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Geometric characterization of tripotents in real and complex JB*-triples ☆

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Abstract

We establish a geometric characterization of tripotents in real and complex JB*-triples. As a consequence we obtain an alternative proof of Kaup's Banach–Stone theorem for JB*-triples.
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1. Introduction

Recently, C.A. Akemann and N. Weaver have established "geometric" characterizations of the partial isometries, unitaries, and invertible elements in C*-algebras in terms of the norm. More precisely, in [1, Theorem 1] the authors proved that a norm-one element x in a C*-algebra $\mathcal{A}$ is a partial isometry if and only if the sets

$$D_1(x) := \{y \in \mathcal{A}\colon \text{ there exists } \alpha > 0 \text{ with } \|x + \alpha y\| = \|x - \alpha y\| = 1\}$$

and

$$D_2(x) := \{y \in \mathcal{A}\colon \ \|x + \beta y\| = \max\{1, \|\beta y\|\} \text{ for all } \beta \in \mathbb{C}\}$$

coincide.

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It is well known that every C*-algebra belongs to the more general class of complex Banach spaces known as JB*-triples (see definition below). Indeed, every C*-algebra is a JB*-triple with respect to the triple product

$$\{a, b, c\} := 2^{-1}(ab^*c + cb^*a)$$

and the same norm. An element e in a JB*-triple $\mathcal{E}$ is said to be a tripotent whenever $\{e, e, e\} = e$. When $\mathcal{A}$ is a C*-algebra regarded as a JB*-triple, then it is also known that partial isometries and tripotents coincide (cf. [18, 2.2.8]). The question clearly is whether the coincidence of $D_1(x)$ and $D_2(x)$ could be applied to characterize the fact that x is a tripotent, when x is a norm-one element in a JB*-triple. In Theorem 2.1 we show that the "geometric" characterization of tripotents elements in C*-algebras obtained by Akemann and Weaver is also valid for JB*-triples. As a consequence, we obtain, in Theorem 2.2, an alternative proof of Kaup's Banach–Stone theorem for JB*-triples (cf. [15, Proposition 5.5]). The references [3] and [13, Theorem 4.8] contain also independent proofs of the above mentioned result, however, the proof developed in this paper is a novelty with respect to the previous ones. In the last part of the paper we establish geometric characterizations of tripotents and complete tripotents in the more general class of real JB*-triples (see Theorem 2.3 and Corollary 2.5). Finally, we describe in terms of the underlying Banach space structure those real JB*-triples which are unital JB-algebras.

The basic "geometric" tool applied in our proofs involves results on M-structure in JB*-triples and JBW*-triples and on the dual L-structure in their duals or preduals. It is worth mentioning that the theory of M-structure in JB*-triples and JBW*-triples has focused the attention of diverse researchers in the last years. For example, the papers [2,6–9,12] and [5] contains results connected with this theory.

Given a Banach space X, we denote by B_X, S_X, and X^* the closed unit ball, the unit sphere, and the dual space of X, respectively.

2. Tripotents in real and complex JB*-triples

A (complex) *JB*-triple* is a complex Banach space $\mathcal{E}$ equipped with a continuous triple product

$$\{\cdot,\cdot,\cdot\} : \mathcal{E} \otimes \mathcal{E} \otimes \mathcal{E} \to \mathcal{E}, \quad (x, y, z) \mapsto \{x, y, z\},$$

which is bilinear and symmetric in the outer variables and conjugate linear in the middle one and satisfies

(a) *Jordan identity*

$$L(x, y)\{a, b, c\} = \big\{L(x, y)a, b, c\big\} - \big\{a, L(y, x)b, c\big\} + \big\{a, b, L(x, y)c\big\}$$

for all $x, y, a, b, c \in \mathcal{E}$, where $L(x, y) : \mathcal{E} \to \mathcal{E}$ is the linear mapping given by $L(x, y)z = \{x, y, z\}$;

(b) The map $L(x, x)$ is an hermitian operator with nonnegative spectrum for all $x \in \mathcal{E}$;

(c) $\|\{x, x, x\}\| = \|x\|^3$ for all $x \in \mathcal{E}$.

