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Little Grothendieck's theorem for real JB*-triples

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Abstract. We prove that given a real JB*-triple E, and a real Hilbert space H, then the set of those bounded linear operators T from E to H, such that there exists a norm one functional $\varphi \in E^*$ and corresponding pre-Hilbertian semi-norm $\|.\|_\varphi$ on E such that

$$\|T(x)\| \leq 4\sqrt{2}\|T\| \, \|x\|_\varphi$$

for all $x \in E$, is norm dense in the set of all bounded linear operators from E to H. As a tool for the above result, we show that if A is a JB-algebra and $T : A \to H$ is a bounded linear operator then there exists a state $\varphi \in A^*$ such that

$$\|T(x)\| \leq 2\sqrt{2}\|T\|\varphi(x^2)$$

for all $x \in A$.

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1 Introduction

It is well known [Gro] that there is a universal constant K such that if Ω is a compact Hausdorff space and T is a bounded linear operator from $C(\Omega)$

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to a complex Hilbert space H, then there exists a probability measure μ on Ω such that

$$\|T(f)\|^2 \leq K^2\|T\|^2\left(\int_\Omega |f|^2 d\mu\right)$$

for all $f \in C(\Omega)$. This result is called “Little Grothendieck’s inequality” or “Little Grothendieck’s Theorem” for commutative C*-álgebras. In the non-commutative case, Pisier ([P1], [P2]) and Haagerup ([H1],[H2]) proved a “Little Grothendieck Theorem” for C*-algebras. That is, if $T : C \to \mathcal{H}$ is a bounded linear operator from a C*-algebra, C, to a complex Hilbert space, $\mathcal{H}$, we can find a state ψ of C such that

$$\|T(x)\| \leq 2\|T\|\psi\left(\frac{1}{2}(xx^* + x^*x)\right)^{\frac{1}{2}} \quad (x \in C).$$

As is pointed out in [CIL], Pisier’s proof of the “Little Grothendieck’s theorem” for C*-algebras [P2, Theorem 9.4] can be verbatim extended for JB*-algebras in the following setting. For every bounded linear operator T from a JB*-algebra $\mathcal{A}$, to a complex Hilbert space $\mathcal{H}$, there exists a state $\varphi \in \mathcal{A}^*$ such that

$$\|T(z)\| \leq 2\|T\|\left(\varphi(z \circ z^*)\right)^{\frac{1}{2}}$$

for all $z \in \mathcal{A}$. For the most general class of complex Banach spaces called JB*-triples (which includes C*-algebras and JB*-algebras) a “Little Grothendieck’s Theorem” is established by Barton and Friedman [BF, Theorem 1.3]. According to the formulation of that Theorem in [BF], for every bounded linear operator T from a complex JB*-triple $\mathcal{E}$ to a complex Hilbert space $\mathcal{H}$ there is a normalized functional $\varphi \in \mathcal{E}^*$ such that

$$\|T(x)\| \leq \sqrt{2}\|T\|\|x\|_\varphi$$

for every $x \in \mathcal{E}$, where $\|x\|_\varphi^2 = \varphi\{x, x, e\}$ for some tripotent $e \in \mathcal{E}^{**}$ with $\varphi(e) = 1$. However, the Barton-Friedman proof contains a gap. Indeed, they assert, that for T as above, T^{**} attains its norm (at a complete tripotent), a fact that is not always true. Indeed, consider the operator S from the complex ℓ_2 space to itself, whose associated matrix is

$$\begin{pmatrix} \frac{1}{2} & 0 & \ldots & 0 & \ldots \\ 0 & \frac{2}{3} & \ldots & 0 & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & \frac{n}{n+1} & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{pmatrix}.$$

It is worth mentioning that, although the operator S above does not attain its norm, it satisfies

$$\|S(x)\| \leq \sqrt{2}\|S\|\|x\|_\varphi$$

for every $x \in \ell_2$ and every normalized functional $\varphi \in \ell_2^*$. Therefore it does not become a counterexample to the Barton-Friedman "Little Grothendieck's Theorem". In fact we do not know if Theorem 1.3 of [BF] is true.

From the proof of [BF, Theorem 1.3], it may be concluded that if T is a bounded linear operator from a complex JB*-triple $\mathcal{E}$ to a complex Hilbert space $\mathcal{H}$ whose second transpose T^{**} attains its norm at a complete tripotent, then there exists a norm one functional $\varphi \in \mathcal{E}^*$ such that

$$\|T(x)\| \leq \sqrt{2}\|T\|\|x\|_\varphi$$

for all $x \in \mathcal{E}$, where $\|x\|_\varphi^2 = \varphi\{x, x, e\}$ and $e \in \mathcal{E}^{**}$ is a tripotent with $\varphi(e) = 1$.

If T^{**} attains its norm, the norm is attained at a complete tripotent (see the proof of Theorem 4.3). Finally, since the set of all operators $T \in BL(\mathcal{E}, \mathcal{H})$ such that T^{**} attains its norm is norm dense in $BL(\mathcal{E}, \mathcal{H})$, (see [L, Theorem 1]), the result of Barton and Friedman can be formulated as follows.

