JORDAN STRUCTURES IN ANALYSIS

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Dedicated to Inés Espinar de la Paz

SUMMARY:

Introduction.

It happens frequently that a reader attracted by the suggestive title of a given paper, after reading it, becomes disappointed because of the peculiar or partial point of view under which the paper was written. To avoid such a situation concerning the survey we are beginning, the reader should be forewarned that it will deal not only with "Jordan structures in Analysis" but also (parodying the title of Koecher's paper [Ko1]) with "Analysis in Jordan structures". This means that sometimes Jordan structures arise normally in the treatment of more or less classical problems in Analysis, but that also analysts have their own right to apply their methods to Jordan structures. In this way they often improve or clarify the abstract purely algebraic theory in a context more familiar for them. Both sides of the topic (frequently closely related) will be collected in this survey.

Of course the first side seems to be more suggestive, so let's try to attract the attention of the uninterested reader by reviewing (in a very rough and partial form) the impressive work of Kaup on the algebraic characterization of bounded symmetric domains on complex Banach spaces: if the open unit ball of a complex Banach space behaves holomorphically like that of a C*-algebra, then the Banach space itself is "almost" a C*-algebra, and there is an intrinsically defined triple product $\{\ldots\}$ on it that behaves algebraically and geometrically like the one obtained from the binary associative product of a C*-algebra by taking $\{xyz\}=\frac{1}{2}(xy^*z+zy^*x)$. This is the nice way of deriving the ternary Jordan identity from purely holomorphic properties of Banach spaces. The resulting mathematical creature, called a JB*-triple, has become a very useful mathematical model leading to a better understanding of some aspects of C*-algebras (for example the order-free aspects, that curiously are more than that was thought). This has been feverishly studied in the last years, most times with points of view not much related to its original holomorphic birth. A large part of Section D in this survey will be devoted to review some recent advances in this field, selected on behalf of a criterion that we shall explain later.

The Spanish school, which we are pleased to represent here, has arrived at the Jordan identity (without premeditation) through a rather different motivation. The Spanish school on Jordan structures can be characterized by its "very nonassociative" point of view, whenever such a point of view becomes possible. Exaggerating somewhat, we can say that the Spanish school has been forced to work on Jordan structures because nonassociative things refuse to be nonassociative as soon as they are suitably flavoured. Thus, if a unital complete normed nonassociative complex algebra is subjected to the geometric Vidav condition characterizing C*-algebras in the associative context, then it is also "almost" a C*-algebra. Moreover, the symmetrization of its product behaves algebraically and geometrically like the symmetrization of the product of a C*-algebra. In this way Vidav's condition in the general nonassociative context gives birth to the binary Jordan identity. As the reader may suspect, mathematical creatures built by this "nonassociative Vidav-Palmer theorem", called noncommutative JB*-algebra, are closely related to Kaup's JB*-triples: every noncommutative JB*-algebra can be seen as a JB*-triple under a suitable triple product naturally derived from its binary product, and, as a partial converse, every JB*-triple can be seen as a JB*-subtriple of a suitable (commutative) JB*-algebra. Because of this close dependence, commutative or noncommutative JB*-algebras have been not too much worked in the last years. In any case the relevant recent results about them, together with the above-mentioned selected ones about JB*-triples, will be reviewed in Section D of our survey. The very recent improvement of the classical structure theory of JB*-algebras (courtesy of Zel'manov's prime theorem) will be considered in Section F.

As the title itself says, Section A will deal with the subject, already mentioned in connection with the nonassociative Vidav-Palmer theorem, of obtaining identities from geometric conditions on general nonassociative normed algebras. It may be convenient to remark that usually "geometric conditions" mean conditions on the norm which are fine enough that they are not preserved by equivalent renorming. (Requirements or results that are preserved under equivalent renorming are called then "of algebraic and topological type" or, simply, "nongeometric".) The most relevant recent results in this direction, such as an improved version of the "nonassociative Gelfand-Naimark theorem", the description of absolute-valued nonassociative one-sided division algebras by representing "smooth normed" Jordan algebras on their prehilbert spaces, or the Blecher-Ruan-Sinclair nonassociative characterization of non-self-adjoint algebras of bounded linear operators on complex Hilbert spaces, will be reviewed. After this characterization, we have the impression that even the associativity of a general normed algebra can be geometrically characterized.

Section B of the survey is devoted to the advances in the nongeometric theory of nonassociative normed algebras and in particular of normed Jordan algebras. Concerning the Jordan theory, most of the advances which have been made involve the existence of a "large" socle in some way. Thus we shall review the spectral characterization of semiprimitive modular annihilator Jordan-Banach algebras, the progress on the problem of the coincidence of the socle and the largest algebraic ideal in semiprimitive Jordan-Banach complex algebras, and the structure theory of prime nondegenerate Jordan-Banach complex algebras with nonzero socle. Concerning nonassociative results, we shall explain in detail the recent treatment of the extended centroid of normed nonassociative algebras. This has led to the important results that ultraprime normed nonassociative complex algebras, as well as primitive Jordan-Banach complex algebras, are centrally closed.

In the short Section C, we shall continue with the ideas of the previous one, reviewing the recent nongeometric theory of Jordan-Banach (*-) triples (*-triple means that the middle variable in the triple product behaves in a conjugate-linear way). Until now, all known results in this field involved the existence of a "large" socle. The results are very fine structure theorems, such as the description of von Neumann regular Jordan-Banach triples, or the classification of prime Jordan-Banach *-triples with nonzero socle and with no nonzero nilpotent elements. The actual concept of the socle of a Jordan triple and a great part of the purely algebraic development of the theory of the socle in Jordan triples have arisen from this analytic motivation.

As we have already commented, Section D will deal with some selected topics in JB*-algebras and JB*-triples. Among the recent advances we shall review, we emphasize here the characterization of the associativity of a JB*-algebra by the absence of nilpotent elements (including also suitable interesting refinements), the description of weakly compact JB*-triples, as well as that of prime JB*-triples with nonzero socle, the recent results on the strong* topology of a JBW*-triple, the progress on the understanding of closed (respectively, w*-closed) inner ideals of JB*-triples (respectively, JBW*-triples), and the study of the continuity with respect to several natural topologies of the one-parameter subgroups of holomorphic automorphisms of the open unit ball of a JB*-triple.

Let's also make a short comment about the material collected in Section E, namely, the H*-theory. H*-theory refers to the reasonably well-behaved binary or ternary products on a real or complex Hilbert space. A complete nonassociative theory of Wedderburn type for H*-things was developed before 1988, including also the classification of "simple" H*-things in some of the most familiar classes of binary or ternary algebras defined by identities, with special emphasis in the complex case. In particular simple complex Jordan H*-algebras and H*-triples were described during this time. (It is interesting to note that the study of Jordan H*-triples, due to W. Kaup, was motivated by the problem of the classification of bounded symmetric domains in complex Hilbert spaces.) Therefore, we shall emphasize in our survey the most recent and important advances in automatic continuity for general nonassociative H*-algebras (automatic continuity of derivations and of dense range homomorphisms in the case of H*-algebras with zero annihilator), as well as the recent classification theorems of simple H*-things in more restrictive and familiar contexts. Among these, the classification of simple Lie H*-algebras (answering affirmatively an old and famous conjecture), that of simple "structurable" complex H*-algebras (which are categorically related to simple complex Lie H*-algebras by means of an extended Kantor-Koecher-Tits construction, that is also reviewed), that of simple real Jordan H*-algebras, and that of simple real or complex Jordan H*-things in a trilinear context rather different to Kaup's *-triple approach.

The concluding section of the survey (Section F) will be devoted to the surprising classification theorems for some classes of prime nondegenerate normed Jordan algebras, obtained very recently by applying the concepts and methods of the proof of Zel'manov prime theorem for Jordan algebras. These are: the classification of prime JB*-algebras (the application of this classification to the answer to some previously unsolved problems in the theory is also referred), that of simple normed Jordan complex algebras with a unit (with the appropriate refinement in the complete normed case), and that of "nondegenerately ultraprime" Jordan-Banach complex algebras. Some relevant advances on the so called "norm-extension" problem, closely related to that of obtaining more general normed versions of Zel'manov prime theorem, will be also reviewed.

Now, let's make some remark on the general planning of the survey. As the reader may have divined, it will be impregnated with the spirit of the Spanish school, emphasizing when it is possible the general nonassociative approach to a given topic, and giving the appropriate importance to nongeometric aspects of the theory of normed nonassociative or Jordan structures and the H*-theory, that are fields recently not much worked outside Spain. In contrast, the field of JB*-triples will be only partially reviewed, considering mainly those aspects of the theory more or less related to the work of the Spanish school. We expect other simultaneous surveys to give a more complete view of this field.

Initially, we thought we would review only the progress made after the 1988 Oberwolfach meeting on Jordan algebras. However, in most of the cases the earlier results seemed to us not to be adequately codified for a relatively nonspecialist reader (this is an important problem to be overcome in the next years). Hence we have included in each section (except in Sections C and F) a summary of pre-1988 results including the definitions of basic concepts without which the nonspecialist reader would not understand the more recent results to be reviewed. Each section (now without exceptions) concludes with a subsection devoted to problems and future directions.

Let's finally made some technical remarks. We have restricted the view of the analytic treatment of Jordan structures to the REAL OR COMPLEX NORMED CASE. Then, when a purely algebraic concept or result is quoted, to be sure about its context of validity it is better to think that we are dealing at least with binary or ternary structures over a field of characteristic zero. Also topological nonnormed real or complex aspects, as well as other interesting analytic results involving exotic base fields, are not included in the survey. Concerning the list of references at the end of the survey, it should be remarked that almost all nonassociative or Jordan results here reviewed have an associative precedent. However, except for very recent works, or when an old result is not easily available in standard books, we have not listed the corresponding preceding paper. On the contrary, we have tried to include in our list all nonassociative references we know at this time, except those concerning the nonassociative C*-creatures (JB-algebras, JB*-algebras, and JB*-triples), and Banach-Lie algebras. Of course we have included the books we know about Jordan structures in Analysis ([Ay2], [Di3], [HSto], [I], [IsSt1], [N1], [Up1], and [Up2]), and also other surveys that may complement the matter reviewed here in one or another direction ([Fe3], [Is2], [Io], [Lo4], [Mc4], [Rod10], and [Rod13]).

A. Some geometric conditions on nonassociative normed algebras giving rise to Jordan algebras.

It is the aim in this section to show how Jordan algebras and other algebras close to them answer some natural questions on the geometry of nonassociative normed algebras.

Summary of results until 1988.

Speaking about Jordan structures (in Analysis), one is tempted to omit the definition of the most familiar such structure, as it is the one of a Jordan algebra, because one thinks that this concept must be well-known for all people interested in the talk, and in any case it doesn't seem convenient to begin a lecture with an unmotivated definition. But, on the other hand, it could be lamentable to immediately discourage some nonspecialist reader because of the absence of a simple definition. A reasonable solution to this doubt may be given by presenting the Jordan identity as a theorem instead of as an axiom, and in fact geometric functional Analysis can provide such a theorem.

To this end, consider a complex nonassociative algebra A endowed with a complete algebra norm $\|.\|$, assume A has a unit $\mathbf{1}$ with $\|\mathbf{1}\|=1$, and that the equality $A=H(A)+iH(A)$ holds. Here $H(A)$ denotes the closed real subspace of A of those elements h in A such that $f(h)$ is a real number whenever f is in the dual Banach space of A and satisfies $\|f\|=f(\mathbf{1})=1$. Algebras A as above are called Vidav algebras (in short V-algebras), and in the associative context, thanks to the Vidav-Palmer theorem (see for example [BoDu]), they are exactly the unital C*-algebras. In the general nonassociative setting we have:

A.1. *Every V-algebra satisfies the following two identities:*

$$a^2(ba)=(a^2b)a \quad (\text{"Jordan axiom"}), \text{ and}$$

$$a(ba)=(ab)a \quad (\text{"flexibility"}).$$

Moreover, if A is a V-algebra, then $H(A)\cap iH(A)=0$, the mapping $h+ik\xrightarrow{*}h-ik$ $(h,k\in H(A))$ is a conjugate-linear algebra involution on A, and the equality $\|U_a(a^*)\|=\|a\|^3$ holds for every a in A (where, for a and b in A, $U_a(b):=a(ba+ab)-a^2b$).

This nonassociative version of the Vidav-Palmer theorem was obtained in [Rod4] after a wide collection of papers, namely [Bo], [Yo1], [Yo3], [Marti2], [Rod3] and [KaMartiRod], and becomes one of the natural ways of introducing the functional analyst to the Jordan world. In fact the variety of algebras arising from this theorem, namely the one of flexible algebras satisfying the Jordan axiom, is known in the literature as that of noncommutative Jordan algebras (see for example [Mc2]). Noncommutative Jordan algebras contain associative (and even alternative) algebras as well as (commutative) Jordan algebras (defined of course as those commutative algebras satisfying the Jordan axiom). Complete normed noncommutative Jordan complex algebras with a conjugate-linear algebra involution * satisfying $\|U_a(a^*)\|=\|a\|^3$ are called noncommutative JB*-algebras, and we will have the opportunity of talking about them later.

Another reasonable approach to nonassociative counterparts of associative C*-algebras may arise by the consideration of complete normed nonassociative complex algebras with a conjugate-linear algebra involution * satisfying the Gelfand-Naimark axiom $\|a^*a\|=\|a\|^2$. In the case of the existence of a unit, these Gelfand-Naimark nonassociative algebras turn out to be very particular cases of the nonassociative Vidav algebras considered above. The fundamental result in this direction was obtained in [Rod3] using a "light" version of A.1, earlier proved in [KaMartiRod], together with a result in [WriYo2] on isometries of JB-algebras, and reads as follows.

A.2. *Let A be a complete normed nonassociative complex algebra with a unit* 1 *and a conjugate-linear vector space involution * satisfying* $1^*=1$ *and* $\|a^*a\|=\|a\|^2$ *for every a in A. Then A is an alternative algebra, and * is an algebra involution on A.*

We recall that an algebra A is said to be alternative if the equalities $a^2b=a(ab)$ and $ba^2=(ba)a$ hold for all a,b in A, and that this requirement is equivalent (via Artin's theorem [Sc1]) to the fact that the subalgebra of A generated by two arbitrary elements of A is associative. Complete normed alternative complex algebras with an algebra involution * satisfying $\|a^*a\|=\|a\|^2$ for every a in the algebra are called alternative C*-algebras. It is easy to see that alternative C*-algebras are noncommutative JB*-algebras. More precisely, alternative C*-algebras are exactly those noncommutative JB*-algebras that are alternative. The algebra of complex octonions can be structured in an essentially unique way as an alternative C*-algebra ([KaMartiRod] and [Bra2]), and it is easy to derive from [ZhSlShShir; Theorem 9 in p. 194] that this alternative C*-algebra is the only prime alternative C*-algebra that is not associative. Then a standard C*-argument reduces the theory of general alternative C*-algebras to the particular cases of primitive associative C*-algebras and of the alternative C*-algebra of complex octonions (see [PaPeRod1] and [Bra2]).

Recall that a smooth normed algebra is a unital normed algebra (i.e.: a normed algebra with a unit **1** satisfying $\|\mathbf{1}\|=1$) whose unit is a smooth point of its closed unit ball, i.e.: there is a unique element ϕ in the dual space of the given normed algebra with $\|\phi\|=\phi(\mathbf{1})=1$. It is well known that $\mathbb{C}$ is the only smooth normed nonassociative complex algebra, and that $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{H}$ (the division algebra of real quaternions), with their usual absolute values as norms, are the only smooth normed associative real algebras. J. I. Nieto [Ni] determined the smooth normed alternative real algebras, using earlier results of E. Strzelecki [Str] on smooth normed power-associative real algebras. But actually general nonassociative smooth normed real algebras are noncommutative Jordan algebras (hence power-associative), as shown by the following theorem that was proved in [Rod4] as a relatively easy consequence of A.1 (see also [Rod10] for a more direct proof that does not involve A.1).

A.3. *Given a real pre-Hilbert space E with a bilinear anticommutative product $\wedge$ satisfying $(x\wedge y|z)=(x|y\wedge z)$ and $\|x\wedge y\|\leq\|x\|\|y\|$ for all x,y,z in E, consider the real normed space $B=\mathbb{R}\mathbf{1}\oplus_2^1 E$ with product defined by*

$$(\alpha\mathbf{1}+x)(\beta\mathbf{1}+y):=[\alpha\beta-(x|y)]\mathbf{1}+\alpha y+\beta x+x\wedge y.$$

Then B is a smooth normed algebra. Moreover all smooth normed real algebras are of this type.

Note that, as a consequence of A.3, every smooth normed real algebra A is a quadratic algebra over $\mathbb{R}$ (i.e.: A has a unit element **1**, and, for every a in A, there are α,β in $\mathbb{R}$ such that $a^2+\alpha a+\beta\mathbf{1}=0$), and that the norm of such an algebra derives from an inner product. Because the commutative smooth normed real algebras are in fact Jordan algebras, we shall call them the "smooth normed Jordan algebras". They correspond to the construction in A.3 by taking $\wedge=0$.

There are nice classical precedents for the above results deriving identities from geometric requirements on general nonassociative normed algebras, such as the Urbanik-Wright theorem [UrWr] describing absolute-valued algebras with a unit. Recall that an absolute-valued algebra is a nonzero algebra A with a norm $\|.\|$ satisfying $\|ab\|=\|a\|\|b\|$ for all a,b in A. There are examples of infinite dimensional complete absolute-valued real or complex algebras, even with a nonreflexive underlying Banach space (see [UrWr], [Ur1], [Bel], and [Rod12; Remark 3(i)]). But, if the absolute-valued algebra has a unit, then the dimension must be finite, as shown by the following result in which we collect the theorems of Urbanik-Wright and Nieto.

A.4. *For a real algebra A, the following assertions are equivalent:*

i) A is an absolute-valued algebra with a unit.

ii) A is a smooth normed alternative algebra.

iii) A equals $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, or $\mathbb{O}$ (the division algebra of real octonions), with its usual absolute value as norm.

An easy consequence of A.4 is the earlier result of F. B. Wright [Wr] on absolute-valued real division algebras. Recall that a left division algebra is a nonzero algebra (say A) with the property that, whenever a and b are in A with $a\neq 0$, there exists a unique x in A satisfying $ax=b$. 'Of course there is an analogous concept of right division algebra, and we say that A is a division algebra if it is at the same time a left division and a right division algebra. Recall also that two absolute-valued algebras A and B are said to be isotopic if there exist linear isometries $\varphi_1,\varphi_2,\varphi_3$ from A onto B satisfying

$$\varphi_1(xy)=\varphi_2(x)\varphi_3(y)$$

for all x,y in A. With this terminology Wright's theorem can be stated as follows.

A.5. *An absolute-valued real algebra is a division algebra if and only if it is finite dimensional, and this is the case if and only if it is isotopic to $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, or $\mathbb{O}$.*

Finally, we cite the following sufficient condition for the finite-dimensionality of absolute-valued algebras obtained by M. L. El-Mallah and A. Micali in [MMi].

A.6. *Every flexible absolute-valued algebra is finite-dimensional.*

Advances since 1988.

Returning to nonassociative Gelfand-Naimark situations, we note that, as a relatively easy consequence of A.2, it was shown in [Rod3] that complete normed nonassociative complex algebras with a unit **1** and a conjugate-linear algebra involution * satisfying $\|a^*a\|=\|a^*\|\|a\|$ are nothing more than unital alternative C*-algebras. In this "weak" Gelfand-Naimark situation one is tempted to consider the possibility of relaxing the requirement $(ab)^*=b^*a^*$ to the weaker one $\mathbf{1}^*=\mathbf{1}$, without perturbing the characterization of alternative C*-algebras (in the same way as this was possible in the "strong" Gelfand-Naimark situation studied in A.2). Actually the answer to this question is "almost" affirmative, as we see from the next theorem. It was proved very recently in [CRod5], using A.1 and A.2 among other tools, and it seems to be new even in the associative setting.

Theorem A.7. *Let A be a complete normed complex nonassociative algebra with a unit* **1** *and a conjugate-linear vector space involution* $\square$ *satisfying* $\mathbf{1}^{\square}=\mathbf{1}$ *and* $\|a^{\square}a\|=\|a^{\square}\|\|a\|$ *for every a in A. Then A is an alternative algebra, and (except possibly in the case when A is isometrically isomorphic to the C*-algebra* $\mathbb{C}^2$*)* $\square$ *is an algebra involution on A and the equality* $\|a^{\square}a\|=\|a\|^2$ *holds for every a in A. For the exceptional case of the C*-algebra* $\mathbb{C}^2$*, the involutions* $\square$ *satisfying the above requirements are exactly the mappings of the form* $a\longrightarrow a^*+\overline{f(a)}\mathbf{1}$*, where * denotes the C*-algebra involution and f is any fixed linear form on* $\mathbb{C}^2$ *such that* $f(\mathbf{1})=0$ *and* $f(a^*)=-\overline{f(a)}$ *for all a in* $\mathbb{C}^2$.

After the above theorem, we don't know any substantial progress about Vidav-Palmer and Gelfand-Naimark nonassociative theorems, except for the paper of A. Bensebah [Ben1], where a Vidav-Palmer type theorem is proved for some nonunital complex Jordan-Banach algebras. Of course, there are classical and recent advances about the structure theory of the models built by the nonassociative Vidav-Palmer theorem (A.1), namely the noncommutative JB*-algebras, but, as we have said before, these results will be reviewed later.

Returning now to absolute-valued algebras, we must say that all the classical examples of infinite-dimensional such algebras we mentioned fail to be one-sided division algebras. Very recently, J. A. Cuenca [Cue3] and A. Rodríguez [Rod12] have proved the existence of infinite-dimensional complete absolute-valued real one-sided division algebras. The Banach spaces of the algebras in these examples are in fact Hilbert spaces, of hilbertian dimension $\aleph_0$ in the Cuenca's nice construction, and of arbitrary infinite hilbertian dimension in the more involved one of Rodríguez. Since the fact that absolute-valued complex one-sided division algebras are isomorphic to the complex field can be considered as folklore, it seems to be reasonable to look for a structure theory of arbitrary absolute-valued real one-sided division algebras. Such a structure theory is provided in [Rod12], and will be reviewed in what follows. The first result in this direction is given by the next proposition.