Every C*-algebra is a JB*-triple with respect to the triple product $\{x, y, z\} = 2^{-1}(xy^*z + zy^*x)$, every JB*-algebra is a JB*-triple with triple product $\{a, b, c\} = (a \circ b^*) \circ c + (c \circ b^*) \circ a - (a \circ c) \circ b^*$, and the Banach space $B(H, K)$ of all bounded linear operators between two complex Hilbert spaces H, K is also an example of a JB*-triple with product $\{R, S, T\} = 2^{-1}(RS^*T + TS^*R)$.

A JBW*-triple is a JB*-triple which is also a dual Banach space. The bidual, $\mathcal{E}^{**}$, of every JB*-triple, $\mathcal{E}$, is a JBW*-triple with triple product extending the product of $\mathcal{E}$ (cf. [4]).

For any JB*-triple $\mathcal{E}$ and a tripotent $e \in \mathcal{E}$ there exist decompositions of $\mathcal{E}$ in terms of the eigenspaces of $L(e, e)$ and $Q(e)$ (where $Q(e)(x) = \{e, x, e\}$) given by

$$\mathcal{E} = \mathcal{E}_0(e) \oplus \mathcal{E}_1(e) \oplus \mathcal{E}_2(e) = \mathcal{E}^0(e) \oplus \mathcal{E}^1(e) \oplus \mathcal{E}^{-1}(e), \tag{1}$$

where $\mathcal{E}_k(e) := \{x \in \mathcal{E}\colon\ L(e, e)x = kx/2\}$ is a subtriple of $\mathcal{E}$ (k: 0, 1, 2), $\mathcal{E}^k(e) = \{x \in \mathcal{E}\colon\ Q(e)(x) = kx\}$ (k: 0, 1, −1). The natural projection of $\mathcal{E}$ onto $\mathcal{E}_k(e)$ and $\mathcal{E}^k(e)$ will be denoted by $P_k(e)$ and $P^k(e)$, respectively. The first decomposition is called the Peirce decomposition with respect to the tripotent e and the natural projections are called Peirce projections. The following rules are also satisfied:

$$\mathcal{E}_2(e) = \mathcal{E}^1(e) \oplus \mathcal{E}^{-1}(e), \quad \mathcal{E}^{-1}(e) = i\mathcal{E}^1(e),$$
$$\{\mathcal{E}_k(e), \mathcal{E}_l(e), \mathcal{E}_m(e)\} \subseteq \mathcal{E}_{k-l+m}(e),$$
$$\{\mathcal{E}_0(e), \mathcal{E}_2(e), \mathcal{E}\} = \{\mathcal{E}_2(e), \mathcal{E}_0(e), \mathcal{E}\} = 0,$$
$$\{\mathcal{E}^p(e), \mathcal{E}^q(e), \mathcal{E}^r(e)\} \subseteq \mathcal{E}^{pqr}(e) \quad (p, q, r\colon 1, -1),$$

where $\mathcal{E}_{k-l+m}(e) = 0$ whenever $k - l + m$ is not in $\{0, 1, 2\}$. It is also known that $\mathcal{E}_2(e)$ is a unital JB*-algebra with respect to the product $x \circ y = \{x, e, y\}$ and involution $x^* = \{e, x, e\}$.

Let x be a norm one element in a Banach space X. The set $D(X, x)$ of all states of X relative to x is define by

$$D(X, x) := \{f \in S_{X^*}\colon\ f(x) = \|x\|\}.$$

The following theorem generalizes [1, Theorem 1] to the setting of JB*-triples. It is worth pointing out that partial isometries and tripotents coincide in the case of a C*-algebra regarded as a JB*-triple.