Theorem 1.1 *Let $\mathcal{E}$ be a complex JB*-triple and let $\mathcal{H}$ be a complex Hilbert space. Then the set of those bounded linear operators T from $\mathcal{E}$ to $\mathcal{H}$ such that there exists a norm one functional $\varphi \in \mathcal{E}^*$ satisfying*

$$\|T(x)\| \leq \sqrt{2}\|T\|\|x\|_\varphi$$

for all $x \in \mathcal{E}$, is norm dense in the set of all bounded linear operators from $\mathcal{E}$ to $\mathcal{H}$.

In this paper we prove a similar result for the most general class of Banach spaces called real JB*-triples.

Complex JB*-triples were introduced by Kaup [K1] in the study of bounded symmetric domains in complex Banach spaces. He shows that every bounded symmetric domain in a complex Banach space is biholomorphically equivalent to the open unit ball of a complex JB*-triple [K2]. Every C*-algebra and every JB*-algebra are JB*-triples with triple product $\{x, y, x\} := xy^*x$ and $\{a, b, c\} := (a \circ b^*) \circ c + (c \circ b^*) \circ a - (a \circ c) \circ b^*$ respectively. See [U], [R], [Ru] and [CM] for the general theory of JB*-triples.

Definitions of real JB*-triples have been introduced in ([U],[IKR],[DR]) and we adopt the definition of [IKR] in this paper. Real JB*-triples are defined as closed real subtriples of complex JB*-triples. The class of real JB*-triples is bigger than the class of complex JB*-triples. Every complex JB*-triple, JB-algebra, real C*-algebra and J*B-algebra is a real JB*-triple (see [IKR], [HS], [G] and [A]). Recently real JB*-triples have been the object of intensive investigations (see for example [D], [CDRV], [IKR], [K3], [CGR], [MP] and [PS]).

The aim of this paper is to obtain a “Little Grothendieck’s Theorem” for real JB*-triples. Section 2 presents some preliminary results. In Sect. 3 we proceed with the study of the “Little Grothendieck Theorem” in the particular case of a JB-algebra. This result will be very useful in the proof of the main result. Finally Sect. 4 provides a detailed proof of the “Little Grothendieck Theorem” for real JB*-triples. In the complex case the proof of the Little Grothendieck Theorem is based in the fact that $it(L(a,b)+L(b,a))$ is a derivation for all $t\in\mathbb{R}$ and $a,b\in\mathcal{E}$ where $\mathcal{E}$ is a complex JB*-triple and so $\exp(it(L(a,b)+L(b,a)))$ is an isometric bijection for every t in $\mathbb{R}$, $a,b\in\mathcal{E}$. In the real case $it(L(a,b)+L(b,a))$ does not make sense but we can use that $\delta(a,b):=L(a,b)-L(b,a)$ is a derivation for all a,b in a real JB*-triple E and then $\exp(t(L(a,b)-L(b,a)))$ is an isometric bijection for every t in $\mathbb{R}$, $a,b\in E$ (see [IKR, Proposition 2.5]). This fact will be the basic idea in the proof of the main result.

2 Background

We recall that a complex JB*-triple is a complex Banach space $\mathcal{E}$ with a continuous triple product $\{.,.,.\}:\mathcal{E}\times\mathcal{E}\times\mathcal{E}\to\mathcal{E}$ which is bilinear and symmetric in the outer variables and conjugate linear in the middle variable, and satisfies:

1. (Jordan Identity) $L(a,b)\{x,y,z\}=\{L(a,b)x,y,z\}-\{x,L(b,a)y,z\}$ $+\{x,y,L(a,b)z\}$ for all a,b,c,x,y,z in $\mathcal{E}$, where $L(a,b)x:=\{a,b,x\}$;
2. The map $L(a,a)$ from $\mathcal{E}$ to $\mathcal{E}$ is an hermitian operator with spectrum ≥ 0 for all a in $\mathcal{E}$;
3. $\|\{a,a,a\}\|=\|a\|^3$ for all a in $\mathcal{E}$.

Following [IKR], a real Banach space E together with a trilinear map $\{.,.,.\}:E\times E\times E\to E$ is called a real JB*-triple if there is a complex JB*-triple $\mathcal{E}$ and an $\mathbb{R}$-linear isometry λ from E to $\mathcal{E}$ such that $\lambda\{x,y,z\}=\{\lambda x,\lambda y,\lambda z\}$ for all x,y,z in E.

Real JB*-triples are essentially the closed real subtriples of complex JB*-triples and, by [IKR, Proposition 2.2], given a real JB*-triple E there exists a unique complex JB*-triple $\widehat{E}$ and a unique conjugation (conjugate linear and isometric mapping of period 2) τ on $\widehat{E}$ such that $E=\widehat{E}^\tau:=\{x\in\widehat{E}:\tau(x)=x\}$. In fact, $\widehat{E}$ is the complexification of the vector space E, with triple product extending in a natural way the triple product of E and a suitable norm. For the rest of the paper, given a real JB*-triple E, we will denote by $\widehat{E}$ its complexification and by τ the canonical conjugation on $\widehat{E}$ such that $E=\widehat{E}^\tau$.