Proposition A.8. *An absolute-valued algebra is a left division algebra if and only if it is isotopic to an absolute-valued algebra with a left unit.*

Note that, as a consequence, absolute-valued algebras with a left unit are left division algebras. Now the attention must be centered on absolute-valued real algebras with a left unit. To state the corresponding structure theorem for such algebras, it is convenient to introduce some natural terminology. Given a Jordan algebra J and a vector space X, a representation of J on X will mean an homomorphism (say ψ) from J onto a Jordan algebra of operators on X. If J has a unit $\mathbf{1}$ and $\psi(\mathbf{1})$ equals the identity operator on X, the representation ψ will be called unital. If X is a pre-Hilbert space, if $*$ is an algebra involution on J, and if the representation ψ satisfies

$$(\psi(x)(\eta)|\xi)=(\eta|\psi(x^*)(\xi))$$

for all x in J and all η,ξ in X, then we will say that ψ is a $*$-representation. Every smooth normed Jordan algebra $J=\mathbb{R}\mathbf{1}\oplus H$ (see A.3) will be considered as an algebra with algebra involution $*$ defined by

$$(\lambda\mathbf{1}+\eta)^*:=\lambda\mathbf{1}-\eta \ .$$

Theorem A.9. *If J is a smooth normed Jordan algebra and ψ is a unital $*$-representation of J on the pre-Hilbert space of J, then the normed space of J with product $\odot$ defined by*

$$x\odot y:=\psi(x)(y)$$

is an absolute-valued real (automatically left division) algebra with a left unit. Moreover, up to isometric isomorphism, all absolute-valued real algebras with a left unit can be constructed in this way.

The first assertion in Theorem A.9 is easy to prove, and has become the common starting idea in [Cue3] and [Rod12] for the construction of the infinite-dimensional examples of absolute-valued left division algebras mentioned above. Concerning the last assertion in Theorem A.9 (the most relevant one), it can be stated without involving Jordan algebras and their representations on vector spaces, as follows: *The norm of any absolute-valued algebra A with a left unit e derives from an inner product $(.|.)$, and, for a,b,c in A with a orthogonal to e, we have*

$$(ab|c)=-(b|ac) \quad \text{and} \quad a(ab)=-\|a\|^2b\ .$$

The proof of this last assertion in Theorem A.9, as well as that of a part of Proposition A.8, relies on a Theorem, also proved in [Rod12], asserting that, *if P is a space of bounded linear operators on a real normed space X containing the identity operator on X and satisfying*

$$\|T(x)\|=\|T\|\|x\|$$

for all T in P and x in X, then P is a Jordan algebra of operators on X which, endowed with the operator norm, is isometrically isomorphic to some of the smooth normed Jordan algebras. The paper [Rod12] also contains interesting results about irreducible *-representations of smooth normed Jordan algebras on real Hilbert spaces, from which one can derive the existence of complete absolute-valued real left division algebras of arbitrary infinite hilbertian dimension and without nonzero proper closed left ideals.

In the line of deriving identities from geometric requirements on a nonassociative normed algebra, even the associative identity may be obtained. Classical examples of this fact are Kadison's paper [K] and Corollary 32 in [Rod3], where the associativity is obtained together with the commutativity. But, as has been observed in [ILouRod; p.284], some results in [PaPeRod1] imply that even unital associative C*-algebras can be nonassociatively characterized as those algebras satisfying the assumptions in A.2, and whose Banach spaces contain no *8*-codimensional "primitive M-ideals" in the sense of [AlfEf]. We shall conclude the exposition of recent advances in this line by reviewing the following nonassociative characterization of unital not necessarily self-adjoint algebras of bounded linear operators on complex Hilbert spaces, recently obtained by D. P. Blecher, Z-J. Ruan, and A. M. Sinclair [BlRuSin].

Theorem A.10. *A unital normed nonassociative complex algebra A is isometrically isomorphic to some unital algebra of bounded linear operators on a suitable complex Hilbert space if (and only if), for each natural number n, there exists an algebra norm $\|.\|_n$ on the algebra $M_n(A)$, of all $n\times n$ matrices with entries in A, satisfying the following conditions:*

i) $\|.\|_1=\|.\|$ (the given norm of A).

ii) For n,m in $\mathbb{N}$, x in $M_n(A)$, and y in $M_m(A)$, we have $\|x\oplus y\|_{n+m}=\max\{\|x\|_n,\|y\|_m\}$, where $x\oplus y$ denotes the $(n+m)\times(n+m)$ matrix with x and y in the diagonal blocks (in that order) and zero in the off-diagonal blocks.

iii) For n in $\mathbb{N}$, x in $M_n(A)$, and α,β in $M_n(\mathbb{C})$, we have $\|\alpha x\beta\|_n\leq\|\alpha\|\|x\|_n\|\beta\|$, where $\|.\|$ denotes the natural C^*-norm on $M_n(\mathbb{C})$.

It is also proved in [BlRuSin] that, when the algebra A satisfies the two equivalent assertions in the above theorem, then actually A can be seen as an algebra of bounded linear operators on a complex Hilbert space H in such a way that, for n in $\mathbb{N}$, the abstract norm $\|.\|_n$ on $M_n(A)$ agrees with the operator norm of elements of $M_n(A)$ when they are naturally regarded as operators on the l_2-sum of n copies of H. Let's finally remark that Theorem A.10, together with the associative Vidav-Palmer theorem, gives rise to another nonassociative characterization of unital associative C*-algebras [BlRuSin; Corollary 3.3].

Problems and future directions.

In our opinion, after A.1, A.2, Theorem A.7, and the structure theory for noncommutative JB*-algebras (that we shall review in Sections D and F), the work on nonassociative Vidav-Palmer and Gelfand-Naimark theorems can be considered as concluded. However, it remains to provide a reasonably available complete proof of these results. To be more precise, a complete proof of A.1 from standard algebra and functional analysis, and following the network of references in the original papers, may need around 500 pages. This is so mainly because the known proofs of the equality $\|U_a(a^*)\|=\|a\|^3$ in A.1 (see [Yo3] and [Rod3]) use the main results in Wright's paper [Wri], which in its turn depends on the large and difficult work of Alfsen-Shultz-Stormer on the structure of JB-algebras [AlfShuSto]. Although this last work was clarified later in [Shu] and [H], and today it is nicely collected in the book [HSto], almost the whole book is necessary for the complete proof of A.1. Therefore a non-structural proof of the equality $\|U_a(a^*)\|=\|a\|^3$ in A.1 would be welcome. In this respect, it must be remarked that, although the classical proof of the equality $\|a^*a\|=\|a\|^2$ in the associative Vidav-Palmer theorem is structural (see [BoDu; Section 38]), a non-structural proof of this fact can be given by using the theory of the so called "hermitian Banach algebras" [BoDu; Section 41]. It seems therefore convenient to study "hermitian Jordan-Banach algebras" in depth, but this will be considered more precisely in the next section.

Fortunately, the proof of A.2, although depending partially on A.1, doesn't involve the equality $\|U_a(a^*)\|=\|a\|^3$. Even, the Wright-Youngson theorem on isometries of JB-algebras involved in this proof has been reproved in a very simpler way by C. H. Chu [Ch]. Consequently the proof of A.2 can be completely given in around 150 pages, and in fact it is my intention to write a short book on this topic, perhaps including also Theorem A.7.

In spite of the naturality of the axiom $\|ab\|=\|a\|\|b\|$, absolute-valued algebras have been considered only by a relatively small number of authors (see previous references together with [Al1], [Al2], [M1], [M2], [M3], [M4], and [Ur2]), and the corresponding theory is far from being finished. After the recent papers of Cuenca [Cue3] and Rodríguez [Rod12] reviewed above, we hope this theory will flourish. We expect classification theorems or at least significant general results. Note that all nontrivial known results on absolute-valued algebras involve additional assumptions on such algebras, so they can be looked on as incomplete classification theorems. (The absolute-valued algebras not satisfying the additional assumption under consideration remain unknown.) An exception is the general result in [Rod12; Theorem 4] asserting that homomorphisms from complete normed algebras into absolute-valued algebras are contractive, hence continuous. As examples of questions that can be posed, consider the following problems.

A.11. *Is the bidual of an absolute-valued algebra (endowed with the Arens product) an absolute-valued algebra?*

A.12. *Let A be an absolute-valued algebra satisfying a nonzero identity. Is A finite-dimensional?* (compare with A.6, and see [Rod12; Remark 3(ii)]).

A.13. *Describe the Banach spaces that are absolute-valued algebras for some product.*

For example, as a consequence of A.5, under the additional assumption of finite dimension, they are only the Euclidean real spaces of dimension *1, 2, 4,* or *8*, while all infinite-dimensional l_p-spaces are in this class of Banach spaces.

After the Blecher-Ruan-Sinclair nonassociative characterization of unital algebras of bounded linear operators on complex Hilbert spaces (Theorem A.10), one is tempted to think that the question of associativity of a general unital normed complex algebra can be settled in a geometrical way. Let's therefore conclude this section with an adventurous conjecture.

Conjecture A.14. *A unital normed complex algebra A is associative if (and only if), for each natural number n, there exists an algebra norm* $\|.\|_n$ *on the algebra* $M_n(A)$ *satisfying conditions (i) and (ii) in Theorem A.10.*

B. Nongeometric theory of normed Jordan algebras.

Now we shall consider the development of the theory of normed Jordan algebras concerning those aspects not involving additional geometric requirements. The possibility of applying "spectral techniques" has allowed a great parallelism with the theory of associative normed algebras. In fact the Jordan approach to normed algebras becomes a natural generalization of the associative approach, because associative normed algebras can be seen as normed Jordan algebras by symmetrizing their associative products. Thus, when an associative result has been extended to the Jordan setting, this means in particular that, in the associative context, this result depends only on the Jordan structure of the associative algebra under consideration. Some results for normed Jordan algebras are simple specifications of more general ones on nonassociative normed algebras that we shall also review.

Summary of results until 1988.

After the early work of M. Koecher [Ko1] introducing Analysis in finite-dimensional Jordan algebras over the real numbers (see also [BrKo]), the first paper we know dealing with normed Jordan algebras is that of V. K. Balachandran and P. S. Rema [BRe], where they showed the uniqueness of the complete algebra norm topology in strongly semisimple Jordan-Banach algebras. Today the main importance of that paper relies on the observation they made that, *for an element a in a complete normed power-associative algebra A, the number* $r(a):=\lim\{\|a^n\|^{1/n}\}$ *is an algebraic invariant, namely r(a) is the maximum of the moduli of the numbers in the spectrum of a relative to any maximal associative subalgebra of A containing a.* Using this observation, either the proof of the main result in [BRe] or Rickart's original associative argument (see [Sin; Theorem 6.18]) can be adapted to obtain the following theorem (see [Rod10; Theorem 7.3] for details).

B.1. *Let A and B be complete normed power-associative algebras, and assume that B is strongly semisimple. Then every dense range homomorphism from A into B is continuous.*

Properly Jordan spectral theory in normed Jordan algebras was first considered in the paper of C. Viola Devapakkiam [Viol]. Spectral methods in Jordan algebras were possible thanks to the Jacobson-McCrimmon concept of inverse in a Jordan algebra with a unit [Ja2], which we now recall. An element x in a Jordan algebra J with a unit **1** is said to be invertible if there exists y in J satisfying $x.y=\mathbf{1}$ and $x.y^2=y$, and, if this is the case, the element y above is unique and it is called the inverse of x (and denoted by x^{-1}). From this concept of invertible element the spectrum of an arbitrary element of a real or complex Jordan algebra is defined as in the associative case [BoDu; Definitions 5.1 and 13.6]. The paper of Viola contains basic facts about the set of invertible elements and the spectrum of arbitrary elements in a Jordan-Banach algebra with a unit. In particular the fundamental Gelfand-Beurling formula is proved in the Jordan setting. It follows that the Gelfand-Mazur theorem holds for normed Jordan-division complex algebras, and this is the germ of the subsequent description of normed Jordan-division real algebras.

Viola's work was continued in the Thesis of J. Martínez [Marti1] (see also [Marti2] and [Marti3]), who showed, as a first remarkable result, that *Jordan-Banach algebras are "locally spectrally" associative*. That means that *every element in such an algebra J can be imbedded in a closed associative subalgebra J' satisfying* $sp(J,x)=sp(J',x)$ *for all x in J' (where sp(.,.) denotes the spectrum of the second coordinate relative to the first one)*. This theorem allows one to develop a holomorphic functional calculus for a single element in a Jordan-Banach algebra. Martinez's Thesis contains also the following result, that become crucial in Kaup's characterization of bounded symmetric domains in complex Banach spaces [Kau3]: *if x is an element in a Jordan-Banach algebra J, and if* L_x *denotes the operator of multiplication by x on J, then* $sp(J,x)\subseteq sp(BL(J),L_x)\subseteq\frac{1}{2}(sp(J,x)+sp(J,x))$ *(where BL(J) denotes the Banach algebra of all bounded linear operators on J)*. In the Thesis of A. Kaidí [Ka1] (see also [Ka2]) general nonassociative normed algebras were considered, and most of the spectral results on Jordan-Banach algebras mentioned above were extended to the setting of noncommutative Jordan-Banach algebras (see [Mc2] for the concept of inverse in this context), giving in this way an unified approach to the associative and Jordan cases. The description given there of normed noncommutative Jordan-division real algebras is remarkable, because it implies that normed (commutative) Jordan-division algebras are nothing but smooth normed Jordan algebras (take $\wedge=0$ in A.3) endowed with an arbitrary algebra norm.

An interesting (and sometimes forgotten) paper on Jordan-Banach algebras is that of P. S. Putter and B. Yood [PuY], where spectral theory is further developed with the aim of making progress on the most fundamental questions of the basic theory, such as automatic continuity, uniqueness of norm, Shiraly-Ford and Pták theorems for hermitian Jordan-Banach algebras, etc.. In fact the paper contains the Jordan version of B.1, as well as weak forms (or particular cases) of the results B.2 and B.4 below. The paper of Putter-Yood also reconsidered the problem [Shi] of whether an associative algebra, which becomes a Jordan-Banach algebra using some norm and the Jordan product arising from the original associative product, is actually an associative Banach algebra for an equivalent norm. The best positive answer we know about this question is that this is so if the associative algebra under consideration is semiprime (see [Rod2] or [Rod7; Proposition 3]).

A revolution in the theory of Jordan-Banach algebras arose when B. Aupetit jordanized the associative methods in his book [Au1] in order to solve several Jordan problems that seemed intractable by classical tools. Thus in Aupetit's paper [Au2], at the same time that a Jacobson-representation-theory-free proof of Johnson's uniqueness-of-norm theorem for Banach algebras was given, the following Jordan version of this theorem was shown.

B.2. *If A and B are Jordan-Banach algebras, and we assume that B is semiprimitive, then every homomorphism from A onto B is continuous. As a consequence, semiprimitive Jordan-Banach algebras have a unique complete algebra norm topology.*

We recall that, given a Jordan algebra J, an element x in J is called quasi-invertible if $\mathbf{1}-x$ is invertible in the unital hull of J, and that K. McCrimmon proved in [Mc1] the existence of a largest ideal of J each element of which is quasi-inversible. This ideal is called the Jacobson-McCrimmon radical of J and is denoted by $Rad(J)$. The condition $Rad(J)=0$ is equivalent to that of J being a subdirect product of Jordan algebras which are "primitive" in a peculiar Jordan sense [HoMc], and, when this is the case, J is therefore called semiprimitive. The reader is referred also to [Mc2□] and [FeRod1] for the noncommutative versions of these ideas. The new tool in the proof of B.2 was the subharmonicity of the spectral radius in Jordan-Banach algebras. This tool was improved in the paper of B. Aupetit and A. Zraïbi [AuZr], where the following theorem was shown.

B.3. *If f is a holomorphic mapping from an open subset* Ω *of* $\mathbb{C}$ *into a Jordan-Banach complex algebra J, then the mapping* $\lambda \longrightarrow sp(J,f(\lambda))$ *is an analytic multivalued function.*

This theorem was applied in [AuZr] to prove that, *if J is a Jordan-Banach complex algebra, then the spectral radius is subadditive and submultiplicative on J if and only if J/Rad(J) is associative.* Another nice application of Aupetit's techniques was the clarification (as well as the simplification of the proof) of an earlier result of H. Behncke [Beh] on hermitian Jordan-Banach algebras, made by Aupetit himself and M. A. Youngson in [AuYo]. A hermitian Jordan algebra is a Jordan complex algebra *J* with a conjugate-linear algebra involution * satisfying $sp(J,x)\subseteq\mathbb{R}$ for every symmetric element *x* in *J*. The commented result about hermitian Jordan-Banach algebras reads as follows.

B.4. *For a Jordan-Banach complex algebra J with conjugate-linear algebra involution *, the following assertions are equivalent:*

i) J is hermitian.

ii) The set of all symmetric elements in J having positive spectrum is convex.

iii) $x^*.x$ *has positive spectrum for all x in J.*

Aupetit's techniques were also useful for the beginning of the development of a nongeometric theory of general nonassociative algebras. This is the case of the paper [Rod5], where the so called "weak radical" of an arbitrary nonassociative algebra was introduced, and, using ¡associative! subharmonic methods in [Au2], the uniqueness of the complete algebra norm topology for complete normed nonassociative algebras with zero weak radical was proved. This result contains the part of B.2 concerning uniqueness of norm, as well as its noncommutative version, which actually arises in a greatly improved form. Thus, as an example, *the Banach-Lie algebra of all derivations on a C*-algebra has zero weak radical, hence a unique complete algebra norm topology* (although it is a noncommutative Jordan algebra agreeing with its Jacobson-McCrimmon radical). This example shows how the uniqueness-of-norm theorem in [Rod5] even gives a satisfactory answer to the corresponding problem for Banach-Lie algebras, posed in [Har2]. The paper [Rod5] contains also a reasonable nonassociative extension of Johnson's theorem, on automatic continuity of surjective homomorphisms into semiprimitive (associative) Banach algebras, implying also Aupetit's theorem (B.2), and that we shall lightly improve later. The weak radical was also the appropriate tool for the statement and proof of a nonassociative complete normed version of the first Wedderburn's theorem [FeRod3], implying in particular that, *if J is a semiprimitive Jordan-Banach algebra, and if the annihilator of every closed proper ideal of J is nonzero, then J is the closure of the direct sum of its minimal closed ideals, and these are topologically simple primitive Jordan-Banach algebras.*

It is well-known that under certain assumptions (von-Neumann regularity, finite spectrum and semiprimitiveness, algebraicness and semiprimitiveness, semiprimeness and agreement with the socle) an associative Banach algebra is finite-dimensional. This is no longer true for Jordan-Banach algebras, but, under the above requirements (in the appropriate Jordan versions), Jordan-Banach algebras are finite direct sums of simple ideals that are either finite-dimensional or quadratic. These results, and even their noncommutative versions, can be seen in the papers of Benslimane-Kaidí [BensKa] and Benslimane-Fernández-Kaidí [BensFeKa]. One of the main tools in the proof of these facts is the concept and basic theory of the socle of a nondegenerate Jordan algebra, first introduced and developed in [OR] (see also [FeRod1] for the noncommutative case). Among the several papers in which the theory of the socle for nondegenerate Jordan algebras was further developed (without getting out of the purely algebraic setting), we cite the one of Fernández-Rodríguez [FeRod2], where an analytic incursion was made, showing that *the socle of a nondegenerate Jordan-Banach algebra agrees with the largest von-Neumann regular ideal.*

In [Fe1], A. Fernández introduced modular annihilator Jordan algebras as those nondegenerate Jordan algebras *J* such that *J/Soc(J)* is radical (where *Soc(J)* denotes the socle of *J*). After developing the basic algebraic theory for such algebras, modular annihilator Jordan-Banach algebras were considered, showing as main results that *nondegenerate compact Jordan-Banach complex algebras are modular annihilator,* and that *JB-algebras are modular annihilator if and only if they are "dual" in the* *sense of* [Bu1]. (Recall that a normed Jordan algebra J is said to be compact if, for every x in J, the operator U_x is compact.) In a subsequent paper ([Fe2]), following the standard terminology in the associative case, Fernández introduced Jordan-Riesz algebras as those Jordan-Banach algebras J such that, for every x in J, zero is the only possible accumulation point of $sp(J,x)$. It was proved that *semiprimitive Jordan-Riesz algebras are modular annihilator Jordan-Banach algebras*, and that the converse is true in several interesting particular cases. In the same paper, it was shown that *the socle of any normed nondegenerate Jordan algebra is an algebraic ideal*. Also the problem of the coincidence of the socle and the largest algebraic ideal in the complete normed case was considered, giving the first steps for an eventual future solution.

Advances since 1988.

A great part of the recent production on nongeometric aspects of normed Jordan algebras has centered in a deeper understanding of the behaviour of the socle of Jordan-Banach algebras. Thus Aupetit's techniques have been tested, by the first time in this field, by Aupetit himself and L. Baribeau in [AuBa], where as the main result the following theorem has been proved.

Theorem B.5. *If J is a semiprimitive Jordan-Banach complex algebra, and if the spectrum of every element in J is at most countable, then the socle of J is nonzero.*

The paper of Aupetit-Baribeau contains also a rather involved structure theorem for separable Jordan-Banach complex algebras with the property that the spectrum of every element is at most countable, that, roughly speaking, reduces their knowledge to that of modular annihilator separable Jordan-Banach algebras [AuBa; Theorem 19]. The proof of this structure theorem relies on Theorem B.5 and the spectral characterization of modular annihilator Jordan-Banach algebras, provided almost at the same time by M. Benslimane and A. Rodríguez [BensRod], and that concludes in a first instance the already commented Fernandez's work in [Fe2]. The proof of this characterization consist of a jordanization of Aupetit's associative methods in [Au3], and reads as follows.

Theorem B.6. *A semiprimitive Jordan-Banach algebra is modular annihilator (if and) only if it is a Jordan-Riesz algebra.*

In fact Theorem B.6 is proved in [BensRod] only for complex algebras, but, as it was pointed out in [BensFeKa], the real case can be easily derived from the complex one. Let's mention the most significant advance we know concerning the problem of the coincidence of the socle and the largest algebraic ideal in Jordan-Banach algebras, provided in the paper of M. Benslimane, O. Jaa, and A. Kaidí [BensJKa], and that can be stated as follows.

Theorem B.7. *Let J be a semiprimitive Jordan-Banach complex algebra, and let I be a spectrum-finite inner ideal of J (for instance, any algebraic ideal of J). Then every element y in I can be written as $y=s+x$, with s in the socle of J and x in J satisfying $x^2=0$.*

Important consequences of the theorem are that *spectrum-finite inner ideals of semiprimitive Jordan-Banach complex algebras are algebraic,* and *the existence, in any (not necessarily semiprimitive) Jordan-Banach algebra, of a largest spectrum-finite ideal.* Let's also comment the paper [BensFeGarKa], improving earlier results in [BensKa], and showing that *normed noncommutative Jordan complex algebras with no nonzero nilpotent elements, and with essential socle, are associative and commutative.*

To conclude the review of recent advances in the Jordan-Banach treatment of the socle, let's explain the structure theory of prime nondegenerate Jordan-Banach complex algebras with nonzero socle, developed in [PeRiRodVi]. The fundamental result there obtained is the following.

Theorem B.8. *A Jordan-Banach complex algebra J is prime nondegenerate with nonzero socle (if and) only if one of the following statements holds:*

i) J is the exceptional Jordan algebra $M_3^8(\mathbb{C})$ of all hermitian 3×3 matrices over the complex octonions.

ii) J is simple quadratic over $\mathbb{C}$.

iii) There are a prime associative Banach complex algebra A with *nonzero socle, and a one-to-one continuous Jordan-homomorphism ϕ from J into A, such that the range of ϕ contains the socle of A.*

iv) There are a prime associative Banach complex algebra A with nonzero socle, a linear algebra involution τ on A, and a one-to-one continuous Jordan-homomorphism ϕ from J into A, such that the range of ϕ is contained in $H(A,\tau)$ and contains $H(Soc(A),\tau)$.