Theorem 2.1. *Let $\mathcal{E}$ be a JB*-triple and let x be a norm-one element in E. Then x is a tripotent if and only if*

$$D_1(x) := \{y \in \mathcal{E}\colon \text{ there exists } \alpha > 0 \text{ with } \|x + \alpha y\| = \|x - \alpha y\| = 1\}$$

coincides with

$$D_2(x) := \{y \in \mathcal{E}\colon\ \|x + \beta y\| = \max\{1, \|\beta y\|\} \text{ for all } \beta \in \mathbb{C}\}.$$

Proof. (⇒) Suppose x is a tripotent in $\mathcal{E}$. The inclusion $D_2(x) \subseteq D_1(x)$ holds for every complex Banach space and every norm-one element x in it. To see the converse inclusion fix $y \in D_1(x)$ and $\alpha > 0$ such that $\|x + \alpha y\| = \|x - \alpha y\| = 1$. Let $f \in D(\mathcal{E}, x)$. It is easy to check that

$$1 = \|x \pm \alpha y\|^2 \geqslant |f(x \pm \alpha y)|^2 = 1 + \alpha^2 |f(y)|^2 \pm 2\alpha \Re f(y).$$

Therefore $f(y)=0$ (for every $f\in D(\mathcal{E},x)$). It is worth remembering that $\mathcal{E}_2(x)$ is a complex JB*-algebra, $P^1(x)(y)$ is an hermitian element in $\mathcal{E}_2(x)$ and, by [7, Proposition 1], for every $f\in D(\mathcal{E},x)$ we have $f=fP_2(x)$. Therefore $D(\mathcal{E},x)=D(\mathcal{E}_2(x),x)$. It is well known that the norm of an hermitian element h in the unital JB*-algebra $\mathcal{E}_2(x)$ can be computed as supreme of the set $\{|f(h)|\colon f\in D(\mathcal{E}_2(x),x)\}$. Since for every $f\in D(\mathcal{E}_2(x),x)$,

$$f(y)=f\big(P_2(x)(y)\big)=f\big(P^1(x)(y)\big)+f\big(P^{-1}(x)(y)\big)$$

with $f(P^1(x)(y))$ and $if(P^{-1}(x)(y))$ in $\mathbb{R}$, we have $|f(P^1(x)(y))|\leqslant|f(y)|$. Therefore, we get

$$\begin{aligned}\big\|P^1(x)(y)\big\|&=\sup\big\{\big|f(P^1(x)(y))\big|\colon f\in D(\mathcal{E},x)\big\}\\&\leqslant\sup\big\{\big|f(y)\big|\colon f\in D(\mathcal{E},x)\big\}=0.\end{aligned}$$

We can then assume $P_2(x)(y)=s\in E^{-1}(x)=iE^1(x)$. Thus $is\in E^1(x)$. The expression

$$\begin{aligned}\|is\|&=\sup\big\{\big|f(is)\big|\colon f\in D\big(\mathcal{E}_2(x),x\big)\big\}\\&=\sup\big\{\big|f\big(P_2(x)(iy)\big)\big|\colon f\in D\big(\mathcal{E}_2(x),x\big)\big\}=\big\{\big|f\big(y\big)\big|\colon f\in D\big(\mathcal{E}_2(x),x\big)\big\}=0,\end{aligned}$$

gives $s=0$. As a consequence, $P_2(x)(y)=0$ and $y=P_1(x)(y)+P_0(x)(y)$. We denote $P_1(x)(y)=y_1$ and $P_0(x)(y)=y_0$.