JBW*-triples (real JBW*-triples resp.) are JB*-triples (real JB*-triples resp.) which are Banach dual spaces [BT] ([MP] resp).

Real and complex JB*-triples are Jordan triples. Therefore, given a tripotent e ($\{e, e, e\} = e$) in a real or complex JB*-triple U, there exist two decompositions of U

$$U = U_0(e) \oplus U_1(e) \oplus U_2(e) = U^1(e) \oplus U^{-1}(e) \oplus U^0(e)$$

where $U_k(e) = \{x \in U : L(e, e)x = \frac{k}{2}x\}$ for $k = 0, 1, 2$ and $U^k(e)$ is the k-eigenspace of the operator $Q(e)x := \{e, x, e\}$ for $k = 1, -1, 0$. It is well known that if $\mathcal{E}$ is a complex JB*-triple and $e \in \mathcal{E}$ is a tripotent then $\mathcal{E}_2(e)$ is a JB*-algebra with product $x \circ y := \{x, e, y\}$ and involution $x^* := \{e, x, e\}$. In the case that E is a real JB*-triple and $e \in E$ is a tripotent, $E^1(e)$ is a JB-algebra with product $x \circ y := \{x, e, y\}$. $E_k(e)$ is called the Peirce k-space of e. For a real or complex JB*-triple U the following rules are satisfied:

1. $U_2(e) = U^1(e) \oplus U^{-1}(e)$ and $U^0(e) = U_1(e) \oplus U_0(e)$
2. $\{U^i(e), U^j(e), U^k(e)\} \subseteq U^{ijk}(e)$ if $ijk \neq 0$
3. $\{U_i(e), U_j(e), U_k(e)\} \subseteq U_{i-j+k}(e)$, where $i, j, k = 0, 1, 2$ and $U_l(e) = 0$ for $l \neq 0, 1, 2$.
4. $\{U_0(e), U_2(e), U\} = \{U_2(e), U_0(e), U\} = 0$.

The last two rules are known as Peirce arithmetic. In particular, Peirce k-spaces are subtriples.

The projection $P_k(e)$ of U onto $U_k(e)$ is called the Peirce k-projection of e. These projections are given by

$$\begin{aligned} P_2(e) &= Q(e)^2; \\ P_1(e) &= 2(L(e, e) - Q(e)^2); \\ P_0(e) &= Id_U - 2L(e, e) + Q(e)^2. \end{aligned}$$

Throughout this paper we will denote by $P^k(e)$ the natural projection $P^k(e) : U \to U^k(e)$ ($k : 1, 0, -1$).

Remark 2.1 Let E be a real JB*-triple, we write $\widehat{E}$ for its complexification and τ for the canonical conjugation on $\widehat{E}$ with $\widehat{E}^\tau = E$. Let us consider

$$\phi : \widehat{E}^* \to \widehat{E}^*$$

by

$$\phi(f)(z) = \overline{f(\tau(z))}.$$

From [IKR] we can assure that ϕ is a conjugation (conjugate-linear isometry of period 2) on $\widehat{E}^*$. Furthermore the map

$$(\widehat{E}^*)^\phi := \{f \in \widehat{E}^* : \phi(f) = f\} \to (\widehat{E}^\tau)^*$$

$$f \mapsto f|_E$$

is an isometric bijection. In the same way if E is a real JBW*-triple and we write $\widehat{E}$ for its complexification (which is a complex JBW*-triple) the predual of E, E_* can be regarded as $(\widehat{E}_*)^\phi := \{f \in \widehat{E}_* : \phi(f) = f\}$.

The construction can be realized one more time to get a conjugation $\widehat{\phi}$ on $\widehat{E}^{**}$ such that

$$E^{**} \cong (\widehat{E}^{**})^{\widehat{\phi}}.$$

It is well known that the surjective linear (resp. conjugate linear) isometries between two complex JB*-triples are exactly the triple linear (resp. conjugate linear) isomorphisms [K2, Proposition 5.5]. Moreover if $\mathcal{E}$ is a JBW*-triple then every surjective linear or conjugate linear isometry on $\mathcal{E}$ is weak* continuous [BT], in particular if we have a JBW*-triple with a conjugation τ then τ is automatically weak* continuous.

We recall [FR, Proposition 2] that if $\mathcal{E}$ is a complex JBW*-triple and $f \in \mathcal{E}_*$ then there exists a unique tripotent $e(f)$ in $\mathcal{E}$ such that $f = fP_2(e)$ and $f|_{\mathcal{E}_2(e)}$ is a faithful normal positive functional on the JBW*-algebra $\mathcal{E}_2(e)$. This tripotent is called the support tripotent of f.

Since the concept of support tripotent is preserved by weak* continuous automorphisms, given a complex JBW*-triple $\mathcal{E}$ with a conjugation τ, we can find a relationship between the support tripotents of f and $\phi(f)$ for every $f \in \mathcal{E}_*$ (Where ϕ is the conjugation constructed from τ like in Remark 2.1).