Here, and in the rest of this survey, for an algebra A with linear algebra involution τ, $H(A,\tau)$ denotes the set of all τ-hermitian elements in A. (Note that, for analysts, the symbol * usually means a conjugate-linear involution.) Simple quadratic Jordan-Banach complex algebras can be constructed from the so called symmetric self-dual complex Banach spaces. These are pairs $(X,<.,.>)$, where X is a complex Banach space, and $<.,.>$ is a continuous, nondegenerate symmetric bilinear form on X. The simple quadratic Jordan-Banach complex algebras are of the form $J(X,<.,.>)$, for such a symmetric self-dual complex Banach space with $dim(X)\neq 1$, where $J(X,<.,.>)$ denotes the topological sum $\mathbb{C}\oplus X$ with the familiar Jordan product $(\lambda+x).(\mu+y):=[\lambda\mu+<x,y>]+(\lambda x+\mu y)$. In order to state the special version of Theorem B.8, also obtained in [PeRiRodVi], under the additional assumption of minimality of norm topology, let's recall that a symmetric self-dual complex Banach space $J(X,<.,.>)$ is called regular if the natural continuous embedding of X in its dual space determined by $<.,.>$ is actually a topological embedding. Recall also that a normed algebra $(A,\|.\|)$ is said to have minimality of norm topology if any algebra norm $|.|$ on A, minoring $\|.\|$, i.e., $|.|\leq\alpha\|.\|$ for some $\alpha>o$, is actually equivalent to $\|.\|$.

Theorem B.9. *Up to bicontinuous isomorphisms, the prime nondegenerate Jordan-Banach complex algebras with nonzero socle and minimality of norm topology are the following:*

i) The Jordan-Banach algebra $M_3^8(\mathbb{C})$.

ii) The Jordan-Banach algebras of the form $J(X,<.,.>)$, where $(X,<.,.>)$ is a regular symmetric self-dual complex Banach space with $dim(X)\geq 2$.

iii) The closed Jordan subalgebras of any prime associative Banach complex algebra A, with nonzero socle and minimality of norm topology, containing the socle of A.

iv) The closed Jordan subalgebras of any prime associative Banach complex algebra A, with linear algebra involution τ, nonzero socle, and minimality of norm topology, contained in $H(A,\tau)$ and containing $H(Soc(A),\tau)$.

Forgetting the socle for the moment, let's refer to the solution of Ostrovski's problem in Jordan-Banach algebras, given by A. M. Slin'ko [Sl]. After the nice simple proof provided by B. Cuartero and J. E. Galé [CuG], it can be stated as follows.

Theorem B.10. *A complete normed algebraic power-associative algebra is of bounded degree.*

Normed Jordan algebras whose sets of quasi-invertible elements are open were considered in the already quoted pioneering paper [Vio1], and, following standard associative terminology, we shall call them normed Jordan Q-algebras. After the relatively recent answer ([P] and [BeOu]) to Wilansy's conjecture [W] on normed associative Q-algebras, normed Jordan Q-algebras have been reconsidered in [PeRiRod], answering as well the Jordan version of Wilansky's conjecture, and providing other interesting characterizations of them. In order to state some of these results, let's recall that, given a Jordan algebra J, a subalgebra of J is said to be a full subalgebra of J if it contains the quasi-inverses of its elements that are quasi-invertible in J.

Theorem B.11. *For a normed Jordan algebra J, the following assertions are equivalent:*

i) J is a normed Jordan Q-algebra.

ii) J is a full subalgebra of some Jordan-Banach algebra.

iii) J is a full subalgebra of its completion.

iv) The maximal modular inner ideals of J (in the sense of [HM]*) are closed in J.*

Normed Jordan Q-algebras can be involved in unusual automatic continuity theorems, as the following one (see [PeRiRod] and [PeRiRodVi]).

Theorem B.12. *Let A be a normed Jordan complex Q-algebra, B a Jordan-Banach complex algebra with minimality of norm topology, $\phi: A \longrightarrow B$ a homomorphism, and assume that either B is semiprimitive and ϕ is surjective or that B is prime nondegenerate with nonzero socle and the range of ϕ contains the socle of B. Then ϕ is continuous.*

Let's conclude our survey on recent advances on nongeometric theory of normed Jordan algebras by reviewing the ones concerning the study of the extended centroid of such algebras. The concept of extended centroid was introduced and developed first by W. S. Martindale ([Mart1] and [Mart2]) in the case of prime associative rings, and later this concept was extended to the more general setting of prime (and even semiprime) nonassociative algebras (see [ErMartO] and [BaxMart]). As its name suggests, the extended centroid is a suitable enlargement of the centroid of a semiprime nonassociative algebra, whose main interest is found in the theory of structure and classification of prime algebras (see for example [Fi] and [CoFe]). The elements in the extended centroid of a nonassociative semiprime algebra A (denoted by $C(A)$) are the so called "maximal essentially defined centralizers" on A. "Essentially defined centralizer" means linear mapping f from an essential ideal of A (say $dom(f)$) into A satisfying $f(ab)=af(b)$ and $f(ba)=f(b)a$ for all a in A and b in $dom(f)$, and "maximal" means that there are no nontrivial extensions of f with the same properties as that of f. For each essentially defined centralizer on A there is a unique maximal essentially defined centralizer on A which extends it, and $C(A)$ is a von Neumann regular associative and commutative ring if we take as sum (respectively, product) of two elements in $C(A)$ the unique maximal essentially defined centralizer on A that extends the usual sum (respectively, composition) of the given elements as partially defined operators on A. Moreover, $C(A)$ is a field if and only if A is prime. In the prime case, the eventual fact that $C(A)$ agrees with the base field has many interesting consequences, and when this is the case it is said that the prime algebra A is centrally closed.

The extended centroid $C(A)$ of a semiprime algebra A contains the usual centroid $\Gamma(A)$ of A, namely the ring of all everywhere defined centralizers on A, and clearly $C(A)=\Gamma(A)$ whenever A is simple. Thus centrally closed simple algebras are nothing more than central simple algebras in the usual meaning. In this way it seems reasonable to ask, as a first question on the extended centroid of normed algebras, if every normed simple complex algebra is central. We give here the affirmative answer to this question, that seems to be new even in the associative context, and that follows almost directly from the following lemma.

Lemma B.13. *Let X be a normed space, B an algebra of continuous linear operators on X, and assume that there exists a cyclic vector x_0 for B (that is, an element x_0 in X such that $Bx_0=X$). Then an algebra norm can be built on the algebra D of all (possibly discontinuous) linear operators on X which commute with every element in B.*

Proof. Defining for f in D $|f|:=\|fx_0\|$, $|.|$ becomes a norm on the vector space of D. Since for any fixed element f in D there exists b in B with $bx_0=fx_0$, for every g in D we have

$$|gf|=\|gfx_0\|=\|gbx_0\|=\|bgx_0\|\leq\|b\|\|gx_0\|=\|b\||g| .$$

It follows that the mapping $T_f:g\longrightarrow gf$ is a bounded linear operator on $(D,|.|)$. It for f in D we define $\|f\|$ to be the operator norm of T_f as a bounded linear operator on $(D,|.|)$, then $\|.\|$ becomes an algebra norm on D, as required. ■

Let A be a normed prime complex algebra with a minimal ideal P. Then, taking in the above lemma $X=P$ and B equal to the multiplication algebra $M(A)$ of A regarded as an algebra of operators on P (note that every nonzero element in X is a cyclic vector for B), we obtain that D can be provided with an algebra norm. Since in this case D is a complex division ring, the Gelfand-Mazur theorem implies $D=\mathbb{C}$. But also in this case $C(A)$ is isomorphic to D [CRod1; Theorem 2], hence we have the following corollary.

Corollary B.14. *Every normed prime complex algebra with a minimal ideal is centrally closed. As a consequence, normed simple complex algebras are central.*

Under the additional assumption of completeness, the above corollary was proved in the already used paper of M. Cabrera and A. Rodríguez [CRod1]. Corollary B.14 also contains the result proved by A. Cobalea and

A. Fernández [CoFe] that *normed prime nondegenerate noncommutative Jordan complex algebras with nonzero socle are centrally closed.* (Incidentally, we note that this fact is the first one we know concerning the extended centroid in normed associative or nonassociative algebras.) In fact Lemma B.13 can be used together with the arguments in [CRod1] to avoid the assumption of completeness in one of the main results in that paper (namely [CRod1; Theorem 4]). In order to state here the definitive version of this theorem, let's recall that, given a normed algebra A, any minimal B-invariant subspace of A for some algebra B of operators on A such that $M(A)\subset B\subset s\text{-}clos(M(A))$ (where $s\text{-}clos$ denotes closure in $BL(A)$ for the strong operator topology) is called an atom of A. The easiest examples of atoms are the minimal ideals.

Theorem B.15. *Let A be a normed complex algebra with a family I of mutually orthogonal atoms such that the annihilator of the sum $\sum_{P\in I} P$ is zero. Then A is semiprime and the extended centroid of A is $\mathbb{C}^I$. More precisely: each element $\gamma=\{\lambda_P\}$ in $\mathbb{C}^I$ defines an essentially defined centralizer f_γ on A with domain the sum of the atoms in I and values $f_\gamma(\sum x_P)=\sum \lambda_P x_P$, and the mapping $\gamma \longrightarrow \tilde{\tilde{f}}_\gamma$ (the unique maximal essentially defined centralizer on A which extends f_γ) is an isomorphism from $\mathbb{C}^I$ onto the extended centroid of A.*

Although the details of the proof are left to the reader, it is convenient to note that Lemma B.13 not only improves the earlier complete version of Theorem B.15 in [CRod1], but also allows to simplify the proof by avoiding the use of the crucial Lemma 7 in that paper. This is not the case for [CRod1; Remark 2] which, although remaining true without assumption of completeness, seems to need lemma 7 in [CRod1]. The paper [CRod1] contains also the following useful theorem relating the extended centroid of normed semiprime algebras with that of some of their essential ideals. The term "self-dense" for a subset N of a normed algebra A, involved in the statement of this result, means that N is dense in A for the initial topology on A relative to the family of mappings of the form $a\longrightarrow ax$ and $a\longrightarrow xa$ from A into $(A,\|.\|)$, when x runs through N.

Theorem B.16. *Let A be a semiprime normed algebra and U a self-dense essential ideal of A. Then U is a semiprime algebra and the extended centroid of U agrees with the extended centroid of A. More precisely, every maximal essentially defined centralizer f on U is an essentially defined centralizer on A and the mapping $f \longrightarrow \tilde{\tilde{f}}$ (the unique maximal essentially defined centralizer on A which extends f) is an isomorphism from $C(U)$ onto $C(A)$.*

As we have pointed out before, even in the associative setting, the extended centroid of normed semiprime (and even prime) algebras has been not considered until recently. One of the first works dealing with this topic is the one of M. Mathieu [Mat1], showing that ultraprime normed associative complex algebras are centrally closed. That a normed algebra A is ultraprime means that there exists a countably incomplete ultrafilter $\mathcal{U}$ on a suitable set such that the corresponding normed ultrapower $A_{\mathcal{U}}$ is a prime algebra. Mathieu's proof relies on his characterization of ultraprime normed associative algebras as those normed associative algebras A for which there exists $k>0$ such that $\|M_{a,b}\| \geq k\|a\|\|b\|$ for all a,b in A, where $M_{a,b}$ denotes the operator on A given by $M_{a,b}(c):=acb$. Although it remains unknown a so easy characterization of ultraprimeness for nonassociative normed algebras, M. Cabrera and A. Rodríguez have shown in [CRod2] that ultraprimeness (strictly) implies a condition rather similar to Mathieu's one, and then they have derived the following "unassociativization" of Mathieu's theorem.

Theorem B.17. *Every ultraprime normed complex algebra is centrally closed.*

Algebraic results in [Mart1] and [Mart2], together with standard theory of Banach algebras, imply that primitive Banach (associative) complex algebras are centrally closed. In order to obtain the appropriate Jordan version of this result, general nonassociative methods have been developed in [Rod11], providing as desired the following theorem.

Theorem B.18. *Every primitive Jordan-Banach complex algebra is centrally closed.*

Nonassociative methods giving rise to the above theorem are part of what we call "elevator technique", that was also successfully exploited in the already quoted papers [Rod5] and [FeRod3]. Roughly speaking, elevator technique consists of an "ascent" from a complete normed nonassociative algebra to a suitable "multiplication algebra", every time that the original argument under unassociativization needs associativity, and of a "descent" from the multiplication algebra to the given nonassociative algebra, when the original argument needs completeness. In the case under consideration an additional problem arises, namely: what must mean the term "primitive" in the general nonassociative setting?. But, as in earlier occasions, elevator technique itself is able to give answer to this question.

Because there is no place here to enter in the details of elevator technique, we put the cart before the horse, and we explain the meaning of the nonassociative primitiveness without further comments. To this end, consider a nonassociative algebra A, let B be any algebra of linear operators on A with $M(A)\subseteq B$, and fix a (left or right) primitive ideal of B (say $\mathcal{P}$). Then the largest B-invariant subspace of A contained in $\{a\in A : L_a, R_a\in\mathcal{P}\}$ (where L_a and R_a denote respectively the left and right multiplication operators by a on A) is a (two-sided) ideal of A. Ideals arising in this way are called (left or right) B-primitive ideals of A, and A is said to be a B-primitive algebra if zero is a B-primitive ideal of A. Now assume additionally that A is complex, and take B equal to the "quasi-full multiplication algebra" of A, that is denoted by $QFM(A)$, and is defined as the subalgebra of the algebra $L(A)$ (of all linear operators on A) which is minimal among the subalgebras C of $L(A)$ subjected to the conditions $M(A)\subseteq C$ and $sup\{|\lambda| : \lambda\in sp(C,c)\}=sup\{|\lambda| : \lambda\in sp(L(A),c)\}$ for every c in C. If A becomes a B-primitive algebra for this choice of B, then we say that A is a quasi-weakly primitive algebra. The main result in [Rod11] asserts that *every complete normed quasi-weakly primitive complex algebra is centrally closed.* It is also proved in [Rod11] that, *if a Jordan complex algebra is primitive* (in the sense of Hogben-McCrimmon [HoMc]), *then it is quasi-weakly primitive,* hence Theorem B.18 follows.

Now that we know something about elevator technique, at least in what concerns terminology, it can be the appropriate time of clarifying some previously cited results. If A is a nonassociative algebra and B is any algebra of linear operators on A containing $M(A)$, let's define the B-radical of A as the intersection of all left (equivalently, right) B-primitive ideals of A, and the ultra-B-radical of A as the sum of all C-radicals when C runs over all algebras of operators on A contained in B and containing $M(A)$. The choice of B equal the "full multiplication algebra" of A (the minimal full subalgebra of $L(A)$ containing $M(A)$) gives rise to the so called in [Rod5] "weak" and "ultraweak" radicals of A, respectively, while, for A complex, the B-radical and the ultra-B-radical corresponding to the choice $B=QFM(A)$ will be called respectively the quasi-weak and ultra-quasi-weak radicals. It is easy to see that the weak radical of a complex algebra is contained in the ultra-quasi-weak radical, that in its turn is contained in the ultra-weak radical. Then, minor changes on the proof of Theorem 3.3 in [Rod5] (whose details are left to the reader) allows us to obtain the following ¡complex! improvement of that theorem.

Theorem B.19. *Let A and B be complete normed nonassociative complex algebras, and assume that the ultra-quasi-weak radical of B is zero. Then every homomorphism from A onto B is continuous.*

An advantage of Theorem B.19 over its precedent in [Rod5] is that now we have the following nice example in which our improved version can be applied. The hints for the proof (whose details are also left to the reader) are that every closed ideal of a normed complex algebra A is invariant under $QFM(A)$ (a consequence of the fact that, as any closed subalgebra of a Banach algebra, the norm-closure of $M(A)$ in $BL(A)$ is a quasi-full subalgebra of $BL(A)$, and hence also of $L(A)$ [Rod5; Remark 1.8]), and arguments close to the ones in Proposition 7.2 and Lemma 7.1 in [Rod10].

Proposition B.20. *Let A be a complete normed complex algebra, and assume that the annihilator of the sum of the minimal ideals of A is zero. Then A has zero ultra-quasi-weak radical. Therefore homomorphisms from complete normed complex algebras onto A are continuous.*

A comparison of Theorem B.15 with the above proposition suggests that there may exist some relation between the well behaviour of the extended centroid of a complete normed algebra and the well behaviour of the algebra concerning automatic continuity. This idea has been successfully taken in a recent nice work of A. R. Villena [Vi2], and has been retaken in [Rod14] following the Villena's methods. Although both papers are mainly devoted to obtain result on automatic continuity for H*-algebras (and therefore they will be reviewed in Section E), we may select here as an example the following nongeometric result in [Rod14].

Theorem B.21. *Let A be a complete normed centrally closed prime algebra, and assume there exists a continuous nondegenerate symmetric associative bilinear form* $<.,.>$ *on A. Then dense range homomorphisms from complete normed algebras into A are continuous.*

Problems and future directions.

After the above discussion about the development of the general theory of normed Jordan algebras, we have the impression that "all" results on normed associative algebras can be suitably extended to the Jordan context, and that "almost all" these results may even have reasonable general nonassociative extensions. As we have seen before, most of these extensions, when provided, have been far from being trivial, and they have encouraged the development of new methods that may even have enriched the associative theory of departure. Concerning future directions of work, the positive side of this planning is that, since the associative theory is very fully developed, many questions concerning Jordan (or nonassociative) normed algebras remain to be considered and answered. It must however said, as a (perhaps fortunate) negative side of this project, that usually a new technique in normed Jordan (or nonassociative) algebras alone only is useful to solve a small number of problems. As an example, the methods in the proofs of the several above-mentioned results on Jordan (or nonassociative) automatic continuity have been inefficient until now to solve the following problem (Jordan version of Johnson-Sinclair theorem [JoSin]).

B.22. *Prove that derivations of semiprimitive Jordan-Banach algebras are continuous.*

We note that the noncommutative-Jordan version of Johnson-Sinclair theorem would follow from the commutative one in view of [FR; Lemma 16], and that, for complete normed alternative algebras, Johnson-Sinclair theorem is true [Vi1].

Other problems that can be attacked, and that are related to previously reviewed results, are the following.

B.23. *Prove B.1 without the assumption of power-associativity, and/or weaken the requirement of strong semisimplicity on the range algebra in such a way that the result can be nontrivially applied to some anticommutative algebras.*

B.24. *Prove that the socle of any semiprimitive Jordan-Banach complex algebra agrees with the largest algebraic (equivalently, spectrum-finite) ideal.*

After Theorem B.7, this is equivalent to show that the socle of a semiprimitive Jordan-Banach complex algebra J is a semiprime ideal of J (i.e.: $J/Soc(J)$ is a semiprime algebra).

Let's recall that, at the conclusion of Section A, we have talked about the interest of the study of Hermitian Jordan-Banach algebras in order to obtain a simplified complete proof of the nonassociative Vidav-Palmer theorem (A.1). Now, it seems we are in the appropriate time to explain this idea in more detail. The implication (i)⇒(iii) in B.4 is a jordanized form of the Shirali-Ford theorem [BoDu; Theorem 41.5], but, in our opinion, it is neither the only possible nor the suitable one for the above-mentioned purpose. To see this, consider a hermitian Banach associative algebra A so that, by the classical Shirali-Ford theorem, for every x in A we have that xx^* and x^*x have positive spectrum. Then, by Gelfand's theory, the operator $L_{xx^*}+R_{x^*x}$ has positive spectrum relative to the Banach algebra $BL(A)$. It is enough to observe that this operator can be expressed in terms of the Jordan product "." of A, namely,

$$(L_{xx^*}+R_{x^*x})(y)=2[x.(x^*.y)-x^*.(x.y)+(x.x^*).y]$$

for all y in A, to feel invited to formulate the following conjecture.

Conjecture B.25. *Let J be a hermitian Jordan-Banach algebra. Then, for every x in J, the operator $x\square x^*: y \longrightarrow x.(x^*.y)-x^*.(x.y)+(x.x^*).y$ has positive spectrum.*

Returning to the particular case $J=A^+$, where A is a hermitian Banach associative algebra, it is easily seen that, for x in A, x^*x and $x\square x^*$ have the same spectral radius. Therefore, we would have a very good

Jordan extension of Pták's theorem [BoDu; Theorem 41.7], if the following conjecture was verified.

Conjecture B.26. *Let J be a hermitian Jordan-Banach algebra. Then the mapping* $x \longrightarrow [r(x\square x^*)]^{1/2}$ *from J into* $\mathbb{R}$ *is a JB*-seminorm on J.*

If Conjectures B.25 and B.26 are right (and if they can be proved without structural methods), we are almost sure that the desired non-structural complete proof of A.1 can be given.

Today, concerning results and methods, the fundamental difference between noncommutative Jordan-Banach algebras and general nonassociative complete normed algebras is the existence in the first case of the spectral theory deriving from the Jacobson-McCrimmon concept of inverse. Many years ago, we have affectively conjectured that this difference could be shortened by introducing, for complete normed flexible power-associative complex algebras with a unit, suitable definitions of "invertible element" and of "inverse" of such an element. Minimum requirements for these definitions are the following:

-These concepts must agree with that of Jacobson-McCrimmon, if the algebra is actually a noncommutative Jordan algebra.

-These concepts must also agree with the usual ones, if the algebra is finite-dimensional (see for example [BrKo] and [Sp]).

-If x is "invertible" with "inverse" x^{-1}, then we must have $xx^{-1}=x^{-1}x=\mathbf{1}$, and x^{-1} must be "invertible" with "inverse" x.

-The set of "invertible" elements must be open.

-Deriving in the natural way the corresponding concept of spectrum, the Gelfand-Beurling formula must be true.

Unfortunately, we don't know any definitive answer to the question of the possibility of introducing such concepts in that context and with the above requirements. We remain feeling that this possibility in fact exists, hence we encourage the interested reader to look for the appropriate definitions. The only hints we know are the observation before B.1, and the interesting papers of H. Essannouni and A. Kaidí [EsKa1] and [EsKa2] (see also [BouzKa] to be sure that no much more conditions on the behaviour of the inverse can be required if we want to go out the context of Jordan algebras).

To conclude this section, let's recall one of the oldest problems in nonassociative normed algebras. Precisely we refer to the question of the nonassociative extension of the Gelfand-Mazur theorem, namely if any normed division (nonassociative) algebra is finite-dimensional (which would imply dimension *1* in the complex case, and *1, 2, 4, 8* in the real one, by a theorem of R. Bott and J. Milnor [BotMil). This problem was explicitly posed by F. B. Wright [Wr] in 1953, who in the same paper gave the partial affirmative answer collected in A.5. Another folklore partial positive result about this question is that one-sided division complete normed complex algebras are isomorphic to the complex field (see [Ka1]). However Wright's problem remains unanswered, and we want to emphasize the following two particular cases of it.

