\pi_F\colon F^{**} \to A_{F^{**}}$, j_E, and j_F be the canonical projections of E^{**} and F^{**} onto their atomic parts and the canonical embeddings of E and F into their biduals, respectively. Since $(\pi_F \circ j_F)\circ\Phi = \Psi\circ(\pi_E \circ j_E)$, it follows, by Proposition 3·1 and the fact that Ψ is a real triple isomorphism, that Φ is a real triple isomorphism.

Remark 3·3. The conclusion of Theorem 3·2 is not always true when E^{**} contains a rank-1 Cartan factor. Indeed, let E and F be a type $I^{\mathbb{R}}_{1,n}$ and a type $IV^{n,0}_n$ real Cartan factor respectively. Then the identity map from E to F, both regarded as n-dimensional real Hilbert spaces, is a surjective linear isometry which is not a triple isomorphism. Another example of this fact can be found in [**6**, remark 2·7].

The following result is an application of our main theorem to the case of a real J*B-algebra extending [**6**, corollary 3·2]. Following [**2**], a J*B-algebra is a real Jordan algebra with unit 1 and an involution $*$ equipped with a complete algebra norm so that $\|U_x(x^*)\| = \|x\|^3$ and $\|x^*\circ x\| \leqslant \|x^*\circ x + y^*\circ y\|$, where $U_x(y) := 2x\circ(x\circ y) - x^2\circ y$. It is shown in [**2**, theorem 4·4] that the complexification of every J*B-algebra is a complex JB*-algebra, and hence a complex JB*-triple, with a norm extending the given one. Therefore, every J*B-algebra is a real JB*-triple with triple product

$$\{a,b,c\} = a\circ(b^*\circ c) + c\circ(b^*\circ a) - (a\circ c)\circ b^*.$$

Corollary 3·4. *Let $\Phi: A \to B$ be a surjective linear isometry between two J*B-algebras. Then Φ is a real triple isomorphism. If Φ is also unital then it is a $*$-algebra-isomorphism.*

Proof. The unit of A is a unitary element of A^{**} when considered as a real JB*-triple. This implies that every factor in the atomic part of A^{**} contains a unitary element. If A^{**} contains no real or complex rank one Cartan factors the proof is provided by Theorem 3·2. Otherwise, let C be a real or complex Cartan factor of rank one contained in the atomic part of A^{**}. If C is a complex Cartan factor or a reduced real Cartan factor, it follows that C coincides with $\mathbb{C}$ or $\mathbb{R}$, since C contains a unitary element and every tripotent is minimal. If C is a non-reduced rank-1 real Cartan factor with a unitary element, it can be seen that C coincides with a type $IV^{n,0}_n$ real Cartan factor with $n \geqslant 3$ (compare, [**26**, 11·9] or [**23**, proposition 5·4]).

Suppose that A^{**} contains a non-trivial rank one real Cartan factor $C_\alpha \equiv IV^{n,0}_n$ ($n \geqslant 3$). Now, by adapting the proof of Theorem 3·2 to this particular case we can show that F^{**} contains another non-trivial rank-1 real Cartan factor $C_\beta \equiv IV^{n,0}_n$ ($n \geqslant 3$), so that $\Phi^{**}(C_\alpha) = C_\beta$. By [**23**, lemma 5·13], we get that $\Phi^{**}|_{C_\alpha} : C_\alpha \to C_\beta$ is a triple isomorphism. Now the proof proceeds as in Theorem 3·2.

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