Lemma 2.2 *Let $\mathcal{E}$ be a complex JBW*-triple, let τ be a conjugation on $\mathcal{E}$, $f \in \mathcal{E}_*$ and let e be the support tripotent of f. Then $\tau(e)$ is the support tripotent of $\phi(f)$. In particular if $\phi(f) = f$ and e is its support tripotent then $\tau(e) = e$ (by the uniqueness of the support tripotent).*

Proof. The proof is immediate from the previous comments. □

Let E be a real JB*-triple and let f be a norm one functional on E. f can be regarded as a norm one functional on the complexification of E, $\widehat{E}$, such that $\phi(f) = f$ (see Remark 2.1). From [FR, Proposition 2] there exists the support tripotent of f in $\widehat{E}^{**}$. By the previous Lemma, this support tripotent of f in $\widehat{E}^{**}$ is in fact in E^{**} and we call it the *support tripotent* of f in E^{**}.

The following Lemma is contained in [PS] and we include here by completeness reasons. It will play a very important role in the proof of the main Theorem.

Lemma 2.3 *Let E be a real JB*-triple, let e be a tripotent of E and $f \in E^*$ such that $\|f|_{E_2(e)}\| = \|f\| = 1$. Then $f = f \circ P_2(e)$. Moreover if $f(e) = 1$ then $f = f \circ P^1(e)$.*

Proof. By [MP, Lemma 2.9] we have $f = f \circ P_2(e)$. Let $y \in E^{-1}(e)$. We may assume without loss of generality $f(y) \geq 0$. Therefore $\{e, e, y\} = y$, $\{e, y, e\} = -y$ and we have the order estimate

$$\begin{aligned}\{e + ty, e + ty, e + ty\} &= \{e, e, e\} + 2t\,\{e, e, y\} + \{e, y, e\} + O(|t|^2) \\ &= e + ty + O(|t|^2)\end{aligned}$$

for $t > 0$ in $\mathbb{R}$. Hence by induction we get

$$(e + ty)^{3^n} = e + ty + O(|t|^2) \qquad (n = 1, 2, \ldots)\,.$$

Therefore, for $t > 0$,

$$\begin{aligned}\|e + ty\| &\geq f(e + ty) = 1 + tf(y) \\ (1 + tf(y))^{3^n} &\leq \|e + ty\|^{3^n} = \|(e + ty)^{3^n}\| \\ &= \|e + ty + O(|t|^2)\| \\ &\leq 1 + t\|y\| + O(|t|^2) \\ 1 + 3^n tf(y) + O(|t|^2) &\leq 1 + t\|y\| + O(|t|^2) \\ 3^n f(y) + O(|t|) &\leq \|y\| + O(|t|).\end{aligned}$$

Thus, for $t \downarrow 0$, we obtain

$$f(y) \leq \frac{1}{3^n}\|y\| \qquad (n = 1, 2, \ldots)\,.$$

It follows $f(y) = 0$ for every $y \in E^{-1}(e)$. Since $E_2(e) = E^1(e) \oplus E^{-1}(e)$ and $f = fP_2(e)$, we conclude $f = f \circ P^1(e)$. $\square$

The next Lemma extends [BF, Proposition 1.2] to real JB*-triples.

Lemma 2.4 *Let E be a real JB*-triple, $f \in E^*$ with $\|f\| = 1$ and let $e \in E$ such that $f(e) = \|e\| = 1$. Then*

$$f\{x, y, e\} = f\,\{y, x, e\}$$

$$f\,\{x, x, e\} \geq 0$$

for all $x, y \in E$, and the Cauchy-Schwartz inequality holds:

$$|f\,\{x, y, e\}\,|^2 \leq f\,\{x, x, e\}\;\; f\,\{y, y, e\}$$

Moreover if $z \in E$ with $f(z) = \|z\| = 1 =$ then

$$f\,\{x, x, e\} = f\,\{x, x, z\}$$

for all $x \in E$ and if we define $\|x\|_f := (f\,\{x, x, e\})^{\frac{1}{2}}\ \forall x \in E$ then

$$\|x\| = Sup\{\|x\|_f : \|f\| = 1\}.$$

Proof. Let $\widehat{E}$ denote the complexification of E. By Remark 2.1 we can see f as an element of $\widehat{E}^*$ with $\|f\| = f(e) = 1$ and $\phi(f) = f$. From [BF, Proposition 1.2]

$$f\{a,b,e\} = \overline{f\{b,a,e\}},$$

$$f\{a,a,e\} \geq 0,$$

$$|f\{a,b,e\}|^2 \leq f\{a,a,e\}\ f\{b,b,e\}$$

$\forall a,b \in \widehat{E}$. Moreover if $z \in \widehat{E}$ with $f(z) = \|z\| = 1 =$ then

$$f\{a,a,e\} = f\{a,a,z\}$$

for all $a \in \widehat{E}$. Now applying that $\phi(f) = f$ $(f \in E^*)$ we have that $f(E) \subseteq \mathbb{R}$ and then we obtain the first three statements.