B.27. *Is every (noncomplete) normed complex division algebra isomorphic to* $\mathbb{C}$*?*

B.28. *Is every complete normed real division algebra finite-dimensional?*

Problem B.27 has affirmative answer in the class of noncommutative Jordan-algebras. The reason is that, for noncommutative Jordan algebras, division implies Jordan-division (relative to the Jacobson-McCrimmon concept of inverse), hence the Jordan version of the complex Gelfand-Mazur theorem applies. The same argument can be used, together with the already cited description of normed (commutative) Jordan-division real algebras in [Ka2], to obtain that the only normed division Jordan real algebras are $\mathbb{R}$ and $\mathbb{C}$. However, Problem B.28 remains unsolved even in the class of noncommutative Jordan algebras. To understand the difficulty of the problem, note that the classification of finite-dimensional division noncommutative Jordan real algebras has been provided only very recently (see [KaRoc] and [Roc]).

C. Jordan-Banach triple systems.

Jordan-Banach triple systems with additional geometric requirements, but without restriction of finite dimension, arose in the literature by the hand of W. Kaup [Kau1], and since then great progress has been made in this field, some aspects of which will be reviewed in Section D. It is however the aim of the present section to remain in the coordinates of the previous one, and consequently to deal with the nongeometric theory of Jordan-Banach triple systems, mainly due to A. Fernández and his coauthors. This theory is very recent (let's cite as an isolated precedent the paper [FeGar1] on strongly regular associative Banach triple systems), so the customary subsection about results until 1988 has no place in this case.

Recent results.

The easier examples of Jordan triple systems (in short, Jordan triples) are the subspaces of Jordan algebras that are closed under the "Jordan triple product" $\{xyz\}:=(x.y).z+x.(y.z)-(x.z).y$. Although not all Jordan triple are of this form, the above examples can help us to enter in the philosophy of general Jordan triples and the methods of work on this field. To be more motivated, note that most of the relevant concepts in Jordan algebras, such as that of von Neumann regularity, nondegeneracy, inner ideal, socle, and Jacobson-McCrimmon radical (see [Mc3]), can be expressed in terms of the Jordan triple product. In fact, for a Jordan algebra with a unit, even the concept of invertible element and of inverse of such an element can be rediscovered from the Jordan triple product, and without involving the unit. But this last fact is not relevant in the theory of Jordan triples because, if a Jordan triple has such an "invertible" element, it is essentially nothing more than a Jordan algebra endowed with its Jordan triple product. This is an intrinsic limitation of the theory that, until now, has prohibited the development of a reasonably satisfactory spectral theory for Jordan triples, even in the complete normed complex case. This limitation may have been the reason why, in the absence of a spectral theory, all known nontrivial nongeometric results on Jordan-Banach triples involve in one or another way the restriction of the existence of a "large" socle. As a compensation, the results are very fine structure theorems. We must note that there is a different approach based in the so called "grid theory" [N1] which is promising.

Because the definition of Jordan triples may be occasionally forgotten even by the specialists, it seems convenient to recall it here. A Jordan triple consists of a vector space J and a trilinear mapping $\{\ldots\}: J\times J\times J \longrightarrow J$ (called the triple product of J) satisfying the following two conditions:

$$\{xyz\}=\{zyx\} \tag{1}$$

$$\{uv\{xyz\}\}-\{xy\{uvz\}\}=\{\{uvx\}yz\}-\{x\{vuy\}z\} . \tag{2}$$

An important variant of this definition, fully justified by the applications to the Analysis, arises when the base field is $\mathbb{C}$ and the triple product $\{\ldots\}$ is supposed to satisfy properties (1) and (2) above, but, instead to be trilinear, it is assumed to be linear in the outer variables and conjugate-linear in the middle variable. Then we say that J is a complex Jordan *-triple. However, the terminology, as well as results, can be unified in some way by noting that complex Jordan *-triple are real Jordan triples. Thus, and unless explicit mention on the contrary, all concepts and results on Jordan triples we shall review apply (sometimes with minor changes) to complex Jordan *-triples.

As we have already pointed out, the theory of the socle in Jordan triples has been the most important tool in the nongeometric study of the structure of Jordan-Banach triples (real or complex Jordan triples with a complete norm making the triple product continuous). It is an honour for analysts that the development of this theory in its purely algebraic setting has been motivated by Analysis. This has happened in the paper of A. Fernández, E. García, and E. Sánchez [FeGarSa1]. In spite of its title "Von Neumann regular Jordan Banach triple systems", analytic methods are only used in its relatively short concluding section and the remaining part of the paper is devoted to algebraic tools for the proof of the concluding analytic result. Let's pay tribute to Algebra by collecting in the next theorem most of the non-structural theory of the socle (defined as the sum of the minimal inner ideals) of a nondegenerate Jordan triple, developed in [FeGarSa1]. We refer also to the paper of O. Loos [Lo2], where further developments, as well as alternative proofs of some results in [FeGarSa1], are provided, and we emphasize that both papers contain interesting new methods and concepts that even clarify the classical theory of the socle for associative and Jordan algebras.

Theorem C.1. *If J is a nondegenerate Jordan triple with nonzero socle, then the socle of J (denoted as usual by Soc(J)) is a direct sum of simple ideals each of which contains a minimal inner ideal. A nondegenerate Jordan triple coincides with its socle if and only if it satisfies the descending chain condition on principal inner ideals. Moreover, the socle of any nondegenerate Jordan triple is von Neumann regular.*

The paper [FeGarSa1] also contains a very fine structure theorem for the so called "reduced" simple Jordan triples over algebraically closed fields, relying on Zel'manov's general classification theorem for prime nondegenerate Jordan triples [Ze2], as well as on earlier results in [FeGar3] and [CueGaMar1] on prime associative triples with nonzero socle. Extending the corresponding result for Jordan-Banach algebras, it is proved that *the socle of a nondegenerate Jordan-Banach triple agrees with the largest von Neumann regular ideal.* With the above-mentioned structure theorem, the main result is then proved. In a summarized form (see [FeGarSa1; Theorem 6.4] for details) it reads as follows.

Theorem C.2. *A von Neumann regular complex Jordan-Banach triple is a direct sum of a finite number of closed simple ideals, each of which is either (i) finite-dimensional, (ii) a Jordan triple coming from an infinite-dimensional simple quadratic Jordan-Banach complex algebra, or (iii) a Jordan triple associated to an infinite-dimensional simple von Neumann regular complex Banach associative triple of the second king.*

Simple von Neumann regular complex Banach associative triples of the first or second kind are also described [FeGarSa1; Theorem 6.5].

Returning to the purely algebraic setting, recall that, by the Zel'manov prime theorem [Ze2] (see also D'Amour's version [Amo]), a prime nondegenerate Jordan triple J is either exceptional, quadratic, or hermitian, and that in this last case J is trapped between the symmetric elements of a *-prime associative triple (of the first kind) T and those of the quotient triple $Q(T)$ [Mc5]. It has been proved in [FeGarSa2] that if the prime nondegenerate hermitian Jordan triple J has nonzero socle, then also T has nonzero socle, so the problem of classifying prime nondegenerate Jordan triples with nonzero socle reduces to determining the quotient triple of a *-prime associative triple of the first kind with nonzero socle and their involutions. Such determinations have been also provided in [FeGarSa2], involving the classical tool of "continuous" σ-linear operators between pairs of dual vector spaces over division algebras.

The results we describe above have been the inspiration for the paper of K. Bouhya and A. Fernández [BouFe] on (complex) nondegenerate Jordan-Banach *-triples with nonzero socle. It is proved there that nonradical nondegenerate compact Jordan-Banach *-triples have nonzero socle and, more precisely, that *nondegenerate compact Jordan-Banach *-triples are modular annihilator.* (As the reader may have divined, a nondegenerate Jordan (*-) triple J is called modular annihilator if $J/Soc(J)$ is radical, and a Jordan-Banach (-*) triple J is called compact if, for every x in J, the operator $y \longrightarrow \{xyx\}$ is compact.) The main result in the paper under review is a classification theorem for prime Jordan-Banach *-triples with nonzero socle and with no nonzero nilpotent elements. Instead of giving here its statement (which is rather long and involves many additional terminology), let's comment that it is the parallel in its setting to Theorem B.8 in the context of Jordan-Banach algebras. Moreover, because of the strong assumption of absence of nilpotent elements in the cited version for triples, the classification is improved, in the sense that the Banach pairings involved in the treatment of the hermitian case are in fact Banach spaces that are self-paired by means of a (continuous) inner product. Note that, when the classical theory of prime associative Banach algebras with nonzero socle is applied to the associative Banach algebra A appearing in cases (iii) (respectively, (iv)) of Theorem B.8, the algebra A may be described in terms of operators on Banach pairings (respectively, self-paired Banach spaces) without any additional property.

The paper [BouFe] contains also the specialization of the above-mentioned structure theorem to the particular cases of prime compact Jordan-Banach *-triples with no nonzero nilpotent elements, and prime JB*-triples with nonzero socle. We shall explain in some detail the result concerning JB*-triples in the next section.

Problems and future directions.

In view of the above collected results, it seems convenient to compare them with the ones in Section B involving the socle of Jordan-Banach algebras. We have seen before that, for a Jordan-Banach triple J, are equivalent: (i) J is von Neumann regular, (ii) J is nondegenerate coinciding with its socle, and (iii) J has finite capacity in the sense of Loos [Lo3]. But for Jordan-Banach algebras there are two further equivalent conditions, namely, algebraicness together with semiprimitiveness, and finite spectrum together with semiprimitiveness again. Certainly, a suitable notion of algebraicness for Jordan triples can be given, by observing that odd (associative) powers of an element are defined. Therefore, the following question can be raised.

C.3. *Prove that semiprimitive Jordan-Banach triples are algebraic (if and) only if they have finite capacity.*

In spite of the absence of a spectrum for elements of Jordan triples, in order to obtain a "finite-spectrum" condition characterizing the finite capacity in the Banach case, we can introduce the following definition. An element x in a Jordan triple J will be said to have "properly finite spectrum" if x has finite spectrum relative to all Jordan algebras of the form $J^{(y)}$ with y in J (where $J^{(y)}$ means the Jordan algebra y-homotope of J, namely, the vector space as J with product $t.z:=\{tyz\}$). Now the problem can be posed as follows.

C.4. *Prove that a semiprimitive Jordan-Banach triple J has finite capacity if (and only if) each element in J has properly finite spectrum.*

Perhaps similar ideas involving homotopes can be useful to palliate the lack of spectral theory in Jordan triples. In particular a Jordan-triple version of Theorem B.5, as well as a "spectral" characterization of semiprimitive modular annihilator Jordan-Banach triples in the line of Theorem B.6, cannot be discarded. It seems also reasonable to look for a Jordan-triple version of Theorem B.7, or, more adventurously, to ask if the socle of a complex semiprimitive Jordan-Banach triple J is the largest algebraic (or properly finite-spectrum) ideal of J (compare with B.24).

In relation with the paper [BouFe] reviewed above, we can pose the following problem (compare with Theorem B.9).

C.5. *Prove a structure theorem for prime Jordan-Banach *-triples with nonzero socle, no nonzero nilpotent elements, and minimality of norm topology.*

We conjecture that, with such a theorem, it can be proved that the Jordan-Banach *-triples considered are essentially nothing more than prime JB*-triples with nonzero socle under equivalent renorming,

Let's finally make a short comment about the possibility of developing the nongeometric theory of Jordan-Banach triples without involving the socle. In this respect we think that results like for example B.2 (respectively, B.19, that contains B.2) have easy extensions for Jordan-Banach triples (respectively, complete normed nonassociative ternary algebras), by simply applying the original techniques with suitable minor changes. (If the extension of B.2 for triples is desired without going over the extension of B.19, it is even enough to use the original result B.2 together with the technique of homotopes we mentioned above.) However, the suggested line of work has no special merit, and therefore it must be only understood as a possible point of departure for further more intrinsic results on Jordan-Banach triples (respectively, complete normed nonassociative ternary algebras).

D. Selected topics in JB^*-algebras and triples.

JB-algebras, JB*-algebras, and JB*-triples, as well as Vidav nonassociative algebras reviewed in Section A, are four different approaches to nonassociative generalizations of C*-algebras, giving birth, at an advanced step of their respective theories, to almost the same mathematical creature. This field has become without doubt the most active one concerning Jordan structures in Analysis in the last fifteen years, and therefore it seems to be almost impossible to make a complete survey of results (even if we are interested only in the last four years). Since the theory of JB-algebras can be considered reasonably finished [HSto] (being therefore recently not too much worked), and Vidav's nonassociative algebras have been already considered, we shall mainly center the attention in JB*-algebras and JB*-triples. Even in this case we shall only consider those aspects of the theory that are close to the work of the Spanish school, expecting other surveys in this meeting to give a more complete view of this so suggestive field.

Summary of results until 1988.

JB-algebras, first introduced and developed by E. Alfsen, F. Shultz, and E. Stormer [AlfShuSto], are defined as those Jordan-Banach real algebras J satisfying $\|x\|^2 \leq \|x^2+y^2\|$ for all x, y in J. Through the proof that the bidual of a JB-algebra (endowed with the Arens product) is also a JB-algebra, and the introduction of JBW-algebras (JB-algebras that are dual Banach spaces) and JBW-factors (prime JBW-algebras), the theory of general JB-algebras is reduced to that of JBW-factors, and it is shown that JBW-factors are either weakly-closed (prime) Jordan algebras of symmetric bounded linear operators on complex Hilbert spaces, or the Albert exceptional Jordan algebra $M_3^8(\mathbb{R})$ of all 3×3 hermitian matrices over the Cayley numbers. In a roughly summarized form, the above facts constitute the outline of the structure theory for JB-algebras and complete details are to be found in [HSto].

(Commutative) JB*-algebras were first considered by J. D. M. Wright

[Wri], who used the structure theory of JB-algebras to prove that *JB-algebras are nothing but the self-adjoint parts of JB*-algebras*. The categorical one-to-one correspondence between JB-algebras and JB*-algebras derived from Wright's result completely reduces the *-theory of JB*-algebras to the theory of JB-algebras. Even this correspondence preserves the algebras which are dual Banach spaces (called JBW*-algebras in the JB*-case) [E2], as well as those that are prime, hence in particular it maps JBW-factors onto JBW*-factors. From the above, it follows that the interest of JB*-algebras centers in the following points:

-They become an alternative approach to JB-algebras, from which sometimes the theory itself of JB-algebras has benefited (see for example Theorem D.11 below and its corollaries).

-As we shall comment later, they have the advantage over JB-algebras of being much more close to JB*-triples.

-There are intrinsic geometric and algebraic (non-*) aspects of JB*-algebras which have their own interest, for example the Russo-Dye type theorem for unital JB*-algebras [WriYo1], or the factorization theorem for non-* isomorphisms between JB*-algebras [PaPeRod1; Theorem 2.9] (see also its precedent [Rod1]).

But, from our point of view, the greatest relevance of JB*-algebras is that their noncommutative natural generalizations, namely noncommutative JB*-algebras, become the answer for the nonassociative Vidav-Palmer theorem (Theorem A.1). (Incidentally we note that the papers dealing with the noncommutative approach to JB*-algebras are at this time the most well-codified references for a JB-algebra-free treatment of commutative JB*-algebras. Also note that noncommutative JB*-algebras are not too much generalizations of commutative ones, because every noncommutative JB*-algebra becomes a commutative one by simple symmetrization of its product.) Let's therefore return to noncommutative JB*-algebras and review their structure theory developed in [PaPeRod1], [PaPeRod2], [AlvJan], and [Bra1]. The planning in these papers is very similar to the above-mentioned one for JB-algebras: noncommutative JBW*-factors are introduced in a natural meaning, abundance of factor representations for an arbitrary noncommutative JB*-algebra is assured, and noncommutative JBW*-factors are classified. However, to be in agreement with the finer structure theory for (commutative) JB*-algebras recently obtained in [FeGarRod], and that we shall explain in Section F, we shall reformulate here the results in [PaPeRod1] and [PaPeRod2] in a new equivalent way.

One of the main techniques in these two papers is that of M-ideals on Banach spaces [AlfEf]. An M-ideal of a Banach space X is a closed subspace (say M) of X for which there exists a subspace P of the dual Banach space X^* satisfying $X^*=M^0\oplus P$ and $\|f+g\|=\|f\|+\|g\|$ for all f in M^0 and g in P. An M-ideal M of the Banach space X is called "primitive" if there is an extreme point ϕ of the closed unit ball of X^* such that M it is the largest M-ideal of X contained in the kernel of ϕ. *Closed ideals of noncommutative JB*-algebras are nothing but M-ideals of their Banach spaces* [PaPeRod1; Theorem 4.3]. Let's therefore say that a noncommutative JB*-algebra is "geometrically primitive" if zero is a primitive M-ideal of its Banach space (equivalently, if there exists an extreme point in the closed unit ball of its dual space whose kernel contains no nonzero ideals of A). An easy consequence of results by C. A. Akemann and B. Russo [ARus], explicitly stated in [PaPeRod1], is that, for C*-algebras, geometrical primitiveness agrees with primitiveness in the usual meaning.

Geometrically primitive noncommutative JB-algebras are prime,* and, to reduce precisely the theory of noncommutative JB*-algebras to the geometrically primitive case, let's also consider the following definition. Given a family $\{A_i\}_{i\in I}$ of noncommutative JB*-algebras, any closed self-adjoint subalgebra B of the noncommutative JB*-algebra $\oplus^{l_\infty}_{i\in I} A_i$ with the property that $\pi_i(B)=A_i$ for all i in I (where π_i denotes the natural projection onto the i-th coordinate) will be called a subdirect l_∞-sum of the given family $\{A_i\}_{i\in I}$. Now we can state the first step in the structure theory of noncommutative JB*-algebras.

D.1. *Every noncommutative JB*-algebra is (totally isomorphic to) a subdirect l_∞-sum of a suitable family of geometrically primitive noncommutative JB*-algebras.*

The second step consists of the following classification theorem for geometrically primitive noncommutative JB*-algebras.

D.2. *The geometrically primitive noncommutative JB*-algebras are the following:*

i) the geometrically primitive (commutative) JB-algebras,*

ii) the simple quadratic noncommutative JB-algebras, and* *iii) the noncommutative JB*-algebras obtained from primitive associative C*-algebras by changing the associative product "ab" by* $a\circ b:=\lambda ab+(1-\lambda)ba$, *where* λ *is a fixed real number with* $0\leq\lambda\leq 1$.

The last step in the structure theory for noncommutative JB*-algebras we are reviewing is the following description of simple quadratic noncommutative JB*-algebras.

D.3. *Given a nonzero real Hilbert space E of dimension* $\neq 1$ *with a bilinear anticommutative product* $\wedge$ *satisfying* $(x\wedge y|z)=(x|y\wedge z)$ *and* $\|x\wedge y\|\leq\|x\|\|y\|$ *for all x,y,z in E, consider the real algebra B whose vector space is* $\mathbb{R}1\oplus E$ *and whose product is defined by*

$$(\alpha 1+x)(\beta 1+y):=[\alpha\beta-(x|y)]1+\alpha y+\beta x+x\wedge y.$$

*Then the complexification of B, with the involution * defined by*

$$[\alpha 1+x+i(\beta 1+y)]^*:=\alpha 1-x-i(\beta 1-y),$$

and norm given by

$$\|b+ic\|^2:=\|b\|^2+\|c\|^2+2[\|b\|^2\|c\|^2-(b|c)^2]^{1/2}$$

(where, in the right hand of the equality, the inner product and norms refer to the natural hilbertian structure of B as l_2*-sum of* $\mathbb{R}1$ *and E), is a simple quadratic noncommutative JB*-algebra. Moreover, all simple quadratic noncommutative JB*-algebras can be constructed in this way.*

It follows from D.1, D.2, and D.3 that, to have a complete structure theory for noncommutative JB*-algebras, it only remains to describe the geometrically primitive (commutative) JB*-algebras. As we have pointed out above, such a description has been provided very recently, and it will be reviewed in Section F.

Now let's briefly comment on other interesting relatively old results in the theory of JB*-algebras. Concerning structure under additional assumptions, let's refer to the description of noncommutative JB*-algebras with reflexive (equivalently, hilbertizable) Banach space [PaPeRod2; Theorem 3.5], as well as that of noncommutative JB*-algebras that are ideals in their biduals (see [Rod10; Corollary 5.8] and [L; Theorem 14]). Concerning non-structural results, we first emphasize the following "contractive projection" theorem (see [RoYo] and [Rod4]).

D.4. *Let A be a unital noncommutative JB*-algebra, and* $P:A\longrightarrow A$ *be a unit-preserving positive linear projection. Then the Banach space P(A) is *-invariant and, with the restriction of * and the product given by* $x\square y:=P(xy)$, *P(A) becomes a noncommutative JB*-algebra.*

The proof of this result used the nonassociative Vidav-Palmer theorem (A.1), and that *unit preserving linear operators on noncommutative JB*-algebras are contractive if (and only if) they are positive.* This fact was derived in [Rod4] from a Gelfand-Naimark type theorem asserting that, *if A is a unital noncommutative JB*-algebra, then there are a complex Hilbert space and a unit-preserving isometric linear mapping from A into BL(H).*

Before passing to present JB*-triples and some classical results about them, let's finally refer to the characterization of JB*-algebras up to equivalent renorming (see [Yo2] and [AuYo]), and the papers of K. Alvermann [Alv1], where an Araki-Elliot type theorem for JB*-algebras was proven, and [Alv2], where an intrinsic characterization of closed selfadjoint real subalgebras of JB*-algebras was obtained.

After the relevant works of M. Koecher ([Ko2] and [Ko3]) and O. Loos [Lo1] on the close dependence between bounded homogeneous circled domains in finite-dimensional complex vector spaces and certain finite-dimensional (complex) Jordan *-triples, L. A. Harris ([Harr2], and [Harr3]) observed, and exploited intensively his observation that the good holomorphic behaviour of the open unit ball of C*-algebras is inherited by the so called J*-algebras. J*-algebras are closed subspaces of C*-algebras that are also closed (in the algebraic meaning) under the natural Jordan *-triple product $\{xyz\}:=\frac{1}{2}(xy^*z+zy^*x)$. Harris's J*-algebras are particular types and the most immediate precedents of JB*-triples. These were introduced by W. Kaup in [Kau1], where he showed that *the open unit ball of a JB*-triple is a bounded symmetric domain,* and obtained some basic non-structural properties of JB*-triples. After classifying bounded symmetric domains in complex Hilbert spaces of arbitrary dimension [Kau2], Kaup's work reached its god in [Kau3], where, using an earlier result of J. P. Vigué [V], it was proved that *every bounded symmetric domain in a complex Banach space is biholomorphically equivalent to the open unit ball of a JB*-triple.* JB*-triples are defined as those Jordan-Banach *-triples J such that, for every x in J, the (bounded linear) operator $P(x):y\longrightarrow\{xxy\}$ on J is hermitian (in the sense of [BoDu; Definition 10.12]), has positive spectrum, and satisfies $\|P(x)\|=\|x\|^2$. In order to stop us worrying about the norm of a given JB*-triple, it is interesting to note that the norm of a JB*-triple J is algebraically determined. Indeed, by Sinclair's theorem [BoDu; Theorem 10.17], for x in J, $\|P(x)\|$ is nothing but the spectral radius of $P(x)$ in the Banach algebra *BL(J)*, hence, by the Gelfand-Beurling formula and Banach isomorphisms theorem (see [Rod5; Remark 1.8]), we have

$$\|x\|^2=sup\{|\lambda| \ : \ \lambda\in sp(L(J),P(x))\}.$$

Here as elsewhere in this survey, *L(J)* denotes the algebra of all (possibly discontinuous) linear operators on *J*.