By [7, Lemma 1.5] the element $\{y_1,y_1,x\}$ is hermitian and positive in $\mathcal{E}_2(x)$. Given $f\in D(\mathcal{E},x)=D(\mathcal{E}_2(x),x)$, by Peirce rules we have

$$f\big(\{y,y,x\}\big)=f\big(\{y_1,y_1,x\}\big).$$

Again by Peirce rules, the positivity of $\{y_1,y_1,x\}$ in $\mathcal{E}_2(x)$, and the inequality

$$\begin{aligned}1=\|x+\alpha y\|^2&\geqslant f\big(\{x+\alpha y,x+\alpha y,x\}\big)\\&=f(x)+\alpha^2f\big(\{y_1,y_1,x\}\big)=1+\alpha^2f\big(\{y_1,y_1,x\}\big),\end{aligned}$$

we conclude that $f(\{y_1,y_1,x\})=0$ (for all $f\in D(\mathcal{E},x)=D(\mathcal{E}_2(x),x)$). This shows that $\{y_1,y_1,x\}=0$ and by [7, Lemma 1.5] we get $y_1=0$. Therefore $y=y_0\in\mathcal{E}_0(x)$ and now [7, Lemma 1.3] ascertains that

$$\|x+\beta y\|=\max\big\{1,\|\beta y\|\big\},$$

and hence $y\in D_2(x)$.

($\Leftarrow$) Suppose that x is not a tripotent in $\mathcal{E}$. Let C be the JB*-subtriple of $\mathcal{E}$ generated by x. It is known that there exists a locally compact subset $S_x\subseteq[0,1]$ such that $S_x\cup\{0\}$ is compact and a surjective triple isomorphism (and hence an isometry) $F:C\to C_0(S_x)$, where $C_0(S_x)$ is the C*-algebra of all complex valued continuous functions on S_x vanishing at 0, and $F(x)(t)=t$ for all $t\in S_x$ (compare [14, 4.8] and [15, 1.15]).

Since $F(x)$ is not a tripotent in $C_0(S_x)$ we have $S_x\cap\,]0,1[\,\neq\emptyset$. Take g in $C_0(S_x)$ given by $g(t):=(t-t^3)^9$. Since the minimum value of $(1-t)(t-t^3)^{-9}$ in $(0,1)$ is strictly greater than 1, we have $(1-t)\geqslant g(t)$ for all $t\in S_x$. Then it follows that $\|F(x)\pm g\|=1$. Since $g\in C_0(S_x)$ there exists $t_0\in S_x$ such that $\|g\|=g(t_0)$. Notice that, since $g>0$ and $g(1)=0$ we must have $0<t_0<1$. Therefore

$$\left\| F(x) + \frac{1}{\|g\|} g \right\| \geqslant \left| t_0 + \frac{g(t_0)}{\|g\|} \right| = 1 + t_0 > 1.$$

Finally, we can take $y = F^{-1}(g)$ to get an element in $D_1(x) \setminus D_2(x)$. □

One of the most celebrated results on the category of JB*-triples is Kaup's Banach–Stone theorem for JB*-triples which assures that the surjective isometries between two JB*-triples coincide with the triple isomorphisms between them (cf. [15]). The geometric characterization of tripotents in JB*-triples obtained in the previous theorem allows us to obtain an alternative proof of Kaup's Banach–Stone theorem, following more or less known arguments.

Theorem 2.2. *Let $\Phi : \mathcal{E} \to \mathcal{F}$ be a surjective isometry between two JB*-triples. Then Φ is a triple isomorphism.*

Proof. The bi-transpose of Φ, $\Phi^{**} : \mathcal{E}^{**} \to \mathcal{F}^{**}$, is also a surjective isometry. $\mathcal{E}^{**}$ and $\mathcal{F}^{**}$ are JBW*-triples with triple products extending the ones of $\mathcal{E}$ and $\mathcal{F}$, respectively. Therefore, except considering Φ^{**} instead of Φ, we can assume that Φ is a surjective isometry between two JBW*-triples.