For the last affirmation we proceed as follows. Let $x \in E$ with $\|x\| = 1$, by the Hahn-Banach Theorem there exists $f \in E^*$ with $\|f\| = f(x) = 1$. We consider $f \in \widehat{E}^*$ with $\phi(f) = f$. Let $u \in \widehat{E}^{**}$ the support tripotent of f. Again by [BF, Proof of Proposition 1.2] $\|x\| = f\{x,x,u\} = \|x\|_f$ in $\widehat{E}$. Since $\phi(f) = f$, Remark 2.1 and Lemma 2.2, assure that the support tripotent u of f is in the bidual of E, i. e. $u \in E^{**}$. Therefore we obtain the last statement. □

From this Lemma, as in the complex case [BF], given a real JB*-triple E and a norm one functional f we can build a pre-Hilbertian seminorm $\|.\|_f$ on E, a real Hilbert space H_f and a natural map $J_f : E \to H_f$ with $\|J_f(x)\| \leq \|x\|$ for all $x \in E$. The real Hilbert space H_f is the completion of E/N_f where $N_f := \{x \in E : \|x\|_f = 0\}$ and J_f is the natural projection.

$$\|J_f x\|_f = \|x\|_f = (f\{x,x,e\})^{\frac{1}{2}} \leq \|x\|$$

where e is the support tripotent of f in E^{**}.

3 JB-Algebras

One of the most important examples of real JB*-triples are JB-algebras. We recall that every JB-algebra is a real JB*-triple with triple product given by $\{x,y,z\} := (x \circ y) \circ z + (z \circ y) \circ x - (x \circ z) \circ y$. This section is devoted to prove a “little Grothendieck’s Theorem” in the case of a JB-algebra.

If $\mathcal{A}$ is a (complex) JB*-algebra, $\mathcal{A}$ can be regarded as (complex) JB*-triple under the triple product $\{x,y,z\} := (x \circ y^*) \circ z + (z \circ y^*) \circ x - (x \circ z) \circ y^*$. The “Grothendieck’s Theorem” for (complex) JB*-algebras (which is a verbatim extension of Haagerup’s proof for C*-algebras [H2]), is stated by Chu, Iochum and Loupias in [CIL, Theorem 2.].

Theorem 3.1 (Little Grothendieck's Theorem for JB*-algebras) *Let $\mathcal{A}$ be a (complex) JB*-algebra, let $\mathcal{H}$ be a complex Hilbert space and $T : \mathcal{A} \to \mathcal{H}$ a bounded linear operator. Then there is a state $\varphi \in \mathcal{A}^*$ such that*

$$\|T(z)\| \leq 2\|T\| \Big(\varphi(z \circ z^*)\Big)^{\frac{1}{2}}$$

for all $z \in \mathcal{A}$.

We can now state the analogue of "Little Grothendieck's Theorem" for (real) JB-algebras.

Theorem 3.2 (Little Grothendieck's Theorem for JB-algebras) *Let A be a JB-algebra, let H be a real Hilbert space and let $T : A \to H$ be a bounded linear operator. Then there is a state $\varphi \in A^*$ such that*

$$\|T(x)\| \leq 2\sqrt{2}\|T\| \Big(\varphi(x^2)\Big)^{\frac{1}{2}}$$

for all $x \in A$.

Proof. We denote by $\widehat{A}$ and $\mathcal{H}$ the complexifications of A and H respectively. $\widehat{A}$ is a JB*-algebra whose self-adjoint part is A and $\mathcal{H}$ is a complex Hilbert space. Consider $\widehat{T} : \widehat{A} \to \mathcal{H}$ the complex linear extension of T. It is easy to check that $\|\widehat{T}\|^2 \leq 2\|T\|^2$. From Theorem 3.1 there exists a state $\psi \in \widehat{A}^*$ such that

$$\|\widehat{T}(z)\|^2 \leq 4\|\widehat{T}\|^2 \psi(z \circ z^*) \leq 8\|T\|^2 \psi(z \circ z^*)$$

for all $z \in \widehat{A}$.

In particular if $x \in A$

$$\|T(x)\|^2 \leq 8\|T\|^2 \psi(x \circ x).$$

Since ψ is a state of $\widehat{A}$, $\psi|_A$ is a state of A, and the proof is concluded. □

4 Main Result

This section will be devoted to the proof of the "Little Grothendieck's Theorem for real JB*-triples". We start introducing some terminology.

Definition 4.1 *If E is a real JB*-triple and H is a real Hilbert space, we will say that a bounded linear operator T from E to H satisfies the "Little Grothendieck's inequality" if there exists a norm one functional $\varphi \in E^*$ with*

$$\|T(x)\| \leq 4\sqrt{2}\ \|T\|\ \|x\|_\varphi$$

for all $x \in E$. Let $LG(E, H)$ denote the set of all operators $T \in BL(E, H)$ satisfying the "Little Grothendieck's inequality".