JB-algebras are JB*-triples under the triple product* $\{xyz\}:=x.(y^*.z)-y^*.(z.x)+z.(x.y^*)$ (see [BraKauUp] and [Yo4]), and, as an important consequence of the structure theory for JB*-triples which we shall summarize below, *every JB*-triple can be found as a JB*-subtriple of a suitable JB*-algebra* [FriRus2; Corollary 2]. Note that, in this relation between JB*-algebras and JB*-triples, *Harris's J*-algebras are nothing more than JB*-subtriples of the JB*-algebras obtained by symmetrization of the products of C*-algebras.*

The first crucial step for a structure theory for JB*-triples was given by S. Dineen (see [Di1] and [Di2]), who, using an ultraproduct version of the principle of local reflexivity and the contractive-projection Theorem (see [St] and [Kau4]), showed that *the bidual of a JB*-triple is also a JB*-triple containing the given JB*-triple as a JB*-subtriple.* Dineen's techniques were later refined in the papers of T. Barton and R. M. Timoney [BarT] and G. Horn [Hor1], showing the uniqueness of the predual for JBW*-triples (JB*-triples that are dual Banach spaces) and, consequently, the separate w*-continuity of the triple product of any JBW*-triple. The Barton-Timoney paper contains also an almost verbatim translation of the M-ideal techniques in [PaPeRod1] to the context of JB*-triples. Thus it is proved that *closed ideals of JB*-triples are nothing but M-ideals of their Banach spaces,* and that *primitive M-ideals of JB*-triples are kernels of* (specially well-behaved) *factor representations,* thus assuring the abundance of factor representations for an arbitrary JB*-triple. Then a concept of "geometrically primitive" JB*-triple can be given, and a result similar to D.1 holds in the context of JB*-triples. But, since, as far as we know, the description of geometrically primitive JB*-triples is still unknown (only certain JBW*-envelopes, called "type I" JBW*-factors, are described), we prefer to explain the outline of the structure theory of JB*-triples with its classical terminology.

Before we enter in this theory, let's refer to the following result in [Hor1], in which the ideas about M-ideals outlined above were based. Also it will be useful later.

D.5. *For every w*-closed ideal P of a JBW*-triple J, there exists another w*-closed ideal Q of J satisfying* $J=P\overset{l_\infty}{\oplus}Q$.

Type I JBW*-factors are defined as those JBW*-factors (prime JBW*-triples) containing minimal tripotents, and it can be deduced from results of Y. Friedman and B. Russo in [FriRus1] that type I JBW*-factors are precisely those JBW*-factors arising in the above-mentioned abundant "well-behaved" factor representations, built by Barton and Timoney for an arbitrary JB*-triple. Therefore *every JB*-triple is isometrically isomorphic to a w*-dense JB*-subtriple of the* l_∞*-sum of a suitable family of type I JBW*-factors* (see also [FriRus2]). Incidentally we note that l_∞-sums of type I JBW*-factors are called "atomic JBW*-triples" in the literature. By the main result of G. Horn in his thesis (see the more accessible reference [Hor3]), *type I JBW*-factors are the* so called *"Cartan factors", namely, the classical finite-dimensional Cartan factors* C^5 *and* C^6, *plus the natural infinite-dimensional generalizations of classical Cartan factors of types* C^1, C^2, C^3, *and* C^4 (see for example Section 1 of [FriRus2] for details). In this way the structure theory of JB*-triples can be considered finished in a first instance.

A rather different approach to the structure of JB*-triples has been made more recently, beginning with the Horn's paper [Hor2] (involving the theory of alternative C*-algebras reviewed in Section A), and concluding with the papers of Horn [Hor3] and Horn-Neher [HorN], where a surprising description of arbitrary JBW*-triples is provided. Of course, the w*-dense inclusion of a JB*-triple in its bidual finishes the structure theory form this point of view.

Now let's review some interesting non-structural results on JB*-triples. We begin with the following consequence of [BarDaHor; Proposition 6].

D.6. *Every JBW*-triple is (isometrically) isomorphic to a w*-closed ideal of its bidual.*

The subsequent result we shall review, the so called "little Grothendieck's theorem" for JB*-triples, is deeper. It was proved by T. J. Barton and Y. Friedman in [BarFri1] as a consequence of more general theorems also stated there (see also [ChILou]), and its formulation itself relies on a nice idea translating to the context of JB*-triples the spirit of the prehilbertian seminorms associated to positive linear forms on C*- and JB*-algebras. To explain this idea, consider a norm-one bounded linear form f on a JB*-triple J, and chose a norm-one element z in J^{**} attaining its norm at f (note that, if J is actually a JBW*-triple, and f is w^*-continuous, then such an element z can be selected in J). Then it was proved in [BarFri1] that $(x,y) \longrightarrow f(\{xyz\})$ is a positive sesquilinear form on J not depending of the chosen supporting element z in J^{**}. The prehilbert seminorm $\|.\|_f$ is then defined by the equality $\|x\|_f^2 := f(\{xxz\})$ for all x in J. Now little Grothendieck's theorem for JB*-triples reads as follows.

D.7. *Let J be a JB*- (respectively, JBW*-) triple, H a complex Hilbert space, and T be a bounded (respectively, w^*-continuous) linear mapping from J to H. Then there exists a norm-one bounded (respectively, w^*-continuous) linear form f on J such that $\|T(x)\| \leq 2^{1/2}\|T\|\|x\|_f$ for all x in J.*

Let's conclude this subsection by referring to the work of H. J. Zettl [Zet] on ternary C*-rings. A ternary C*-ring is a complex associative *-triple A of the second kind endowed with a complete norm $\|.\|$ satisfying $\|\{xyz\}\| \leq \|x\|\|y\|\|z\|$ and $\|\{xxx\}\| = \|x\|^3$ for all x,y,z in A. Natural examples of ternary C*-rings are the so called "ternary rings of operators", namely, norm-closed subspaces of bounded linear operators between two complex Hilbert spaces that are (algebraically) closed under the triple product RS^*T. The main result in [Zet] asserts that *every ternary C*-ring is the direct l_∞-sum of two closed ideals each of which is isomorphic (the second one up to multiplication of the triple product by -1) to a ternary ring of operators.* Note that ternary rings of operator become J*-algebras (hence JB*-triples) by symmetrization of the triple product in its outer variables.

Advances since 1998.

Because the structure theory of JB*-algebras and JB*-triples was basically concluded before 1988, the recent advances in these field have been essentially of non-structural type. Also there have been some interesting applications of the theory of JB*-algebras to obtain new results concerning the Jordan structure of C*-algebras, as for example the following result in [Rod7].

Theorem D.8. *Let A be an associative complex algebra and assume that the Jordan algebra A^+, obtained by symmetrization of the product of A, is a JB*-algebra for suitable norm and involution. Then A with the same norm and involution is a C*-algebra.*

This result, together with the Dauns-Hofmann theorem, has been later used in [Rod6] to prove that *noncommutative JB*-algebras that are split quasiassociative over their centroids are nothing but f-mutations of C*-algebras, with f an element in the centroid of the C*-algebra under consideration and satisfying $0 \le f \le 1$.*

Centroids and extended centroids of JB*-algebras have been considered in the paper of A. Rodríguez and A. R. Villena [RodVi]. By using that *the centroid of a JB*-algebra coincides with the "centralizer" of its Banach space* [DiT1] (see [Behr] for the concept of the centralizer), a Dauns-Hofmann type theorem for JB*-algebras is proved, the cores of maximal modular inner ideals (in the sense of [HoMc]) playing the role of the primitive ideals in the original C*-case. With the same philosophy, the recent Ara's description of extended centroids of C*-algebras [Ar1] is generalized to the JB*-context. As a consequence, the following result is obtained.

Proposition D.9. *Every prime noncommutative JB*-algebra is centrally closed.*

The theory of normed Jordan Q-algebras developed in [PeRiRod], and already commented in Section B, has been applied in the same paper [PeRiRod] (see also [Ben2]) to obtain the JB*-extension of a relatively old result of S. B. Cleveland [Cl] on the topology of the norm of a C*-algebra. The proof involves Theorem B.12, which in its turn follows the pattern of the recent new proof given in [Rod8] (again relying on Aupetit's subharmonicity methods) of Cleveland's result, and the precise formulation is the following.

Theorem D.10. *The topology of an arbitrary algebra norm on a noncommutative JB*-algebra is stronger than the topology of the JB*-norm.*

The minimality of norm topology contained in the above theorem has been applied in [PeRiRod], together with the nonassociative Vidav-Palmer theorem (A.1) to obtain that *noncommutative JB*-algebras have "minimality of norm", namely,* $|.|=\|.\|$ *whenever* $|.|$ *is any algebra norm satisfying* $|.|\le\|.\|$. Also the minimality of norm topology (respectively, minimality of norm) for JB*-algebras and Theorem D.8 have been used to show that *the normed associative complex algebras that are ranges of continuous (respectively, contractive) Jordan homomorphisms from C*-algebras are bicontinuously (respectively, isometrically isomorphic) to C*-algebras.* It has been also proved that *weakly compact Jordan homomorphisms from C*-algebras have finite rank,* and a description of the ranges of weakly compact homomorphisms from noncommutative JB*-algebras has been given (see also [G]).

Let's conclude the survey on recent advances in JB*-algebras by reviewing the paper of B. Iochum, G. Loupias, and A. Rodríguez [ILouRod] on the JB*-extension of the well-known theorem of Kaplansky asserting that noncommutative C*-algebras have nonzero nilpotent elements. In fact it has been proved in [ILouRod] that, *if a noncommutative JB*-algebra J has no nonzero nilpotent elements, then J is associative and commutative.* But, for (commutative) JB*-algebras, this result has even been nicely refined in the same paper in the way we shall state in the next theorem. Let Q_3 denote the three-dimensional (automatically simple) quadratic JB*-algebra (take $dim(E)=2$ and $\lambda=0$ in the construction D.3), and, for a normed algebra A, let $\mathcal{C}_0([0,1],A)$ denote the algebra of all continuous A-valued functions on the closed interval $[0,1]$ of $\mathbb{R}$ vanishing at zero, endowed with the supremum norm. Then we have:

Theorem D.11. *A JB*-algebra is not associative (if and) only if it contains (as a JB*-subalgebra) either* Q_3 *or* $\mathcal{C}_0([0,1],Q_3)$.

Theorem D.11 was obtained in [ILouRod] from the JB*-version of Kaplansky's result mentioned above and a C*-version of the theorem itself, also provided in [ILouRod], and that seems to have been previously unknown. This C*-version of Theorem D.11 asserts that *A C*-algebra is not commutative (if and) only if it contains (as a C*-subalgebra) either* $M_2(\mathbb{C})$ *or* $\mathcal{C}_0([0,1],M_2(\mathbb{C}))$. Theorem D.11 has the following more or less direct corollaries: (i) *A JBW*-algebra is not associative if and only if it contains* Q_3, (ii) *A JB-algebra is not associative if and only if it contains either* S_3 *or* $\mathcal{C}_0([0,1],S_3)$, *where* S_3 *denotes the three-dimensional spin factor*, and (iii) *A JBW-algebra is not associative if and only if it contains the three-dimensional spin factor.*

As a transition between JB*-algebras and JB*-triples, let's refer to a result in the paper of J. Arazy and B. Solel [AraSo] that, although dealing with JB*-algebras, has been obtained from Jordan-triple-product techniques closely related to Kaup's holomorphic approach to JB*-triples (see [KauUp]). The Arazy-Solel result was already announced by Arazy in the 1988 Oberwolfach meeting on Jordan algebras, and reads as follows.

Theorem D.12. *Let* ϕ *be a unit-preserving surjective linear isometry between unital closed (not necessary self-adjoint) subalgebras of suitable JB*-algebras. Then* ϕ *is an (algebra-) isomorphism.*

Now, entering properly in recent advances in the theory of JB*-triples, let's first review the paper of T. J. Barton and Y. Friedman [BarFri2], some results of which were also announced (by Barton) in the 1988 Oberwolfach meeting on Jordan algebras. Essentially the paper is devoted to providing the following affirmative answers to two questions raised by Upmeier in [Up2].

Theorem D.13. *If the domain of a partially defined derivation* D *on a JB*-triple* J *contains, with each of its elements* y, *also the only element* z *in* J *satisfying* $y=\{zzz\}$, *then* D *is closeable. As a consequence, everywhere defined derivations on JB*-triples are continuous.*

Theorem D.14. *The set of inner derivations on a JB*-triple* J *is dense in the set of all derivations on* J *with respect to the strong operator topology.*

To prove Theorem D.14, Barton and Friedman introduced a new and powerful tool in the theory of JB*-triples, namely, the so called "strong* topology" (in short, *s**-topology) of a JBW*-triple. Using again the prehilbert seminorms $\|.\|_f$ associated to bounded linear forms f, built by themselves in connection with Grothendieck's theorem for JB*-triples (D.7), they defined the s^*-topology of a JBW*-triple J (denoted by $s^*(J,J_*)$, when some confusion can occur) as the topology in J generated by the family of seminorms $\|.\|_f$, when f runs over the norm-one elements of the predual J_* of J, and they proved the following theorem.

Theorem D.15. *The s^*-topology of a JBW*-triple J is compatible with the standard duality (J,J_*). Moreover, if J' is a w*-dense subtriple of J, then the closed unit ball of J' is s^*-dense in the closed unit ball of J.*

After showing that, for each fixed element x in an atomic JBW*-triple J, the mapping $y \longrightarrow \{yyx\}$ from J into J is $s^* \times w^*$ continuous on bounded sets, Barton and Friedman asked if the triple product of any JB*-triple is in fact jointly $s^* \times s^*$ continuous on bounded sets. The answer to this question has been provided in [Rod9], where the following result has been shown.

Theorem D.16. *The triple product of a JBW*-triple is jointly s^*-continuous on bounded sets.*

The proof of the above theorem given in [Rod9] consists of three subsidiary results with have their own interest. First the JBW*-version of the little Grothendieck theorem (D.7) is applied to "spacialize" the s^*-topology in the following way.

Proposition D.17. *The s^*-topology of a JBW*-triple J coincides with the topology on J generated by the family of seminorms $x \longrightarrow \|T(x)\|$, when T runs over all w*-continuous linear mappings from J into arbitrary complex Hilbert spaces.*

The above proposition and the Bishop-Phelps-Bollobas theorem are then applied to obtain:

Proposition D.18. *Let K be a JBW*-subtriple of a JBW*-triple J. Then $s^*(J,J_*)$ agrees with $s^*(K,K_*)$ on bounded subsets of K.*

The third component of the proof of Theorem D.16 is the following.

Proposition D.19. *The s*-topology of a JBW*-algebra (regarded as a JBW*-triple) coincides with its usual algebra-s*-topology.*

We recall that the algebra-s^*-topology of a JBW*-algebra J is defined as the topology on J generated by the family of seminorms $x \longrightarrow (\rho(x^*.x))^{1/2}$, when ρ runs over all w^*-continuous positive linear forms on J. The proof of Theorem D.16 follows easily from Propositions D.18 and D.19 and the result (of folklore type at this time) that *every JBW*-triple can be regarded as a JBW*-subtriple of a suitable JBW*-algebra.* Using intensively the earlier quoted structure theory for JBW*-triples by Horn [Hor3] and Horn-Neher [HorN], C-H. Chu and B. Iochum [ChI] have proved that *every JBW*-triple J can be regarded as a subtriple of a suitable JBW*-algebra J' in such a way that J is the range of a contractive linear projection on J'*. We think that a careful reading of the proof of this fact given in [ChI] would show that the contractive projection there built is actually w^*-continuous, thus improving the above cited folklore result. However, we prefer to start with the Chu-Iochum result in its original form, and then to complete the proof of the following theorem in a non-structural way.

Theorem D.20. *Every JBW*-triple J can be regarded as a subtriple of a suitable JBW*-algebra A in such a way that J is the range of a w*-continuous contractive linear projection Π on A.*

Proof. By the Chu-Iochum theorem there are a JB*-algebra B containing J as a subtriple, and a contractive linear projection π on B such that $\pi(B)=J$. Then the bipolar J^{oo} of J in B^{**} is a subtriple of the JBW*-algebra B^{**} and $\pi^{**}(B^{**})=J^{oo}$. Identifying J^{oo} naturally with J^{**}, and applying D.6 and D.5, we obtain $J^{oo}=P\overset{1}{\oplus}{}^{\infty}Q$, where P and Q are w^*-closed ideals of J^{oo}, and P is a copy of J. To conclude the proof, take A equal to B^{**}, and Π equal to the composition of π^{**} with the projection from J^{oo} onto P deriving from the decomposition $J=P\overset{1}{\oplus}{}^{\infty}Q$. ∎

The above theorem will allow us to obtain here a new result for the s^*-topologies of JBW*-triples extending a well-known one of C. A. Akemann [A] for W*-algebras. Recall that the Mackey topology $m(X,X_*)$ of a dual Banach space X is defined as the largest locally convex topology on X compatible with the standard duality (X,X_*).

Theorem D.21. *The strong* and Mackey topologies of a JBW*-triple J agree on bounded subsets of J.*

Proof. Accordingly to Theorem D.20, there exists a JBW*-algebra A containing J as a w^*-complemented JBW*-triple. As a consequence $m(J,J_*)$ is nothing but the restriction to J of $m(A,A_*)$. Now, since $m(A,A_*)$ agrees with the algebra-s^*-topology of A (equal to $s^*(A,A_*)$ by Proposition D.19) on bounded subsets of A [AlvJan; Theorem 5.13], and $s^*(A,A_*)$ agrees with $s^*(J,J_*)$ on bounded subsets of J (Proposition D.18), it follows that $m(J,J_*)$ and $s^*(J,J_*)$ agree on bounded subsets of J, as required. ∎

A direct consequence of Theorems D.16 and D.21 is the following.

Corollary D.22. *The triple product of a JBW*-triple J is jointly $s^*(J,J_*)\times m(J,J_*)$ continuous on bounded sets.*

Now let's partially review the impressive work of L. J. Bunce and C-H. Chu in [BuCh1] and [BuCh2] concerning weakly compact JB*-triples, together with some interesting complements to this work given in the already commented paper of K. Bouhya and A. Fernández [BouFe] (see Section C). *Every Cartan factor is the bidual of a unique* (perfectly describable) *JB*-triple* [BuCh1; Lemma 3.2]. These JB*-triples that are unique double preduals of Cartan factors are called elementary JB*-triples, and they can be characterized intrinsically in several other ways that we collect in the next proposition. Recall that a Jordan-Banach (*-) triple J is called weakly compact if, for every x in J, $P(x)$ is a weakly compact operator on J.

Proposition D.23. *A JB*-triple is elementary if and only if it is topologically simple and satisfies one of the following five conditions:*

i) J is weakly compact.

ii) J has nonzero socle.

iii) J is modular annihilator.

iv) J is an inner ideal of its bidual.

v) J is an ideal of its bidual.

Actually Proposition D.23 is a direct consequence of the following more general theorem (see [BuCh2; Lemma 3.3 and Theorem 3.4], together with [BouFe; Theorem 17]).

Theorem D.24. *For a JB*-triple J, the following six assertions are equivalent:*

i) J is weakly compact.

ii) J has dense socle.

iii) J is modular annihilator.

iv) J is an inner ideal of its bidual.

v) J is an ideal of its bidual.

vi) J is the c_0-sum of a suitable family of elementary JB-triples.*

A consequence of the theorem is that *compact JB*-triples are those weakly compact JB*-triples with no infinite-dimensional direct summands of type C^4* (see [BuCh2; Theorem 3.6]). The Bunce-Chu papers contains many other interesting ideas among which, in view of the line of our survey, we emphasize the development of a peculiar concept of spectrum for elements of JB*-triples providing the corresponding "spectral characterization" of modular annihilator JB*-triples (see [BuCh2; Proposition 4.5(ii)] and compare with Theorem B.6), as well as the consideration of those JB*-triples on which the operators $y \longrightarrow \{xxy\}$ are (weakly) compact for all x [BuCh2; Theorems 3.4 and 3.7]. Let's conclude our review on this circle of ideas by formulating the following characterization of prime JB*-triples with nonzero socle obtained in [BouFe].

Theorem D.25. *A JB*-triple J is prime and has nonzero socle (if and) only if there exists an elementary JB*-triple E such that J is a JB*-subtriple of E** containing E.*

In several papers [E1], [ERü1], [ERü2] C. M. Edwards and G. T.

Rüttimann studied the facial structure of the closed unit balls in JBW-algebras and their preduals and in JB-algebras and their duals. In [ERü3] these authors went on to investigate the facial structure of the unit ball in a JBW*-triple and its predual.

Let X be a Banach space with dual Banach space X^*. As a basic tool to study the facial structure of the unit ball in X and in X^* they introduced the following pair of polarities $E \longrightarrow E^{\square}$ and $F \longrightarrow F_{\square}$ between the subsets of the closed unit balls X_1 and X^*_1:

$$E^{\square} := \{a \in X^*_1 \ : \ a(x)=1 \ \forall x \in E\}, \qquad F_{\square} := \{x \in X_1 \ : \ a(x)=1 \ \forall a \in F\}.$$

Let J be a JBW*-triple and let $U(J)$ be the set of tripotents in J. Let $U(J)^{\sim}$ be the disjoint union of $U(J)$ with a one point set $\{\omega\}$ and define a binary relation on $U(J)^{\sim}$ by writing $u \le v$ if u and v lie in $U(J)$ and $\{uvu\}=u$ or if u is an element in $U(J)^{\sim}$ and v equals ω. It follows that $\le$ is an order relation on $U(J)^{\sim}$.

Theorem D.26. *Let J be a JBW*-triple and let J_* be its predual Banach space. Let J_1 be the unit ball in J and let J_{*1} the closed unit ball in J_*.*

(i) The mapping $u \longrightarrow \{u\}_{\square}$ (where $\{\omega\}_{\square} := J_{*1}$) is an order isomorphism from the partially ordered set $U(J)^{\sim}$ defined above and the complete lattice $\mathcal{F}(J_{*1})$ of norm closed faces of J_{*1}. The partially ordered set $U(J)^{\sim}$ is a complete lattice.