By Theorem 2.1 we know that Φ preserves tripotents. We claim that Φ also preserves orthogonal tripotents. Indeed, two tripotents e_1, e_2 in $\mathcal{E}$ are orthogonal if and only if $e_1 \pm e_2$ is a tripotent of $\mathcal{E}$ (compare [13, Lemma 3.6]). Therefore, $\Phi(e_1)$, $\Phi(e_2)$, and $\Phi(e_1) \pm \Phi(e_2)$ are tripotents in $\mathcal{F}$. This shows that $\Phi(e_1)$ and $\Phi(e_2)$ are orthogonal tripotents in $\mathcal{F}$.

Let $a \in \mathcal{E}$ be an algebraic element, i.e., $a = \sum_{j=1}^{m} \lambda_j e_j$, where $\lambda_j \in \mathbb{C}$ and $e_1, \ldots, e_m$ are orthogonal tripotents in $\mathcal{E}$. Since Φ preserves orthogonal tripotents we can see that $\Phi(\{a, a, a\}) = \{\Phi(a), \Phi(a), \Phi(a)\}$. By [11, Lemma 3.11], for every element $x \in \mathcal{E}$ there is a sequence of algebraic elements converging in norm to x. Since the triple product is jointly norm-continuous and Φ preserves cubes of algebraic elements, it can be concluded that Φ preserves cubes. The expression

$$\{x, y, x\} = 4^{-1} \sum_{k=0}^{3} (-i)^k \{x + i^k y, x + i^k y, x + i^k y\} \quad (x, y \in \mathcal{E})$$

allows us to assure that Φ preserves triple products of the form $\{x, y, x\}$ (x, y in $\mathcal{E}$). Finally, since the triple product is symmetric in the outer variables, we get that Φ is a triple isomorphism. □

Following [13] we define a real JB*-triple as a real closed subtriple of a JB*-triple. Clearly every JB*-triple is a real JB*-triple regarded as a real Banach space. Another examples of real JB*-triples are the real C*-algebras and the Banach space $B(H, K)$ of all bounded real linear operator between two real Hilbert spaces H and K, with respect to the triple product $\{a, b, c\} = (1/2)(ab^*c + cb^*a)$.

Let E be a real JB*-triple. It is known (cf. [13]) that there exists a unique complex JB*-triple structure on the complexification $\hat{E} = E \oplus iE$ and a unique conjugation (conjugate linear isometry of period 2) τ on $\hat{E}$ such that

$$E = \hat{E}^{\tau} = \{z \in \hat{E} \colon \tau(z) = z\}.$$

By Kaup's Banach–Stone theorem we can assure that τ is a conjugate linear triple isomorphism on $\hat{E}$. Given a tripotent e in a real JB*-triple E, then the decompositions and rules described in (1) are also satisfied by E except perhaps $E^{-1}(e) = iE^1(e)$.

In the light of the geometric characterization of tripotents provided by Theorem 2.1, the question clearly is whether the geometric characterization can be also obtained for tripotents in real JB*-triples. Let E be a real JB*-triple. The first observation that we should make is that the set called $D_2(x)$ in Theorem 2.1 must be changed by

$$D_2'(x) = \bigl\{y \in E\colon\ \|x + \beta y\| = \max\bigl\{1, \|\beta y\|\bigr\} \text{ for all } \beta \in \mathbb{R}\bigr\},$$

since E is only a real Banach space. We can state now the geometric characterization of tripotents in a real JB*-triple.

Theorem 2.3. *Let E be a real JB*-triple and let x be a norm-one element in E. Then x is a tripotent if and only if*

$$D_1(x) := \bigl\{y \in \mathcal{E}\colon\ \textit{there exists } \alpha > 0 \textit{ with } \|x + \alpha y\| = \|x - \alpha y\| = 1\bigr\}$$

coincides with

$$D_2'(x) := \bigl\{y \in \mathcal{E}\colon\ \|x + \beta y\| = \max\bigl\{1, \|\beta y\|\bigr\} \textit{ for all } \beta \in \mathbb{R}\bigr\}.$$