We have seen (Lemma 2.4) that if E is a real JB*-triple, and f is a norm one functional on E, we can define a pre-Hilbertian seminorm $\|.\|_f$ on E given by $\|x\|_f^2 = f\{x,x,e\}$ where e is the support tripotent of f in E^{**}. Suppose that e is a complete tripotent ($E_0(e) = 0$) of E such that $f(e) = 1$. The following Lemma states that the projections associated with e, $P_k(e)$ $(k : 0,1,2)$ and $P^k(e)$ $(k : 1,-1,0)$ are $\|.\|_f$-contractive.

Lemma 4.2 *Let E be a real JB*-triple, and let e be a complete tripotent of E. Suppose that f is a norm one functional on E such that $f(e) = 1$ then*

1. $\|x\|_f^2 = \|P_1(e)x\|_f^2 + \|P_2(e)x\|_f^2$ *($x \in E$).*
2. $\|P_2(e)x\|_f^2 = \|P^1(e)x\|_f^2 + \|P^{-1}(e)x\|_f^2$ *($x \in E$).*

In particular $P_k(e)$ $(k : 0,1,2)$ and $P^k(e)$ $(k : 1,-1,0)$ are $\|.\|_f$-contractive.

Proof. Let $x \in E$ and let us denote by $x^k := P^k(e)x$ and $x_k := P_k(e)x$. Since e is complete $P_0(e) = 0$ ($x = x_1 + x_2$ $\forall x \in E$). Using Lemma 2.4, Peirce Arithmetic and Lemma 2.3 we can check that

$$\begin{aligned}\|x\|_f^2 = \|x_1+x_2\|_f^2 &= f\{x_1+x_2, x_1+x_2, e\} \\ &= f\{x_1,x_1,e\} + f\{x_2,x_2,e\} + 2f\{x_1,x_2,e\} \\ &= f\{x_1,x_1,e\} + f\{x_2,x_2,e\} = \|x_1\|_f^2 + \|x_2\|_f^2.\end{aligned}$$

Similar considerations show that $\left\{x^1, x^{-1}, e\right\} \in E^{1(-1)1}(e) = E^{-1}(e)$ hence applying Lemma 2.3 again

$$\begin{aligned}\|P_2(e)x\|_f^2 &= \|x^1 + x^{-1}\|_f^2 \\ &= \|x^1\|_f^2 + \|x^{-1}\|_f^2 + 2f\left\{x^1, x^{-1}, e\right\} = \|x^1\|_f^2 + \|x^{-1}\|_f^2.\end{aligned}$$

This completes the proof. □

We can now state the analogue of [BF, Theorem 1.3] for real JB*-triples. As we have mentioned in the introduction this is a ”Little Grothendieck's Theorem” with an additional hypothesis for T^{**}. Concretely we are going to prove that if T is a bounded linear operator from a real JB*-triple E to a real Hilbert space H such that T^{**} attains its norm, then $T \in LG(E,H)$.

Theorem 4.3 *Let E be a real JB*-triple, let H be a real Hilbert space and let $T : E \to H$ be a bounded linear operator. Suppose that T^{**} attains its norm. Then there exists a norm one functional φ on E such that*

$$\|T(x)\| \leq 4\sqrt{2}\|T\|\|x\|_\varphi$$

for all $x \in E$.

Proof. We can suppose that $\|T\| = 1$. We first prove that, in fact, T^{**} attains its norm at a complete tripotent $e \in E^{**}$. By hypothesis, T^{**} attains its norm, so we know that $\|T^{**}\| = \|T^{**}(c)\| = \|T\| = 1$ for $c \in E^{**}$. Let us consider $\rho(x) =< T^{**}(x)|T^{**}(c) >$. It is clear that ρ is a norm one and weak*-continuous functional on E^{**}, so by Alaoglu's Theorem, the Krein-Milman Theorem and the characterization of the complete tripotents, there exists a complete tripotent $e \in E^{**}$ such that

$$\begin{aligned}\|T^{**}\| = \rho(e) = \langle T^{**}(e)|T^{**}(c)\rangle &\leq \|T^{**}(e)\| \, \|T^{**}(c)\| \\ = \|T^{**}(e)\| \leq \|T^{**}\|,\end{aligned}$$

thus

$$\|T^{**}(e)\| = \|T^{**}\|.$$

Now we suppose that E is a real JBW*-triple and T is norm one and $w*$-continuous (we can consider $T^{**} : E^{**} \to H$) and there is a complete tripotent $e \in E$ such that $\|T\| = \|T(e)\|$. Let us define

$$\xi(x) :=< T(x)/T(e) > \quad (x \in E).$$

It is clear that $1 = \|\xi\| = \xi(e)$.