(ii) The mapping $u \longrightarrow (\{u\}_{\square})^{\square}$ is an anti-order isomorphism from the complete lattice $U(J)^{\sim}$ onto the complete lattice $\mathcal{F}_{w*}(J_1)$ of weak closed faces of J_1. If u lies in $U(J)$ then $(\{u\}_{\square})^{\square}$ coincides with $u+J_0(u)_1$ (where $J_0(u)$ denotes the Peirce 0-subspace of J relative to u).*

It follows that every norm closed face of J_{*1} is norm exposed and that every weak* closed face of J_1 is weak* "semi-exposed" (i.e., the intersection of a suitable family of weak* exposed faces).

These results are then applied to W*-algebras and weak* closed J*-algebras. As a consequence, *the set of partial isometries in a W*-algebra A in the ordering $u \le v \Leftrightarrow u=uv^*u$ and with greatest element adjoined forms a complete lattice and every non-empty weak* closed face of the closed unit ball A_1 of A is of the form*

$$u + (\mathbf{1}-uu^*)J(\mathbf{1}-u^*u)_1$$

for some unique partial isometry u.

In connection with Theorem D.26, let's refer to the paper by M. Battaglia [Bat2], where, for a JBW*-triple J, the lattice $U(J)^{\sim}$ is studied in depth, and significant results concerning the structure and classification of JBW*-triples are derived. Let's also comment that the fact that norm closed faces of the closed unit ball of the predual of a JBW*-triple are norm exposed has become one of the crucial results in the project of Y. Friedman and B. Russo of characterizing intrinsically the Banach spaces which are preduals of JBW*-triples (see [FriRus3], [FriRus4], [FriRus5], and [FriRus6]).

The study of JB*-triples and JBW*-triples would not be complete without some understanding of their ideal structure. In a series of papers [ERü4], [ERü5], [ERü6], [ERüVa1] and [ERüVa2] the same authors, with later collaboration by S. Yu. Vasilowsky, investigated the structure of the complete lattice $\mathcal{J}(J)$ of weak* closed inner ideals in a JBW*-triple J. The first main result, provided in [ERü4] (see also [Bu3]), served the program for a W*-algebra. A pair (e,f) of elements of the complete lattice $\mathcal{P}(A)$ of projections in a W*-algebra A is said to be centrally equivalent if the central supports of e and f coincide. They showed that *the mapping* $(e,f)\longrightarrow eAf$ *is an order isomorphism from the complete lattice of centrally equivalent pairs of projections in* A *onto* $\mathcal{J}(A)$.

In the paper [HorN] already quoted, G. Horn and E. Neher showed that a continuous JBW*-triple J is isomorphic to one of the form $pC \overset{1}{\oplus}{}^{\infty} H(B,\tau)$ where B and C are continuous W*-algebras, p is a projection in C, τ is a *-involution on B (i.e., an involutive *-anti-automorphism of B), and $H(B,\tau)$ is the JBW*-algebra of elements of B invariant under τ. The weak* closed right ideal pC in C is an example of a weak* closed ternary subalgebra of C. The weak* closed inner ideals in such ternary algebras were identified in [ERü5] as were those in $H(B,\tau)$ in [ERüVa2] thereby yielding the following structure theorem.

Theorem D.27. *Let* I *be a weak* closed inner ideal in the continuous JBW*-triple* $pC \overset{1}{\oplus}{}^{\infty} H(B,\tau)$. *Then there exist projections* f *and* g *in* C *with* $f\leq p$ *and a projection* e *in* B *such that*

$$I=fCg \overset{1}{\oplus}{}^{\infty} eH(B,\tau)\tau(e).$$

Quite naturally the structure of the norm closed inner ideals in a JB*-triple are less readily identifiable that those in a JBW*-triple. However, a recent remarkable result by Edwards and Rüttimann [ERü7] shows that the norm closed inner ideals can be identified by a simple geometric property. A subspace L of a Banach space X is said to have the unique Hahn-Banach extension property if every bounded linear functional on L possesses a unique norm preserving linear extension to X.

Theorem D.28. *Let B be a norm closed subtriple of a JB*-triple J. Then B is an inner ideal in J if and only if B enjoys the unique Hahn-Banach extension property.*

When J is chosen to be a C*-algebra by using the results of [ERü6] the following new piece of information about C*-algebras is obtained.

Theorem D.29. *Let B be a norm closed subtriple of a C*-algebra A. Then B has the unique Hahn-Banach extension property if and only if it is of the form $eA^{**}f \cap A$ where (e,f) is a centrally equivalent pair of open projections in the second dual A^{**} of A.*

Since we are writing this survey under the general aim of putting special emphasis on those topics related to the work of the Spanish school in Jordan structures in Analysis, we become pleasingly obliged to review here the outline of the recent work of J. M. Isidro. To our knowledge he is the only representative of the Spanish school in the holomorphic approach to JB*-triples, and mainly his recent collaborations with W. Kaup [IsKau] and L. L. Stachó ([StIs], [Is3], and [IsSt2]) seem to have provided very interesting advances in this field. Because we are not specialists in holomorphy, we must apologize for the possible lack of enthusiasm in reviewing these advances.

J. M. Isidro has provided some results on the nonholomorphic approach to JB*-triples, for example the determination of all surjective linear isometries of Cartan factors of type C^4 [HeIs]. But, as we have pointed out above, his main contributions are of holomorphic type. Thus recall the fundamental result of Kaup stating that every bounded symmetric domain in a complex Banach space is biholomorphically equivalent to the open unit ball B_J of a JB*-triple J. Also take into account of the relevance (concerning the structure of the Lie group $Aut(B_J)$ of all biholomorphic automorphisms of B_J) of those one-parameter subsemigroups of (biholomorphic) automorphisms of B_J given by $t\in\mathbb{R}\longrightarrow exp(tX)$, where X is a complete holomorphic vector field on B_J of the form $x\longrightarrow a-\{xax\}$ for some a in J, and, for x in B_J, $exp(tX)(x)$ denotes the value at t of the unique solution $y:\mathbb{R}\longrightarrow B_J$ of the initial value problem

$$y'(t)=X(y(t)), \qquad y(0)=x \tag{*}$$

(see [KauUp] and [Ara]). (Perhaps it could be the suitable time to remark that vector fields X of the form indicated above are the only complete holomorphic vector fields on B_J satisfying $X'(0)=0$, and that this fact becomes the nice determination of the triple product of a JB*-triple by the geometry of its unit ball.) It is therefore important, in the treatment of bounded symmetric domains, to integrate the equation (*) as well as to study in depth the subgroups of automorphisms mentioned above. In this direction, Isidro has obtained a formula giving explicitly the solution of (*) in the particular case of J being a J*-algebra [Is1]. But the most original idea arises in his collaborations with Kaup and Stachó which we have already mentioned, when the continuity of vector fields X as above with respect to several topologies is considered and studied. To summarize with some precision the results obtained, consider a JBW*-triple J, let τ be any locally convex topology on J such that $w^*\leq\tau\leq n$ (n denoting the norm topology), denote by $Cont_\tau(B_J)$ the subgroup of $Aut(B_J)$ consisting of those automorphisms that are $\tau\times\tau$ continuous on B_J, and by $cont_\tau(J)$ the set of those elements a in J for which the mapping $x\longrightarrow\{xax\}$ is $\tau\times\tau$ continuous on B_J. Then we have the following theorem.

Theorem D.30. *The set $cont_\tau(J)$ is a norm-closed ideal of J. Moreover, an element a in J lies in $cont_\tau(J)$ if and only if, for every t in $\mathbb{R}$, $exp(tX)$ belongs to $Cont_\tau(B_J)$, where X denotes the vector field on B_J given by $X(x)=a-\{xax\}$. Furthermore, if J is the l_∞-sum of a family $\{J_\lambda\}_{\lambda\in\Lambda}$ of JBW*-triples, then*

$$cont_{w^*}(J)=\overset{l_\infty}{\underset{\lambda\in\Lambda}{\oplus}}cont_{w^*}(J_\lambda) \quad \text{and} \quad cont_w(J)=\overset{c_0}{\underset{\lambda\in\Lambda}{\oplus}}cont_w(J_\lambda)\ .$$

Other interesting results are that $cont_{w^*}(J)=0$ for every *infinite-dimensional Cartan factor J of type* C^4, while, *for Cartan factors J of type* C^1, C^2, *and* C^3 *acting naturally on a complex Hilbert space H, and* τ *equal to either the w*- or the w-topology, the equality* $cont_{\tau}(J)=J\cap\mathcal{K}(H)$ *holds, where* $\mathcal{K}(H)$ *denotes the ideal of compact operators on H.* This characterizes the compactness of an operator *a* in *J* in purely structural terms, i.e., without any reference to the action of the operator *a* on the space *H*. Indeed, this characterizes the property of being compact in terms of the triple product (algebraic structure of *J*) and *w**- or *w*-continuity (topological structure of *J*).

To conclude the review of the recent work of J. M. Isidro, let's refer to his paper [Is4] devoted to the study of those subsets of the unit sphere of a JB*-triple which completely determine the Lie group of biholomorphic automorphisms of the open unit ball.

Before passing to the habitual subsection about problems and future directions, we refer to some minor recent advances on Zettl's ternary C*-rings. These are the structure of compact ternary C*-rings obtained in [FeGar2], and the application of Theorem D.13 made in [CMaMartiRod2] to show that "generalized derivations" of ternary C*-rings are automatically continuous.

Problems and future directions.

In the same way as we have commented in Section B that "all" nongeometric results on associative normed algebras can be suitably extended to the Jordan context, we may say here that "all" results on C*-algebras must have a nonassociative version in some of the four sides of the nonassociative C*-creature (JB-algebras, JB*-algebras, JB*-triples, and nonassociative V-algebras). It seems clear that today, and mainly concerning the Banach-space approach, the most appropriate side is that of JB*-triples, as shown for example by the very partially reviewed paper of Bunce-Chu [BuCh1], where some Banach-space aspects of C*-algebras have been not only "unassociativized" but also improved in their original associative setting. As we have already commented, the Bunce-Chu paper is also interesting from the algebraic point of view, because of the development of the "spectral theory" in JB*-triples there made. That would inspire similar techniques in the general nongeometric theory of Jordan-Banach (*-) triples reviewed in Section C. As an example, let's comment the result in [BuCh1; Theorem 3.4] that *the dual* J^* *of a* JB^**-triple* J *has the Radon-Nikodym property* (a Banach-space property for J) *if and only if the "spectrum" of every element in* J *is at most countable* (an almost algebraic property for J). This result, as well as [BuCh1; Corollary 3.6], should be compared with the content of the Aupetit-Baribeau paper [AuBa] partially reviewed in Theorem B.5. We think such a comparison should inspire a new characterization (involving the socle) of JB*-triples whose duals have the Radon-Nikodym property.

We hope also that the theory of the s^*-topology on JBW*-triples will have a flourishing development in the near future. Let's give here an unorthodox project in this line. Proposition D.17 (that in the case of associative W*-algebras, after Pisier's associative precedent of D.7, can be considered as folklore) suggests the possibility of introducing the s^*-topology of any dual Banach space X with (not necessarily unique, but fixed) predual X_*. This would be the topology on X generated by the family of seminorms $x \longrightarrow \|T(x)\|$, when T runs over all w^*-continuous linear mappings from X into arbitrary Hilbert spaces. It is easy to see that the s^*-topology defined in this way (and denoted, as in the case of JB*-triples, by $s(X,X_*)$) is compatible with the standard duality (X,X_*) (compare with the first assertion in Theorem D.15), a fact that can also be derived more transparently from the folklore (?) characterization of the Mackey topology $m(X,X_*)$ as the topology on X generated by the family of seminorms $x \longrightarrow \|T(x)\|$, when T runs over all w^*-continuous linear mappings from X into arbitrary reflexive Banach spaces. The project of work would consist of a systematic treatment of the s^*-topology of a dual Banach space X, perhaps under some (or all) additional conditions that are known to be true in the case of JBW*-triples, such as the uniqueness of the predual, coincidence of the Mackey and s^*-topologies on bounded subsets, or the fact that X_* is an L-summand of X^* [BarT] (note also that, in the case of X being a JBW*-triple all conditions above are inherited by the bidual of X and also by all even duals). This project involves as a first question the one of the eventual dependence among these purely Banach-space automatic properties of JBW*-triples, but, in any case, we think it will clarify several aspects of the theory of JB*-triples. Thus, for example, *the agreement of Mackey and strong* topologies on bounded subsets of a given dual Banach space* X *is* easily seen to be *equivalent to the following condition:*

-For every w-continuous linear mapping F from X into a reflexive Banach space, there exists a w*-continuous linear mapping G from X into a Hilbert space such that* $\|F(x)\|\leq\|G(x)\|+\|x\|$ *for all x in X.*

But, with a little more of effort, it can be shown that *the above condition is also equivalent to the following one:*

-For every w-continuous linear mapping F from X into a reflexive Banach space, there exist a w*-continuous linear mapping G from X into a Hilbert space and a mapping* $N:(0,\infty)\longrightarrow(0,\infty)$ *such that* $\|F(x)\|\leq N(\varepsilon)\|G(x)\|+\varepsilon\|x\|$ *for all x in X and* ε *in* $(0,\infty)$. Indeed, for each n in $\mathbb{N}$, there are a Hilbert space H_n and a w^*-continuous linear mapping $G_n:X\longrightarrow H_n$ such that $\|n^2F(x)\|\leq\|G_n(x)\|+\|x\|$ for all x in X, and it is enough to consider the Hilbert space $H:=\oplus^{l_2}_{n\in\mathbb{N}}H_n$, the ($w^*$-continuous linear) mapping $G:x\longrightarrow\{\frac{1}{n^2\|G_n\|}G_n(x)\}$ from X into H, and the mapping $N:\varepsilon\longrightarrow\|G_{n(\varepsilon)}\|$ (where $n(\varepsilon)$ denotes the smallest natural number satisfying $n^2\geq\frac{1}{\varepsilon}$), to obtain $\|F(x)\|\leq N(\varepsilon)\|G(x)\|+\varepsilon\|x\|$ for all x in X and ε in $(0,\infty)$.

From Theorem D.21 and the above observations, most of results in [ChI], as well as their associative precedents due to H. Jarchow, follow easily.

Now, let's conclude this section with another project of research that, among other things, shows the advantage of the several sides of the nonassociative C*-creature. Recall that an associative algebra A with an involution * is said to be a Rickart *-algebra if, for each x in A, the right annihilator $ran(x)$ is generated by a projection, i.e., $ran(x)=eA$ for some $e=e^2=e^*$ in A. The reader is referred to Berberian's book [Ber] for general results on Rickart *-algebras. Examples of these algebras are W*-algebras and *-regular algebras with a unit. It is not difficult to prove that, if A is a Rickart *-algebra then the Jordan algebra J of all symmetric elements in A has the following property: for every x in J, $ann_J(x)=U_e(J)$ for some idempotent e in J, where $ann_J(x)$ denotes the Zelmanov annihilator of x (see [Fe4] for general results on annihilators in Jordan algebras). This suggests to say that a Jordan algebra J is a "Rickart Jordan algebra" if the annihilator of every element in J is generated (as an inner ideal) by an idempotent. With this definition, JBW-algebras and strongly regular Jordan algebras with a unit become examples of Rickart Jordan algebras (see [Bat1; Theorem 5.6] and

[FeGarSaSi1; Theorem 6], together with [Fe4; 2.11]).

Now consider some associative precedents that, through the concept just introduced, gives rise to corresponding Jordan questions. D. Handelman [Ha] constructed for a "finite" ($xx^*=1 \Rightarrow x^*x=1$) Rickart C*-algebra A, a *-regular ring R containing A and containing no new projections. Later P. Ara and P. Menal [ArMe] showed that the Handelman's ring R is nothing but the classical ring of quotients of A. Moreover they proved that every *-regular ring is finite. In fact some jordanizations of these results were shown in the paper of P. Jiménez and A. Rodríguez [JiRod], where it was shown that every "finite" (in a peculiar JB-sense) JBW-algebra J is contained in a von Neumann regular Jordan algebra K containing no new idempotents (see also [Ay1]), and that K is the (unique) total ring of quotients of J. But the complete jordanizations of the associative results cited above remain as unanswered questions and we state them with some precision here.

Question D.31. *Is every strongly regular Jordan algebra with a unit finite? (for a suitable notion of finitess to be specified).*

Question D.32. *Has every finite Rickart JB-algebra a classical ring of quotients* (in the sense of Zelmanov [Ze3]) *which is strongly regular?*

E. Results in H^*-theory.

The reasonably well-behaved coexistence of a binary or ternary product and a Hilbert space is the essence of the concept of an H*-thing. Since the pioneering paper of W. Ambrose [Am] considering the associative binary side of this concept, the H*-theory has shown to be the appropriate medium to extend almost verbatim (only minor topological variants should be involved) the Wedderburn's type finite-dimensional structure theorems to the infinite-dimensional case. We shall review here this suggestive field from a general nonassociative perspective, but putting special emphasis in those more familiar nonassociative cases closely related to Jordan structures.

Summary of results until 1988.

Beginning with binary H*-things (more usually known by the name of H*-algebras), let's present here their precise definition. An H*-algebra is a real or complex algebra A, with an algebra involution * (that is linear in the real case, conjugate-linear in the complex one, and in any case is called the H*-algebra involution of A), which is a Hilbert space relative to an inner product $(.|.)$ satisfying

$$(xy|z)=(x|zy^*)=(y|x^*z) \qquad (H^*)$$

for all x, y, z in A. The theory of general nonassociative H*-algebras begins with the papers of J.A. Cuenca and A. Rodríguez [CueRod1] and [CueRod2] (see also [Cue1]). An elemental but important result is the following.

E.1. *The product of every H*-algebra is continuous.*

Hence, by changing the inner product by a suitable positive multiple if necessary, every H*-algebra becomes a (complete) normed algebra in the usual sense that the norm is submultiplicative. An elementary but powerful tool in the structure theory of H*-algebras is that, if B is an ideal of an H*-algebra A, by axiom (H*) its orthogonal complement $B^{\perp}$ is a closed ideal of A. As a consequence, thanks to the orthogonal projection theorem for Hilbert spaces, *every closed ideal of A is a direct summand.* This result applies in particular to obtain the following improved version of "Wedderburn's principal theorem".

E.2. *Every H*-algebra A is the orthogonal sum of two closed ideals*

$$A=B\oplus Ann(A) \ ,$$

where Ann(A) denotes the annihilator of A given by

$$Ann(A):=\{x\in A \ : \ xA=0\}=\{x\in A \ : \ Ax=0\} \ .$$

Clearly *Ann(A)* is *-invariant (so an H*-algebra with zero product), while, although the closed ideal *B* in E.2 need not be *-invariant because of the eventual discontinuity of the H*-algebra involution of *A*, a suitable involution can be defined on *B* with which *B* becomes an H*-algebra with zero annihilator in itself (see [BoDu; Theorem 34.10], where *A* is assumed to be associative and complex, but these assumptions are not used). Moreover, by (H*), *B* is the closure of the linear hull of the set $\{xy \ : \ x,y\in A\}$. These facts reduce the theory of H*-algebras to the consideration of H*-algebras with zero annihilator, giving even the following characterization.

E.3. *An H*-algebra A has zero annihilator if and only if it is the closure of the linear hull of the set* $\{xy \ : \ x,y\in A\}$.

The H-algebra involution of any H*-algebra A with zero annihilator is isometric* [CueRod2; Proposition 2(ix)]. Moreover, *the topology of the Hilbert norm on A is the unique complete normable topology on A making its product continuous* (a consequence of the nonassociative uniqueness-of-norm theorem in [Rod5] quoted in Section B, although a more direct proof for this particular H*-case can be given [Cue1]). As a consequence, *isomorphisms and antiisomorphisms between H*-algebras with zero annihilator are automatically continuous.* Also we enjoy the following well behaviour of closed ideals.

E.4. *Closed ideals of an H*-algebra A with zero annihilator are automatically *-invariant (so new H*-algebras)* [CueRod2; Proposition 2(v)]. *Moreover, for B and C closed ideals of A,* $B\cap C=0$ *if and only if B and C are mutually orthogonal (with respect to the inner product)* [CuR2; Proposition 2(vii)].

Until now, the proofs of all commented results on (nonassociative) H*-algebras are more or less easy adaptations of the original associative arguments. This is not the case for the next theorem, first proved in [CueRod2] by geometric methods, and that today can also be obtained from the nonassociative Wedderburn theorem in [FeRod3] quoted in Section B (together with E.4). We note that, although the original proof of this theorem is given only for the complex case, it works also without problems in the real context.

E.5. *Every H*-algebra with zero annihilator is the closure of the orthogonal sum of its minimal closed ideals, and these are topologically simple H*-algebras.*

Since topologically simple H*-algebras have zero annihilator, the result above reduces the theory of general H*-algebras with zero annihilator to the particular case of topologically simple H*-algebras. This reduction is involved in the proof of most of the results we will review in what follows. Note also that, if A is an H*-algebra with zero annihilator and $\{A_\lambda\}_{\lambda\in\Lambda}$ denotes the family of its minimal closed ideals (regarded as topologically simple H*-algebras), then A can be totally reconstructed from the knowledge of this family in the suggestive way $A = \overset{l_2}{\underset{\lambda\in\Lambda}{\oplus}} A_\lambda$.

The subsequent significant result on H*-algebras is the "essential uniqueness of the H*-structure" on topologically simple H*-algebras, given by the following theorem. It was proved in [CueRod1] for the complex case and later it was extended to the real context in [CMartiRod2].

E.6. *Isomorphic H*-algebras with zero annihilator are in fact *-isomorphic. Moreover isomorphic topologically simple H*-algebras are, up to multiplication of the inner product by a suitable positive number, totally isomorphic.*

As it is usual, by isomorphism between given H*-algebras we mean isomorphism of algebras without any reference to the H*-algebra structure, while by *-isomorphism we mean an isomorphism preserving H*-algebra involutions. A *-isomorphism preserving also the norm (equivalently, the inner product) will be called a total isomorphism. The last result we shall review concerning relatively old results on general nonassociative H*-algebras is the following one (see [CMartiRod2]), relating real with complex H*-algebras in the topologically simple case. Recall that a linear algebra involution τ on a complex algebra A with conjugate-linear involution * is said to be a *-involution if τ commutes with *.

E.7. *Let A be a topologically simple real H*-algebra. Then either A is a topologically simple complex H*-algebra regarded as a real algebra,* *or there exists a topologically simple complex H*-algebra B with *-involution τ such that* $A=\{b\in B : \tau(b)=b^*\}$.

The theory of H*-algebras began with the work of W. Ambrose [Am] on associative complex H*-algebras. Ambrose's paper has inspired the subsequent study of H*-algebras in the most familiar classes of nonassociative algebras and even the treatment of general nonassociative H*-algebras commented above. In fact Ambrose proved the associative complex version of E.5 and determined all topologically simple complex associative H*-algebras. In an equivalent formulation more suitable for our purposes, this determination reads as follows.