Proof. ($\Rightarrow$) Suppose x is a tripotent in E; then x is also a tripotent in $\hat{E}$, the complexification of E. By Theorem 2.1 we have

$$D_1^{\mathbb{C}}(x) = \bigl\{y \in \hat{E}\colon \text{ there exists } \alpha > 0 \text{ with } \|x \pm \alpha y\| = 1\bigr\} \tag{2}$$

$$= D_2^{\mathbb{C}}(x) = \bigl\{y \in \hat{E}\colon\ \|x + \beta y\| = \max\bigl\{1, \|\beta y\|\bigr\} \text{ for all } \beta \in \mathbb{C}\bigr\}. \tag{3}$$

Take $y \in D_1(x)$ it is clear that $y \in D_1^{\mathbb{C}}(x) = D_2^{\mathbb{C}}(x)$ and hence

$$\|x + \beta y\| = \max\bigl\{1, \|\beta y\|\bigr\}$$

for all $\beta \in \mathbb{R}$, which shows that $y \in D_2'(x)$. Therefore, we have $D_1(x) \subseteq D_2'(x)$. The converse inclusion is always true for any norm-one element x in a real Banach space.

($\Leftarrow$) Suppose now that x is not a tripotent in E. Let $\hat{E}$ denote the complexification of E and τ the canonical conjugation satisfying $\hat{E}^\tau = E$. Since x neither is a tripotent in $\hat{E}$, it follows from the last part of the proof of Theorem 2.1 that taking $y = \{\{z,z,z\},\{z,z,z\},\{z,z,z\}\}$, where $z = x - \{x,x,x\}$, we have $\|x \pm y\| = 1$ and $\|x + y/\|y\|\| \neq 1$. Finally, since τ preserves the triple products, and $\tau(x) = x$, we obtain that $\tau(y) = y$, which gives $y \in D_1(x) \setminus D_2'(x)$. $\square$

Remark 2.4. Since every JB*-triple is a real JB*-triple when is regarded as a real Banach space and the concept of tripotent does not depend on the base field, the above Theorem is also valid for JB*-triples.

Let e be a tripotent in a real JB*-triple E. Since the Peirce projections associated to e on E coincide with the restrictions of the corresponding Peirce projections associated to e on its complexification, it follows, by [7, Lemma 1.3], that $E_0(e) \subseteq D_2'(e)$.

In [16, Proposition 3.5], W. Kaup and H. Upmeier proved that the extreme points of the unit ball of a complex JB*-triple $\mathcal{E}$ are nothing but the complete tripotents of $\mathcal{E}$ (cf. [16, Proposition 3.3]). In [13, Lemma 3.3], J.M. Isidro et al. proved that the same conclusion holds for real JB*-triples. It is worth mentioning that a tripotent e in a real or complex JB*-triple E is called complete if $E_0(e) = 0$.

We can see now how our geometric characterization of tripotents provides an alternative proof of the above fact. Let e be a norm-one element in a real or complex JB*-triple E. Then e is an extreme point of the unit ball of E if and only if $D_1(e) = \{0\}$ (see comments preceding [1, Theorem 2]). Let us suppose that e is a complete tripotent in E. From the proofs of Theorems 2.1 and 2.3 it may be concluded that $D_1(e) = E_0(e)$. Since e is complete he have $E_0(e) = \{0\}$ and consequently e is an extreme point of the unit ball of E. We assume now that e is an extreme point of the unit ball of E. Then $\{0\} \subseteq D'_2(e) \subseteq D_1(e) = \{0\}$. Now Theorem 2.3 implies that e is a tripotent of E. Since we also have $E_0(e) \subseteq D'_2(e) = \{0\}$, we deduce that e is a complete tripotent of E. We have thus proved the following corollary.

Corollary 2.5. *Let E be a real or complex JB*-triple and let e be a norm-one element in E. The following are equivalent*:

(a) *e is a complete tripotent.*
(b) *e is an extreme point of the unit ball of E.*
(c) $D_1(e) = \{0\}$.