Let $a \in E$ and let us denote $a^k := P^k(e)a$ and $a_k := P_k(e)a$. It is well known [IKR, Proposition 2.5] that $\exp(t(L(a,e) - L(e,a)))$ is an isometric bijection for all $t \in \mathbb{R}$ and $a, e \in E$. Then

$$\begin{aligned}1 &\geq \|T(\exp(t(L(a,e) - L(e,a)))e)\|^2 \\ &= \|T(e) + tT((L(a,e) - L(e,a))e) \\ &\quad + \frac{t^2}{2}T((L(a,e) - L(e,a))^2 e)\|^2 + O(|t|^3)\end{aligned}$$

for all $t \in \mathbb{R}$. Therefore

$$\begin{aligned}&\|T(e) + tT((L(a,e) - L(e,a))e) + \frac{t^2}{2}T((L(a,e) - L(e,a))^2 e)\|^2 \\ &\quad \leq 1 + O(|t|^3) \\ &\|T(e) - tT((L(a,e) - L(e,a))e) + \frac{t^2}{2}T((L(a,e) - L(e,a))^2 e)\|^2 \\ &\quad \leq 1 + O(|t|^3)\end{aligned}$$

Now from the parallelogram law we obtain that

$$\begin{aligned}&\|T(e) + \frac{t^2}{2}T((L(a,e) - L(e,a))^2 e)\|^2 + \|tT((L(a,e) - L(e,a))e)\|^2 \\ &\quad \leq 1 + O(|t|^3) \qquad (t.1)\end{aligned}$$

Since

$$\begin{aligned}&\left\|T(e)+\frac{t^2}{2}T((L(a,e)-L(e,a))^2e)\right\|^2\\&\geq\left\langle T(e)+\frac{t^2}{2}T((L(a,e)-L(e,a))^2e)/T(e)\right\rangle^2\\&=\left(1+\frac{t^2}{2}\xi((L(a,e)-L(e,a))^2e)\right)^2\end{aligned}$$

$(t.1)$ shows that

$$t^2\|T((L(a,e)-L(e,a))e)\|^2\leq t^2\xi(-(L(a,e)-L(e,a))^2e)+O(|t|^3)$$

and

$$\|T((L(a,e)-L(e,a))e)\|^2\leq\xi(-(L(a,e)-L(e,a))^2e)+O(|t|)\ \ (t\in\mathbb{R})$$

And letting $t\to 0$ we obtain that

$$\|T((L(a,e)-L(e,a))e)\|^2\leq\xi(-(L(a,e)-L(e,a))^2e)\qquad (t.2)$$

Now we must compute $\xi(-(L(a,e)-L(e,a))^2e)$. In this part of the proof Lemma 2.4 and Peirce Arithmetic play a very important role. $-(L(a,e)-L(e,a))^2e=-\{a,e,\{a,e,e\}\}+\{a,e,\{e,a,e\}\}+\{e,a,\{a,e,e\}\}-\{e,a,\{e,a,e\}\}$. By Peirce Arithmetic $\{\{e,a,e\},a,e\}=\{e,\{a,e,a\},e\}$. Now using Peirce Arithmetic, Lemma 2.4 and Lemma 2.3

$$\begin{aligned}\xi(\{e,a,\{e,a,e\}\})&=\xi(\{e,\{a,e,a\},e\})\\&=\xi(\{\{a,e,a\},e,e\}\\&=\xi(\{\{a_1,e,a_1\},e,e\})+2\xi(\{\{a_1,e,a_2\},e,e\})\\&\quad+\xi(\{\{a_2,e,a_2\},e,e\})\\&=\xi(\{\{a_2,e,a_2\},e,e\})\\&=\xi(\{a_2,e,a_2\})\\&=\xi(\{a,e,a\})\end{aligned}\qquad (t.3)$$

By the same method

$$\begin{aligned}\xi(\{a,e,\{e,a,e\}\})&=\xi(\{a_1,e,\{e,a_2,e\}\}+\{a_2,e,\{e,a_2,e\}\})\\&=\xi(\{a_2,e,\{e,a_2,e\}\})\\&=2\xi(\{a_2,a_2,e\})-\xi(\{e,\{a_2,a_2,e\},e\})\\&=2\xi(\{a_2,a_2,e\})-\xi(\{\{a_2,a_2,e\},e,e\})\\&=\xi(\{a_2,a_2,e\})\end{aligned}\qquad (t.4)$$

$$\begin{aligned}\xi(\{e,a,\{a,e,e\}\})&=\xi(\{\{a,e,e\},a,e\})\\&=\xi(\{a,\{a,e,e\},e\})\\&=\xi(\{a_2,a_2,e\})+\frac{1}{2}\xi(\{a_1,a_1,e\})\end{aligned}\qquad (t.5)$$

and

$$\xi(\{a,e,\{a,e,e\}\}) = \xi(\{a,e,a\}) \qquad (t.6)$$

We conclude from (t.3),(t.4),(t.5) y (t.6) that

$$\begin{aligned}
&\xi(-(L(a,e)-L(e,a))^2 e) \\
&\quad = -2\xi\{a,e,a\} + 2\xi\{a_2,a_2,e\} + \frac{1}{2}\xi\{a_1,a_1,e\} \\
&\quad = 2\xi\{\{e,e,a\},\{e,e,a\},e\} - 2\xi\{a,e,a\}
\end{aligned}$$

Finally from (t.2) we have

$$\begin{aligned}
&\|T(\{a,e,e\}-\{e,a,e\})\|^2 \\
&\quad \leq 2\xi(\{\{e,e,a\},\{e,e,a\},e\} - \{a,e,a\})\ (a \in E) \qquad (t.7)
\end{aligned}$$