E.8. *Up to a positive multiple of the inner product, every topologically simple associative complex H*-algebra is totally isomorphic to the H*-algebra* $\mathcal{HS}(H)$ *of all Hilbert-Schmidt operators on a suitable complex Hilbert space H.*

The corresponding description of topologically simple associative real H*-algebras was obtained by I. Kaplansky [Kap] in analogous terms that of E.8, but involving real, complex, or quaternionic Hilbert spaces. This result has been rediscovered several times (see [BSw], [CMartiRod3], and [CueS]), and in fact it follows from E.7 and E.8, once the *-involutions on $\mathcal{HS}(H)$ (for H a complex Hilbert space) are determined. Because such a determination will be useful for a simplified statement of structure theorems for Jordan and Lie complex H*-algebras, we formulate it here (see [CMartiRod3]).

E.9. *Given a complex Hilbert space H, the *-involutions on the topologically simple associative H*-algebra* $\mathcal{HS}(H)$ *are the mappings of the form* $F\longrightarrow JF^*J^{-1}$ *for a suitable conjugation or anticonjugation J on H.*

We recall that a conjugation (respectively, anticonjugation) on the complex Hilbert space H is a conjugate-linear isometry J from H to H with $J^2=1$ (respectively, $J^2=-1$). Conjugations always exist, while anticonjugations exist if and only if the dimension of H is either an even number or infinite. In any case, when they exist, they are essentially unique [HSto; Lemma 7.5.6].

Topologically simple alternative H*-algebras that are not associative are easily determined using that *the algebra of complex octonions can be regarded* (in an essentially unique way, in view of E.6) *as a complex H*-algebra* (see for example [Per]), together with Slater's theorem on prime alternative algebras [ZhSlShShir; Theorem 9 in p. 194], and the fact that *the centroid of a topologically simple complex* H^**-algebra is* $\mathbb{C}$ [CueRod1] (for the real case use also E.7). As it can be expected, *the topologically simple alternative nonassociative* H^**-algebras are: the algebra of complex octonions, in the complex case; and the same algebra regarded as a real algebra, together with the two real octonions algebras, in the real case.*

Jordan complex H^*-algebras were studied first by C. Viola Devapakkiam and P. S. Rema in [Vio2] and [VioRe]. They proved that *every semisimple finite-dimensional Jordan complex algebra* (hence in particular the simple exceptional Jordan algebra $M_3^8(\mathbb{C})$) *can be structured as an* H^**-algebra.* Moreover they showed that *Jordan complex* H^**-algebras with zero annihilator have dense socle* and gave the first steps for the classification of separable topologically simple Jordan complex H^*-algebras. The results in these papers laid the foundations for the definitive classification of topologically simple Jordan complex H^*-algebras later obtained by J. A. Cuenca and A. Rodríguez in [CueRod2].

In order to formulate this classification theorem in a nice way, let's introduce some natural concepts and facts that will be also useful in other contexts. Given an algebra A, we will denote by A^+, A^-, and A^o the algebras with the same vector space that of A, and products given respectively by

$$(x,y)\longrightarrow x.y:=\frac{1}{2}(xy+yx)$$
$$(x,y)\longrightarrow [x,y]:=xy-yx$$
$$(x,y)\longrightarrow yx\ .$$

If A is an H^*-algebra, A^+, A^-, and A^o can and will be seen also as H^*-algebras under the same inner product and H^*-algebra involution that of A. If τ is a linear (algebra) involution on an algebra A, we will denote by $H(A,\tau)$ and $S(A,\tau)$ the sets of all τ-hermitian and τ-skew elements of A, respectively. That is:

$$H(A,\tau)=\{a\in A\ :\ \tau(a)=a\}\ ,\ S(A,\tau)=\{s\in A\ :\ \tau(s)=-s\}\ .$$

$H(A,\tau)$ and $S(A,\tau)$ are subalgebras of A^+ and A^-, respectively. In case A is an H^*-algebra, it is clear that, for a continuous $*$-involution on the H^*-algebra A, $H(A,\tau)$ and $S(A,\tau)$ are $*$-invariant closed subalgebras of the H^*-algebras A^+ and A^-, respectively, and in this way they will be considered as new H^*-algebras. Note that, if A has zero annihilator, then the assumption of continuity of τ is superfluous. *Every isometric linear algebra involution on an* H^**-algebra with zero annihilator is a*

*-*involution* (see for example [CMartiRod4; Proposition 1.7]), and the converse is true if the H*-algebra is topologically simple (a consequence of E.6). By a topologically τ-simple H*-algebra we mean an H*-algebra with linear algebra involution τ, nonzero product, and no nonzero proper τ-invariant closed ideals. Of course, topologically τ-simple H*-algebras have zero annihilator, and every topologically simple H*-algebra is topologically τ-simple for every linear involution τ. Moreover, we have the following easy result (see for example [CMartiRod4;Theorem 1.5]).

E.10. *Let (A,τ) be a topologically τ-simple H^*-algebra with isometric linear involution, and assume that A is not topologically simple. Then*

$$A = B \overset{l}{\oplus}{}^2 B^0 ,$$

for a suitable topologically simple H^*-algebra B, and τ is the exchange involution.

In this case we have natural isomorphisms

$$H(A,\tau)\cong B^+ \quad \text{and} \quad S(A,\tau)\cong B^- .$$

Given a complex Hilbert space H with a conjugation $\square$, the Hilbert space $\mathbb{C} \overset{l}{\oplus}{}^2 H$, with Jordan product

$$(\lambda+\xi).(\mu+\eta):=(\lambda\mu+(\xi|\eta^\square)) + (\lambda\eta+\mu\xi)$$

and H*-algebra involution $(\lambda+\xi)^*:=\bar{\lambda}+\xi^\square$, is a Jordan H*-algebra called the Jordan H*-algebra of the involutive Hilbert space $(H,\square)$, and denoted by $J(H,\square)$.

Now we can state the classification theorem for topologically simple Jordan complex H*-algebras [CueRod2] as follows.

E.11. *Up to a positive multiple of the inner product, the topologically simple Jordan complex H^*-algebras are $M_3^8(\mathbb{C})$, the Jordan H^*-algebras $J(H,\square)$ of an involutive Hilbert space $(H,\square)$ with $dim(H)\geq 2$, and the Jordan H^*-algebras $H(A,\tau)$ of all τ-hermitian elements in a topologically τ-simple associative H^*-algebra A with isometric linear involution τ.*

The paper [CueRod2] contains also a classification theorem for topologically simple noncommutative Jordan complex H*-algebras asserting that these are either anticommutative, commutative (cf. E.11), simple quadratic, or real mutations of topologically simple associative complex H*-algebras (cf. E.8). The (nonassociative) quadratic complex H*-algebras are relatively well-described (see [Cue2] and [CMartiRod1]), they are automatically noncommutative Jordan algebras, and, except for the nonsimple two-dimensional case, they are very similar to simple quadratic noncommutative JB*-algebras constructed in D.3 (it is enough to relax there the assumption $\|x\wedge y\|\leq\|x\|\|y\|$ to the simple continuity of $\wedge$, and take as concluding norm in the construction the natural Hilbert norm on a complexification of a real Hilbert space).

The papers of J. R. Schue [Sch1] and [Sch2] on Lie H*-algebras became the first incursions in the study of H*-algebras from a nonassociative point of view. It was proved in [Sch1] the particular Lie version of E.5, and it was also shown there the following result.

E.12. *Every finite-dimensional Lie complex algebra can be structured as an H*-algebra.*

Schue's papers contain also the classification of separable topologically simple Lie complex H*-algebras, as well as a fine Cartan decomposition for an arbitrary Lie complex H*-algebra with zero annihilator. Schue's work has become the key tool for the recent classification theorem of topologically simple Lie H*-algebras, obtained in [CueGaMar2] (see also [N2]), and that will be reviewed in the suitable place. Since Schue's paper [Sch1] until 1972, there was a great activity in the field of Lie H*-algebras (see [Har2] and references therein). We only cite here the almost simultaneous papers of V. K. Balachandran [B2], P. de la Harpe [Har1], and I. Unsain [U] (where, in the spirit of E.7, the classification of real topologically simple Lie H*-algebras is reduced to that of complex ones), and the following reformulated version of results in Sections 3 and 4 of Balachandran's paper [B1].

E.13. *If (A,τ) is an infinite-dimensional topologically τ-simple associative complex H*-algebra with isometric linear algebra involution, then the Lie complex H*-algebra $S(A,\tau)$ is topologically simple.*

Schue's Cartan decomposition for Lie H*-algebras has become also useful in the treatment of Malcev H*-algebras (see [CMartiRod1]), allowing to show that *every topologically simple Malcev complex H*-algebra is either a Lie algebra or the Malcev H*-algebra of all trace-zero elements in the alternative H*-algebra of complex octonions.* Topologically simple Malcev non-Lie real H*-algebras are then easily obtained by applying E.7.

There are several options for the treatment of nonassociative ternary H*-things. Because the first ternary H*-things arising in the literature are particular Hilbert versions of (complex) Jordan-Banach *-triples already considered in this survey since Section C, we begin by considering their natural nonassociative counterparts and by introducing the appropriate terminology. A (nonassociative complex) "*-triple" will be a complex vector space A together with a triple product $\{...\}:A\times A\times A\longrightarrow A$ that is linear in the outer variables and conjugate linear in the middle variable, while an "H*-triple" will be a *-triple A endowed with a complete inner product $(.|.)$ satisfying

$$(\{xyz\}|w)=(x|\{wzy\})=(z|\{yxw\})$$

for all x,y,z,w in A. With more or less effort, the adequate variants for H*-triples of E.1, E.2, E.3, and even of the crucial result E.5, can be shown, thus centering the interest in the topologically simple case.

As we have pointed out before, the only ternary H*-things arisen in the literature until 1988 are the particular Jordan cases of the H*-triples defined above. These Jordan H*-triples were introduced (under the name of Hilbert triple systems) and studied in detail by W. Kaup in his already quoted work [Kau2] on the classification of bounded symmetric domains in complex Hilbert spaces. He proved the peculiar variants of E.2 and E.5 in his context, and gave the following description of topologically simple Jordan H*-triples (see also [N1], where concepts and results are also suitably extended to the case of real Jordan triples).

E.14. *Up to a positive multiple of the inner product and up to a multiplication of the triple product by ±1, topologically simple Jordan H*-triples are the "Hilbert variants" of Cartan factors, namely, the (JBW*-) Cartan factors of type* C^4, C^5, *and* C^6 *(with their equivalent natural Hilbert norms), and the Jordan H*-triples of the form* $\mathcal{HS}(H,K)$, $\{x\in\mathcal{HS}(H) : Jx^*J=-x\}$, *and* $\{x\in\mathcal{HS}(H) : Jx^*J=x\}$, *where H and K are complex Hilbert spaces,* $\mathcal{HS}$ *denotes Hilbert-Schmidt operators, and J is a conjugation on H.*

We note that Jordan complex H*-algebras become Jordan H*-triples under the triple product $\{xyz\}:=x.(y^*.z)-y^*.(z.x)+z.(x.y^*)$, and that the above Kaup's theorem shows ostensibly that *every topologically simple "positive" Jordan H*-triple can be seen as a closed *-subtriple of a suitable Jordan complex H*-algebra.* With a little care in using Kaup's theorem and the ternary versions of E.2 and E.5, it can be shown that actually *every positive Jordan H*-triple can be seen as a closed *-subtriple of a Jordan complex H*-algebra.*

Perhaps we cannot hide our predilection for those aspects of the matter under review that do not involve any assumption on identities (to which we are referring as "general nonassociative" aspects). Thus we begin our survey on recent results on H*-theory by reviewing the few but important advances in these line concerning H*-algebras. The first one has been provided by M. Cabrera and A. Rodríguez in the already quoted paper [CRod1] on the extended centroid of normed algebras. It is a consequence of E.5 and (the complete version of) Theorem B.15.

Theorem E.15. *The extended centroid of a complex H*-algebra A with zero annihilator is* $\mathbb{C}^I$, *where I denotes the cardinal number of the family of all minimal closed ideals of A. Hence topologically simple complex H*-algebras are centrally closed.*

Much more recently, the fact that topologically simple complex H*-algebras are centrally closed (that has been reproved in a more transparent way in [CRod2]) has unexpectedly become a fundamental tool in automatic continuity theory for H*-algebras. This has happened by the hand of A. R. Villena [Vi2], who, looking at a Jacobson density type theorem for centrally closed prime algebras in [ErMart0], has begun by proving the following purely algebraic lemma.

Lemma E.16. *Let A be a centrally closed prime algebra such that* $\dim(T(A))>1$ *for all nonzero T in the unital multiplication algebra M(A) of A. Then there exist sequences* $\{a_n\}$ *in A and* $\{T_n\}$ *in M(A) such that* $T_n\dots T_1 a_n \neq 0$ *and* $T_{n+1}T_n\dots T_1 a_n = 0$ *for all n in* $\mathbb{N}$.

With this lemma, E.5, Theorem E.15, and standard arguments in automatic continuity, Villena has built a clever proof of the following theorem (compare with problem B.22).

Theorem E.17. *Derivations on (real or complex) H*-algebras with zero annihilator are continuous.*

Later Villena's techniques have been adapted in [Rod14] to obtain Theorem B.21, from which the following result follows easily.

Theorem E.18. *Let A be a complete normed algebra, B an H^*-algebra with zero annihilator, and Φ be a homomorphism from A into B with dense range. Then Φ is continuous.*

With an earlier result in [CueRod1], Theorem E.18 implies easily the next corollary, asserting as a consequence that *no dense subalgebra of a topologically simple H^*-algebra, other than the whole algebra, can be structured as an H^*-algebra.*

Corollary E.19. *Let A and B be real or complex H^*-algebras, and assume that B is topologically simple. Then every dense range homomorphism from A into B is surjective.*

Particular versions of Theorem E.17 for alternative, Jordan, and Malcev algebras have been shown almost at the same time by B. Zalar ([Z3] and [Z1]), while the associative version of Theorem E.18 can be considered as folklore (see for example [AlbD; Theorem 3.2]). It must be remarked at this respect that the problem of the automatic continuity of dense range homomorphisms from (associative) Banach algebras into semiprimitive Banach algebras seems to be still open, even in the particular case of C*-algebras.

Now, passing to review recent structural results on H*-algebras, in our opinion the most important one is without doubt the following definitive classification of topologically simple Lie complex H*-algebras (take into account of E.12 and E.13), obtained by J. A. Cuenca, A. García, and C. Martin [CueGaMar2], and reproved later by E. Neher [N2] using different methods.

Theorem E.20. *Every infinite-dimensional topologically simple Lie complex H^*-algebra is of the form $S(A,\tau)$ for some topologically τ-simple associative complex H^*-algebra A with isometric linear algebra involution* τ (cf. E.8, E.9, and E.10).

The Cuenca-García-Martín proof of this theorem has introduced a new interesting tool in the field of H*-structures, namely the algebraic ultrafilter method, that has shown its usefulness in dealing with the problem of "approximating" an infinite-dimensional H*-algebra by a "growing" net of separable H*-subalgebras whose structure was previously known. The ultrafilter methods can be applied in many different context within H*-structures, because the sort of defining identities (Jordan, Lie, etc.) or the aridity of the structure (binary or ternary) is irrelevant to the ultrafilter technic.

Another interesting work on H*-algebras satisfying more or less familiar identities is that of M. Cabrera, J. Martínez, and A. Rodríguez [CMartiRod4] on "structurable" complex H*-algebras, that was already announced by Rodríguez in the 1988 Oberwolfach meeting on Jordan algebras. A structurable algebra is an algebra A with a linear algebra involution τ satisfying

i) $(s,x,y)=-(x,s,y)=(x,y,s)$

ii) $(a,b,c)-(c,a,b)=(b,a,c)-(c,b,a)$

iii) $\frac{2}{3}[[a^2,a],b]=(b,a^2,a)-(b,a,a^2)$

for all x,y in A, s in $S(A,\tau)$, and a,b,c in $H(A,\tau)$, where $(.,.,.)$ denotes the associator on A. Structurable algebras were introduced by B. N. Allison in [All1], who provided a fine classification theorem for finite-dimensional structurable algebras that are simple with respect to the linear algebra involution (see also [Sm]). Examples of structurable algebras are alternative algebras with any linear algebra involution and Jordan algebras with the identity operator as involution. The main interest of structurable algebras relies on the fact that, in the finite-dimensional case, they give, by means of an extended Kantor-Koecher-Tits construction, all isotropic simple Lie algebras [All2]. The Cabrera-Martínez-Rodríguez paper begins with a systematic study of nonassociative complex H*-algebras with a linear (algebra) involution τ. After showing the appropriate version of E.5 in their context (thus reducing the general case of algebras with zero annihilator to the particular topologically τ-simple one), they prove that *every complex H*-algebra with zero annihilator and linear involution is (bicontinuously) isomorphic, as algebra with linear involution, to an H*-algebra with (zero annihilator and) isometric (hence *-) linear involution.* Moreover the essential uniqueness of the H*-structure on topologically τ-simple complex H*-algebras with isometric linear involution τ is obtained. As a consequence, the theory of structurable complex H*-algebras reduces to that of topologically τ-simple ones with τ an isometric linear involution. Then they prove the following theorem (see also [AllF], where the part of the theorem concerning the so called "Smirnov's structurable algebra" is reproved in a simpler way).

Theorem E.21. *Every finite-dimensional τ-simple structurable complex algebra (A,τ) can be seen as an H^*-algebra in such a way that τ becomes isometric.*

Now, with the Allison-Smirnov classification of finite-dimensional τ-simple structurable algebras, the interest must be centered in the infinite-dimensional case. Certainly we already know the examples provided by the cases (A,τ), where either A is a topologically τ-simple associative complex H*-algebra with isometric linear involution τ (cf. again E.8, E.9, and E.10), or A is a topologically simple Jordan complex H*-algebra (cf. E.11) and τ is the identity operator on A. In order to build further infinite-dimensional examples, the relatively forgotten Saworotnow's theory of Hilbert modules over associative complex H*-algebras with zero annihilator (see [Saw1], [Gi], and [Smi]) is recaptured in the paper we are reviewing. If we wanted to state with some precision the construction made in [CMartiRod4] of certain structurable H*-algebras from some Hilbert modules, then we would give a quite long list of concepts and result, a fact that seems unsuitable for the philosophy of this survey. It is therefore better to appeal to the imagination and, consequently, to think that:

-Hilbert modules are the natural generalizations of complex Hilbert spaces when the base field $\mathbb{C}$ is replaced by an associative complex H*-algebra with zero annihilator,

-"Involutive Hilbert modules" over associative complex H*-algebras with zero annihilator and an isometric linear involution, introduced in [CMartiRod4], are then reasonable noncommutative variants of involutive complex Hilbert spaces (note that the only linear involution on $\mathbb{C}$ is the identity), and

-Structurable complex H*-algebras constructed in [CMartiRod4] from involutive Hilbert modules correspond in this setting to Jordan complex

H*-algebras of an involutive complex Hilbert space (see definition before E.11).

Now, with the lack of precision accepted above, and with minor improvements provided in [CMaMartiRod1], the main result in [CMartiRod4] can be formulated as follows.

Theorem E.22. *If (A,τ) is a topologically τ-simple structurable complex H*-algebra, and if τ is isometric on A, then one of the following assertions holds:*

i) A is finite dimensional (cf. Theorem E.21).

ii) A is associative (cf. E.8, E.9, and E.10).

iii) A is a Jordan algebra and τ is the identity on A (cf. E.11).

iv) (A,τ) is a structurable H-algebra constructed from an involutive Hilbert module over a topologically τ-simple associative complex H*-algebra $\mathcal{E}$ with isometric linear involution τ.*

Involutive Hilbert modules arising in case (iv) of the theorem are also precisely described in [CMartiRod4].

Theorem E.22, together with nice ideas in Schafer's paper [Sc2], has become the key tool for the proof in [CMaMartiRod1] of the following infinite-dimensional version of the Allison-Kantor-Koecher-Tits construction.

Theorem E.23. *There is a "natural" correspondence $(A,\tau)\longrightarrow\mathcal{K}(A,\tau)$ from the category $\mathcal{S}$ of structurable complex H*-algebras with zero annihilator and isometric linear involution onto the category $\mathcal{L}$ of Lie complex H*-algebras with zero annihilator. Moreover, for (A,τ) in $\mathcal{S}$, this correspondence induces an order-isomorphism from the complete lattice of all τ-invariant closed ideals of A onto that of closed ideals of $\mathcal{K}(A,\tau)$. As a consequence, for (A,τ) in $\mathcal{S}$, A is topologically τ-simple if and only if $\mathcal{K}(A,\tau)$ is topologically simple.*

The proof of the above theorem has needed (and hence encouraged) further developments of the theory of Hilbert modules (over associative complex H*-algebras with zero annihilator). These have been provided in [CMartiRod6], where the notion of an orthonormal basis for a Hilbert module has been introduced, orthonormal bases have been characterized among "orthonormal systems" by means of "Parseval's equality" or "Fourier's expansion", and the existence of orthonormal bases, as well as the coincidence of the cardinal numbers of all these bases, has been proved. Then, as needed for the proof of Theorem E.23, a satisfactory theory of operators of Hilbert-Schmidt type on Hilbert modules has been developed.

In the same line of developing further the theory of Hilbert modules, the paper [CMartiRod5] provides a non-structural proof of the existence of a natural categorical one-to-one correspondence between Hilbert modules and associative H*-triples (of the second kind) with zero annihilator, obtaining as a consequence the structure of topologically simple associative H*-triples, becoming these in essence of the form $\mathcal{HS}(H,K)$, for H and K complex Hilbert spaces (see also [Z2] for a more direct proof of this last result).

By Theorem E.22, every nontrivial problem in structurable complex H*-algebras reduces in essence to the consideration of the problem in structurable H*-algebras constructed from involutive Hilbert modules. In this line, derivations of such structurable H*-algebras commuting with the linear involution have been studied in [CMaMartiRod2], showing the existence of a natural bicontinuous Lie-algebra isomorphism from the Banach-Lie algebra of these derivations onto some (perfectly determined) Banach-Lie algebra of bounded "differential operators" on the parameterizing Hilbert module. The notion of a differential operator arising in this result is the natural version for modules of that appearing for vector spaces over division algebras in the study of derivations on prime associative algebras with nonzero socle (see [Ja1]). The proof of the result above turns again over the correspondence between Hilbert-modules and associative H*-triples with zero annihilator, through which bounded differential operators turn in (automatically continuous) "generalized derivations", that are fully described. We remark that, thanks to [CMaMartiRod2; Proposition 2.2], generalized derivations of associative H*-triples with zero annihilator can be seen as "derivation pairs" in the sense of [Z4], and then we refer to this last paper for further information about the automatic continuity of derivation pairs on alternative and Jordan H*-triples.