By a real JBW*-triple we mean a real JB*-triple E whose underlying Banach space is a dual Banach space in such a way that the triple product of E is separately weak*-continuous. It is known that the separate weak*-continuity of the triple product can be dropped (cf. [17]). The bidual of a real JB*-triple is a real JBW*-triple [13, Lemma 4.2]. It is also known that the algebraic elements in a real JBW*-triple are norm dense (cf. [13, proof of Theorem 4.8, (i) $\Rightarrow$ (ii)]). Therefore, when in the proof of Theorem 2.2, Theorem 2.3 replaces Theorem 2.1, we arrive at the following result.

Theorem 2.6. *Let $\Phi : E \to F$ be a surjective isometry between two real JB*-triples. Then Φ preserves cubes, i.e., $\Phi\{a, a, a\} = \{\Phi(a), \Phi(a), \Phi(a)\}$.*

The conclusion of the above theorem is the best result we could have expected for surjective isometries between real JB*-triple (compare [13, Example 4.12]).

The geometric characterization of the partial isometries in a C*-algebra A given by C. Akemann and N. Weaver was accompanied by similar characterizations of the unitaries and invertible elements in A, where only the structure of Banach space is needed (cf. [1, Theorems 2 and 4]). The techniques developed in [1] could be analogously applied to get a geometric characterization of the unitary elements in a complex JB*-triple. Nevertheless, a shorter proof of the geometric characterization of the unitary elements in a C*-algebra (and in a complex JB*-triple) has been recently obtained by A. Rodríguez-Palacios in [20]. The following theorem establishing the just quoted geometric characterization of the unitary elements in a JB*-triple is included for completeness reasons. We recall that an element u in a real or complex JB*-triple E is called unitary if and only if $L(u,u) = \mathrm{Id}_E$.

Theorem 2.7 [20, Theorem 2.1]. *Let $\mathcal{E}$ be a (complex) JB*-triple and let u be a norm-one element in E. The following are equivalent*:

(a) *u is a unitary element in $\mathcal{E}$.*
(b) *$D(\mathcal{E},u)$ spans $\mathcal{E}^*$.*
(c) *u is a vertex of the closed unit ball of $\mathcal{E}$.*

It is worth mentioning that a norm-one element x in a Banach space X is a vertex of the closed unit ball of X if and only if $D(X,x)$ separates the points of X. It is well known that in the case of a real JB*-triple the above theorem is false in general. More concretely, if u is a unitary element in a real JB*-triple then conditions (b) and (c) in the above theorem need not be satisfied. In the setting of real JB*-triples we can establish the following result.

Proposition 2.8. *Let E be a real JB*-triple and let u be a norm-one element in E. The following conditions are equivalent*:

(a) *$D(E,u)$ spans E^*.*
(b) *u is a vertex of the closed unit ball of E.*
(c) *E is a JB-algebra with unit u and product $x \circ y := \{x,u,y\}$.*

Moreover any of the above conditions implies that u is a unitary element in E.

Proof. The implication (a) $\Rightarrow$ (b) follows straightforwardly even in a general Banach space. To see (b) $\Rightarrow$ (c), let us suppose that u is a vertex of E. Since every vertex of the closed unit ball of a Banach space is an extreme point of the closed unit ball, Corollary 2.5 ascertains that u is a complete tripotent in E. Therefore, $E = E^1(u) \oplus E^{-1}(u) \oplus E_1(u)$. By [19, Lemma 2.7], we have $f(y) = 0$ for every $f \in D(E,u)$ and $y \in E^{-1}(u) \oplus E_1(u)$. Finally, since u is a vertex of the closed unit ball we conclude that $E = E^1(u)$, which is a JB-algebra with unit u and product $x \circ y := \{x,u,y\}$. The implication (c) $\Rightarrow$ (a) is known to be true (cf. [10, Lemmas 3.6.8 and 1.2.6]). $\square$

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