Since e is a complete tripotent, $L(e,e)$ is a bijection. Hence if we denote $x = \{e,e,a\}$, Peirce Arithmetic and (t.7) show that

$$\|T(x - \{e,x,e\})\|^2 \leq 2\xi(\{x,x,e\} - \{x,e,x\})\ (x \in E) \qquad (t.8)$$

In particular, as $x_1 \in E_1(e)$ by Peirce Arithmetic and Lemma 2.3 $\{e,x_1,e\} = \{x_1,e,x_1\} = 0$ then from (t.8)

$$\|T(x_1)\|^2 \leq 2\xi\{x_1,x_1,e\} = 2\|x_1\|_\xi^2 \qquad (t.9)$$

Similarly as $x^{-1} \in E^{-1}(e)$ ($\left\{e,x^{-1},e\right\} = -x^{-1}$) then

$$\|T(x^{-1})\|^2 \leq \xi\left\{x^{-1},x^{-1},e\right\} = \|x^{-1}\|_\xi^2 \qquad (t.10)$$

The problem is that from $(t.8)$ we are unable to estimate $\|T(x^1)\| \leq M\|x^1\|_\xi$ for all x^1 in the JBW-algebra $E^1(e)$ (with unit e), and some positive constant M, as we have made before for $x_1 \in E_1(e)$ and $x^{-1} \in E^{-1}(e)$. At this point we apply Theorem 3.2 to obtain a state ψ of $E^1(e)$ such that

$$\begin{aligned}
\|T(x^1)\|^2 \leq 8\psi(x^1 \circ x^1) &= 8\psi\left\{x^1,x^1,e\right\} \\
&= 8\|x^1\|_\psi^2\ \ (x^1 \in E^1(e)) \qquad (t.11)
\end{aligned}$$

We can see $\psi = \psi P^1(e)$ as a linear functional on E using Lemma 2.3.

Let $x \in E$ from (t.9), (t.10) and (t.11) $\|T(x)\| \leq \|T(x_1)\| + \|T(x^{-1})\| + \|T(x^1)\| \leq \sqrt{8}\|x^1\|_\psi + \|x^{-1}\|_\xi + \sqrt{2}\|x_1\|_\xi$. Hence Lemma 4.2 shows that

$$\|T(x)\| \leq \sqrt{8}\|x\|_\psi + \|x\|_\xi + \sqrt{2}\|x\|_\xi$$

$$= \sqrt{8}\|x\|_\psi + (1+\sqrt{2})\|x\|_\xi \leq \sqrt{8}(\|x\|_\psi + \|x\|_\xi)$$

$$\|T(x)\|^2 \leq 8(\|x\|_\psi^2 + \|x\|_\xi^2 + 2\|x\|_\psi\|x\|_\xi) \leq$$

$$\leq 16(\|x\|_\psi^2 + \|x\|_\xi^2) = 16(\psi\{x,x,e\} + \xi\{x,x,e\}) =$$
$$= 32\frac{\psi+\xi}{2}\{x,x,e\} = 32\,\varphi\{x,x,e\} = 32\,\|x\|_\varphi^2$$

where $\varphi = \frac{\psi+\xi}{2}$ is a norm one functional on E and $\varphi(e) = 1$. □

Remark 4.4 In the setting of the proof of the previous Theorem, we can see that if we can estimate $\|T(x^1)\|^2 \leq M^2\|x^1\|_\xi^2$ for $x^1 \in E^1(e)$ (where $\xi(x) :=< T(x)/T(e) >$) then it is easy to obtain that $\|T(x)\| \leq (1+\sqrt{2}+M)\|x\|_\xi$. It is trivial to estimate $\|T(x^1)\|^2 = \|x^1\|_\xi^2$ when e is a minimal tripotent ($E^1(e) = \mathbb{R}e$).

So if E is a real JB*-triple and e is a minimal tripotent of E. From [PS] $E_2(e)$ is a real Hilbert space (with inner product $< a,b >:= \frac{1}{4}(\{a+b,a+b,e\} - \{a-b,a-b,e\})$). $Q(e) : E \to E_2(e)$ is a bounded linear operator with $\|Q(e)\| = 1 = \|Q(e)e\|$ so from the previous Remark

$$\|Q(e)x\| \leq (2+\sqrt{2})(\xi\{x,x,e\})^{\frac{1}{2}} \ \ (x \in E)$$

where

$$\xi(x) =< Q(e)x/e >$$
$$= \frac{1}{4}(\{Q(e)x+e, Q(e)x+e, e\} - \{Q(e)x-e, Q(e)x-e, e\}).$$

From the previous Theorem 4.3 we can now prove the analogous of Theorem 1.1 for real JB*-triples which is the main result of the paper.

Theorem 4.5 *Let E be a real JB*-triple and let H be a real Hilbert space. Then the set $LG(E,H)$ is norm dense in the set of all bounded linear operator from E to H.*

Proof. The proof straightforward from Theorem 4.3 and the norm denseness of the set of all bounded linear operators whose second transpose attains its norm [L]. □

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