Talking about H*-triples, we encourage the reader to look at the paper of B. Zalar [Z2], where, thorough many nice concepts, and providing elegant complete proofs, the most part of the theory of general nonassociative H*-algebras is translated to the setting of nonassociative H*-triples. Let's also cite the characterization up to equivalent renorming of Jordan H*-triples having finite capacity (or "finite rank", in a more familiar terminology in our context) in the sense of [Lo3]. This characterization has been provided recently in [FeGarSaSi2], and contains an earlier result in [DiT2] extending in its turn one in [Harr1; Theorem 6.2].

Theorem E.24. *A Jordan-Banach *-triple is bicontinuously isomorphic to a Jordan H*-triple of finite rank if and only if it is strongly regular and the set of its division generalized tripotents is bounded.*

In reviewing recent results on binary H*-structures, we have arrived without premeditation to H*-triples (in the sense of the definition given before E.14) by the hand of Hilbert modules needed for the classification of structurable H*-algebras. Even we have reviewed all the few recent more or less relevant advances on H*-triples we know at this time. Before passing to the consideration of other ternary H*-structures introduced and feverishly studied by J. A. Cuenca and his coauthors in the last years, let's return to H*-algebras to complement their review and, in its turn, to motivate such new ternary H*-structures.

J. A. Cuenca and A. Sánchez [CueS] have recently classified topologically simple Jordan real H*-algebras. Although this classification can be reduced, via E.7, to the determination of *-involutions on topologically simple Jordan complex H*-algebras described in E.11, such a determination is very hard, and they have opted for a more direct argument leading to a very precise complete list of possible cases, most of which are described in terms of l_2-summable infinite matrices in agreement with the original statements of Ambrose and Kaplansky classification theorems for complex and real (respectively) topologically simple associative H*-algebras. In a less precise form, the Cuenca-Sánchez theorem can be stated as follows.

Theorem E.25. *Topologically simple Jordan real H*-algebras fall into one of the following cases:*

i) The Jordan H-algebra obtained by symmetrization of the product* *of a topologically simple associative real H*-algebra.*

ii) The Jordan H-algebra $H(A,\tau)$ of hermitian elements of a topologically simple associative real H*-algebra A with respect to a *-involution τ on A.*

iii) A simple Jordan H-algebra of quadratic type over $\mathbb{R}$ or $\mathbb{C}$.*

iv) A (finite-dimensional) simple exceptional Jordan real H-algebra.*

*Moreover, the *-involutions τ in (ii) are determined.*

The Cuenca-Sánchez paper contains also a classification of topologically simple noncommutative Jordan real H*-algebras, whose statement is an almost verbatim translation to the real case of that commented for the complex case after E.11 (only some care should be taken in accepting in the real case quadratic algebras over $\mathbb{R}$ or $\mathbb{C}$). The transition from H*-algebras to certain ternary H*-structures, different from the H*-triples considered above, can be made through the study of two-graded H*-algebras provided by J. A. Cuenca and C. Martín in [CueMar]. Two graded H*-algebras are H*-algebras A which splits into an orthogonal direct sum $A=A_0\oplus A_1$ of self-adjoint (closed) subspaces A_i $(i=0,1)$ such that $A_iA_j\subseteq A_{i+j}$ (sum in module two). The study of two-graded H*-algebras may be reduced to the study of the topologically simple ones (in a graded sense) in a similar way as it is done for ungraded H*-algebras. It is easy to prove that *the only topologically simple two graded associative H*-algebras over $\mathbb{K}$ (=$\mathbb{R}$ or $\mathbb{C}$) are of one (and only one) of the following two types:*

i) $B\overset{1}{\oplus}{}^{2}B$, with B a topologically simple associative H-algebra over $\mathbb{K}$, the product given by $(x,y)(u,v):=(xu+yv,xv+yu)$, H*-algebra involution $(x,y)^*:=(x^*,y^*)$, even part $B\oplus\{0\}$, and odd part $\{0\}\oplus B$.*

ii) An (a priori ungraded) topologically simple associative H-algebra A over $\mathbb{K}$, with even part the hermitian elements relative to an involutive *-automorphism σ of A, and odd part the skew-hermitian elements.*

Furthermore the grading involutive *-automorphisms σ can be precisely determined, arising a number of nonisomorphic H*-algebras which are not worth to describe here. The interesting matter here is that *the odd part of any two-graded topologically simple associative H*-algebra is* *a topologically simple associative "ternary H*-algebra" (of the first kind), and moreover any topologically simple associative "ternary H*-algebra" arises in this way (up to the sign of the "ternary H*-algebra involution")* (see [CaCusMar] for the precise classification of topologically simple associative "ternary H*-algebras"). In this way binary methods can be used to deal with ternary structures in an associative context, and this idea can be also exploited in a Jordan setting. In a purely algebraic context, the same link appears between prime associative two-graded algebras with nonzero socle and prime associative ternary algebras with nonzero socle (see [CueGaMar1] and [CueGaMar3]). Now let's formulate in a condensed form the classification of topologically simple two-graded Jordan H*-algebras obtained in [CueMar].

Theorem E.26. *Topologically simple two-graded Jordan H*-algebras over $\mathbb{K}$ (=$\mathbb{R}$ or $\mathbb{C}$) are of one of the following types:*

i) $J \overset{1}{\oplus}{}^{2} J$, *with* J *a topologically simple H*-algebra over* $\mathbb{K}$, *the product given by* $(x,y).(u,v):=(x.u+y.v,x.v+y.u)$, *H*-algebra involution* $(x,y)^*:=(x^*,y^*)$, *even part* $J\oplus\{0\}$, *and odd part* $\{0\}\oplus J$.

ii) The two-graded Jordan H-algebra obtained by symmetrization of the product of a two-graded associative H*-algebra A over* $\mathbb{K}$ *that is topologically simple without reference to the grading of A.*

iii) The Jordan H-algebra obtained by symmetrization of the product of a topologically simple associative H*-algebra A over* $\mathbb{K}$ *with a *-involution* τ, *with even part* $H(A,\tau)$, *and odd part* $S(A,\tau)$.

iv) The Jordan two-graded H-algebra* $H(A,\tau)$, *where A is as in (ii), and* τ *is a *-involution on A preserving the grading of A.*

v) A two-graded Jordan H-algebra of quadratic type.*

vi) A two-graded Jordan H-algebra of exceptional type.*

Since associative "ternary H*-algebras" (of the first kind) have already arisen in our recent comments, and even we have outlined their structure theory, it seems suitable to formulate the concept of a ternary H*-algebra with some precision, allowing it to work also outside the associative environment. Let's therefore say that a (nonassociative) ternary H*-algebra is a real or complex Hilbert space *A* together with a trilinear triple product $\{...\}:A\times A\times A\longrightarrow A$, and an involutive mapping $A\overset{*}{\longrightarrow}A$, which is linear in the real case, conjugate-linear in the complex case, and satisfies $\{xyz\}^*=\{y^*z^*x^*\}$ and

$$(\{xyz\}|w)=(x|\{wz^*y^*\})=(z|\{y^*x^*w\})=(y|\{x^*wz^*\})$$

for all x,y,z,w in A. Jordan ternary H*-algebras (i.e., Jordan triples that are ternary H*-algebras) have been first considered and studied by A. Castellón and J. A. Cuenca in [CaCue1]. Later, a complete structure theory of Jordan ternary H*-algebras has been achieved by A. Castellón, J. A. Cuenca and C. Martín, and it will appear in a forthcoming joint paper. We can announce here this theory in a very condensed form, leaving to the imagination of the reader the technical notion, involved in the statement, of a "polarized" topologically (*-) simple Jordan ternary H*-algebra.

Theorem E.27. *Every real unpolarized topologically simple Jordan ternary H*-algebra, that is neither finite-dimensional nor of quadratic type, is, up to a positive factor of the inner product, and up the sign of the involution and the triple product, isometrically *-isomorphic to the odd part of a real topologically simple two-graded Jordan H*-algebra* (cf. Theorem E.26). *Moreover, the simple finite-dimensional Jordan ternary H*-algebras, as well as those of quadratic type, are fully described.*

J. A. Cuenca and his coauthors have considered another type of a ternary H*-structure, by imposing on a Hilbert space A with a trilinear triple product $\{...\}$ and an involutive mapping * (linear in the real case, and conjugate-linear in the complex one), the axioms $\{xyz\}^*=\{x^*y^*z^*\}$ and

$$(\{xyz\}|w)=(x|\{wz^*y^*\})=(z|\{y^*x^*w\})=(y|\{x^*wx^*\})$$

for every x,y,z in A (see also [Z5], and note that, if the triple product is symmetric in the outer variables -as it fortunately happens in the Jordan case-, then nothing but particular ternary H*-algebras arise). In this setting they have proved the essential uniqueness (up to the sign of the involution) of the "H*-structure" in the topologically simple case [CaCue4], and have shown fine classification theorems in the associative (of the second kind) and alternative context. The interested reader is referred to the Castellón-Cuenca papers [CaCue2] and [CaCue3] for the precise versions of these classification theorems, as well as for further information about these topics.

Concerning results, let's conclude this section with a new one on general nonassociative H*-algebras. The proof consists of an adaptation of the arguments in the proof of Theorem 1.3 in the already commented paper [CMartiRod4].

Theorem E.28. *Let A be a real algebra with zero annihilator and assume that its complexification $A_{\mathbb{C}}$ can be structured as an H*-algebra. Then A can also be structured as an H*-algebra.*

Proof. Consider the mapping $\tau: x+iy \longrightarrow x-iy$ which is a conjugate-linear automorphism of $A_{\mathbb{C}}=A\oplus iA$. Denoting by $A_{\mathbb{C}}^{\square}$ the H*-algebra having the same structure of $A_{\mathbb{C}}$, except for the product by complex numbers (defined by $\lambda\square x:=\bar{\lambda}x$) and the inner product (defined by $(x|y)_{\square}:=(y|x)$), and regarding τ as a (linear) isomorphism from $A_{\mathbb{C}}$ onto $A_{\mathbb{C}}^{\square}$, we can apply [CueRod1; Theorem 3.3] to obtain the existence of a unique conjugate-linear *-automorphism T of $A_{\mathbb{C}}$ and a unique (linear) automorphism ψ of $A_{\mathbb{C}}$ such that $\tau=T\psi$, $(\psi^{-1})^*=\psi$, and $sp(\psi)\subset\mathbb{R}^+$. Since $\tau=\tau^{-1}$, we have $\tau=\psi^{-1}T^{-1}=T^{-1}(T\psi^{-1}T^{-1})$, and clearly T^{-1} is a conjugate-linear *-automorphism of $A_{\mathbb{C}}$ and $T\psi^{-1}T^{-1}$ is an automorphism of $A_{\mathbb{C}}$ with the same properties of ψ (note that $\mathbb{R}^+$ is invariant under the mappings $z\longrightarrow z^{-1}$ and $z\longrightarrow\bar{z}$). From the uniqueness of the decomposition for τ we obtain $T=T^{-1}$ (so T is actually an involutive conjugate-linear *-automorphism of $A_{\mathbb{C}}$) and $\psi^{-1}=T\psi T^{-1}$. Now the mapping $\Phi: F\longrightarrow TFT^{-1}$ is a conjugate-linear automorphism of the complex algebra $BL(A_{\mathbb{C}})$, hence, for any F in $BL(A_{\mathbb{C}})$, the equality $sp(\Phi(F))=\overline{sp(F)}$ holds, and, for any complex valued rational function with poles outside $sp(F)$, we have $\Phi(R(F))=R^{\#}(\Phi(F))$, where, for a complex valued holomorphic mapping f on an open subset Ω of $\mathbb{C}$, $f^{\#}$ denotes the (holomorphic) function on $\bar{\Omega}$ given by $z\longrightarrow\overline{f(\bar{z})}$. Since $\psi^{-1}=\Phi(\psi)$, it follows that $R^{\#}(\psi^{-1})T=TR(\psi)$ for every rational function R with poles outside $sp(\psi)$. Therefore, by Runge's theorem and [BoDu; Lemma 7.2 and Theorem 7.4(iii)], for every holomorphic function f on an open subset Ω of $\mathbb{C}$ containing $sp(\psi)$ we have $f^{\#}(\psi^{-1})T=Tf(\psi)$. Taking in particular $\Omega=\mathbb{C}\backslash\mathbb{R}_0^-$ and f the principal determination of the square root on Ω, and writing

$\vartheta := f(\psi)$, since $f^{\#}=f$ we obtain $\vartheta^{-1}T=T\vartheta$. Therefore $\tau=T\psi=T\vartheta^2=(T\vartheta)\vartheta=\vartheta^{-1}T\vartheta$. Since ϑ is an automorphism of $A_{\mathbb{C}}$ (see [CueRod1; Corollary 2.4]), from the equality $\tau=\vartheta^{-1}T\vartheta$ we deduce that the real algebras $\{\alpha\in A_{\mathbb{C}} : \tau(\alpha)=\alpha\}$ and $\{\alpha\in A_{\mathbb{C}} : T(\alpha)=\alpha\}$ are isomorphic. The proof is concluded by observing that $A=\{\alpha\in A_{\mathbb{C}} : \tau(\alpha)=\alpha\}$, while $\{\alpha\in A_{\mathbb{C}} : T(\alpha)=\alpha\}$ is a closed *-invariant real subalgebra of the complex H*-algebra $A_{\mathbb{C}}$, hence a real H*-algebra. ■

Problems and future directions.

The powerful classical methods of Hilbert spaces, as well as that of operators in these spaces, have allowed to prove very relevant results on general nonassociative H*-algebras (see E.5, E.6, E.15, E.17, E.18, and E.19), and, unless the appearance of a new revolutionary idea like Villena's one in their proof of Theorem E.17, not much more can be expected concerning the development of this general nonassociative point of view. Even the extensions made until now of most of these result to general nonassociative ternary H*-structures, although sometimes nice, have provided no essentially new techniques in this field. On the other hand, the classification of topologically simple binary or ternary H*-structures has been achieved in all familiar classes of binary or ternary algebras defined by identities (see E.8, E.11, E.14, E.20, E.22, E.25, E.26, and E.27). Therefore, in our opinion, after the advances made in the last years, the H*-theory can be considered finished and, consequently, we do not enthusiastically encourage any people to work in it.

In spite of our adventurous opinion, it remains to provide a reasonably available unified codification of results in H*-theory. While general nonassociative aspects have already arisen relatively enough well-ordered in the original papers, this is not the case for the particular aspects of familiar binary or ternary structures defined by identities, each one requiring its peculiar algebraic and analytic treatment. As an example, it would be interesting to obtain a proof of the Allison-Kantor-Koecher-Tits construction for Lie H*-algebras (Theorem E.23) not involving the structure of such algebras (Theorem E.20). If such a proof was provided, then, since Theorem E.20 can be obtained with minor difficulties from Theorems E.22 and E.23, the theory of Lie H*-algebras would be liberated from its very peculiar isolation.

In any case, if some people insist in working in H*-theory, we recommend mainly its general nonassociative aspect. Questions as, if every topologically simple H*-algebra has a minimal ideal, or, more ambiguously, if a "nonassociative socle" can be defined in such a way that H*-algebras with zero annihilator have dense "socle", would be points of departure for future interesting developments. Another almost unexplored side of nonassociative H*-algebras is the suitable nonassociative translation of the Saworotnow-Friedell theory of trace-class in associative H*-algebras ([SawFr] and [Saw2]). In this direction, the paper [Marti4] seems to be promising.

F. Looking for normed versions of Zel'manov prime theorem.

Zel'manov prime theorem for Jordan algebras, though received enthusiastically by algebraists since its appearance, has taken a relatively long time to be assimilated by analysts in order to obtain new structure theorems for normed prime nondegenerate Jordan algebras, results that cannot be attacked by the familiar technique of the existence of a nonzero socle or by duality methods in JBW- and JBW*-theory. The reason could be (as we have commented earlier in other places) that the formulation of Zel'manov prime theorem, in order to attains a nice simple form, perhaps conceals some crucial information that is needed in the applications. This means for the analyst the necessity of finding out the deep and very difficult proof of Zel'manov theorem, and of course this takes time. Fortunately this time has been already taken, and we may present in this last section of our survey relevant examples of the application of Zel'manov theorem to the structure of normed Jordan algebras that, in the case of JB- and JB*-algebras, refine in a very nontrivial way the classical theory reviewed in Section D. Since, as far as we know, Zelmanovian methods in normed Jordan algebras have been applied only very recently, the habitual summary of results until 1988 has no place in this case.

Recent results.

We began this survey giving birth to the binary Jordan identity, via the nonassociative Vidav-Palmer theorem (A.1). Later, in Section D, we have reviewed the structure theory of noncommutative JB*-algebras, built by that theorem, in a deliberately reformulated way so that it became incomplete. Certainly the description of an arbitrary noncommutative (or even commutative) JB*-algebra J as a subdirect l_∞-sum of geometrically primitive noncommutative JB*-algebras (D.1) is in a trivial way structurally better than the classical embedding of J in an l_∞-sum of type I noncommutative JBW*-factors, because the coordinate projections acting on J are surjective in the first case, while in the second case they only have w^*-dense range. But noncommutative type I JBW*-factors were well-known many years ago, while this was not the case for geometrically primitive noncommutative JB*-algebras. After D.2 and D.3, the problem centers in the familiar commutative case, and then it was known only that geometrically primitive JB*-algebras are prime and, even more, that they are primitive in the sense of Hogben-McCrimmon (and, of course, that they are the JB*-algebras having a faithful type I factor representation, so each one of them is w^*-dense in a suitable type I JBW*-factor). But, if not much was known about prime JB*-algebras, not much more information was provided about primitive JB*-algebras (nor even for the ranges of type I factor representations). Then it is tempting to appeal to Zelmanov's prime theorem [Ze1], which we recall here.

Theorem F.1. *The prime nondegenerate Jordan algebras over a field* $\mathbb{K}$ *are the following:*

i) Central orders in a central simple 27-dimensional exceptional Jordan algebra over a field extension of $\mathbb{K}$.

ii) Central orders in a simple quadratic Jordan algebra over a field extension of $\mathbb{K}$.

iii) Jordan subalgebras of $Q(A)$ *containing* A *as an ideal, where* A *is a prime associative algebra over* $\mathbb{K}$, *and* $Q(A)$ *denotes the symmetric Martindale ring of quotients for* A.

iv) Jordan subalgebras of $Q(A)$ *contained in* $H(Q(A),\tau)$ *and containing* $H(A,\tau)$ *as an ideal, where* A *is a prime associative algebra over* $\mathbb{K}$ *with a linear algebra involution* τ.

Since prime JB*-algebras are central (they are actually centrally closed, D.9), in the case of prime JB*-algebras, cases (i) and (ii) above lead directly to $M_3^8(\mathbb{C})$ and to simple quadratic JB*-algebras (take $\wedge=0$ in the construction D.3), respectively. Things don't behave so nicely in cases (iii) and (iv) of Zel'manov theorem, and, as we have commented before, it has been necessary to delve (lightly in this case) into the proof of the theorem, extracting some arguments that can be summarized in the following proposition (see also [McZe]).

Proposition F.2. *Let* B *be an associative algebra with a linear* *algebra involution τ, J a prime nondegenerate Jordan subalgebra of B contained in $H(B,\tau)$, and assume that J is not a central order in a simple quadratic Jordan algebra. Then there exists a τ-invariant subalgebra A of B such that $H(A,\tau)$ is a nonzero ideal of J.*

Now, if the prime JB*-algebra J is not in cases (i) and (ii) of Zel'manov theorem, we may appeal to the classical theory of JB*-algebras in order to select a specially well-behaved associative envelope B for J, to which Proposition F.2 will be applied. The contribution of the classical JB*-theory is the following (at this time folklore) Proposition.

Proposition F.3. *For every special JB*-algebra J, there exists a C*-algebra B with *-involution τ such that J is a closed *-invariant Jordan subalgebra of B contained in $H(B,\tau)$.*

The arguments we are reviewing are nothing but the first observations in the paper by A. Fernández, E. García, and A. Rodríguez [FeGarRod], where a fine JB*-version of the Zel'manov prime theorem has been provided. In the search for this result they have been inspired by a recent one of P. Ara. He showed in [Ar2] that, for a prime C*-algebra A, the symmetric Martindale ring of quotients $Q(A)$ coincides with the "symmetric algebra of bounded quotients" $Q_b(A)$. Since $Q_b(A)$ is a pre-C*-algebra, its completion $Q_b(A)^\wedge$ became affectively an ideal candidate for playing the role of $Q(A)$ in a JB*-version of Zel'manov's theorem in cases (iii) and (iv) of the original formulation. In fact, Fernández, García, and Rodríguez have been able to replace $Q_b(A)^\wedge$ by the smaller and more familiar C*-algebra $Mult(A)$ of multipliers on A. Recall that, for a semiprime associative algebra A, the symmetric Martindale ring of quotients $Q(A)$ (of all "maximal essentially defined double centralizers") contains the subring of multipliers $Mult(A)$ (of all "everywhere defined double centralizers"), which in its turn contains A as an ideal. Recall also that, in the case of A being a C*-algebra, $Mult(A)$ is in a natural way a C*-algebra containing A as a C*-subalgebra (in fact $Mult(A)$ can be recognized as the largest C*-subalgebra of the C*-algebra A^{**} containing A as an ideal). For the proof of Zel'manov's prime theorem for JB*-algebras some advances in the classical JB*-theory have been made in [FeGarRod] concerning JB*-algebras that contain closed essential ideals of classical type. We state these results in the following two propositions.

Proposition F.4. *Let J be a JB*-algebra containing a closed essential ideal that, regarded as a JB*-algebra, is of the form A^+ for a suitable C*-algebra A. Then J can be viewed as a closed *-invariant Jordan subalgebra of the C*-algebra Mult(A) containing A.*

Proposition F.5. *Let J be a JB*-algebra containing a closed essential ideal of the form $H(A,\tau)$ for a suitable C*-algebra A with *-involution τ, and assume A is generated as a C*-algebra by $H(A,\tau)$. Then J can be regarded as a closed *-invariant Jordan subalgebra of Mult(A) contained in $H(Mult(A),\tau)$ and containing $H(A,\tau)$.*

When all above ingredients are put in the cocktail shaker, and the cocktail shaker is suitably shaken, the cocktail is awaited. Let's taste it.

Theorem F.6. *The prime JB*-algebras are the following:*

i) $M_3^8(\mathbb{C})$.

ii) The simple quadratic JB-algebras.*

*iii) The closed *-invariant Jordan subalgebras of Mult(A) containing A, where A is a prime C*-algebra.*

*iv) The closed *-invariant Jordan subalgebras of Mult(A) contained in $H(Mult(A),\tau)$ and containing $H(A,\tau)$, where A is a prime C*-algebra with *-involution τ.*

Easy consequences of the theorem are the next corollaries (see [FeGarRod] for details).

Corollary F.7. *The prime JB-algebras are the following:*

i) The Albert algebra $M_3^8(\mathbb{R})$.

ii) The spin factors.

iii) The closed Jordan subalgebras of Mult(R) contained in the self-adjoint part of Mult(R) and containing the self-adjoint part of R, where R is a REAL C-algebra.*

Corollary F.8. *The topologically simple JB*-algebras are the following:*

i) $M_3^8(\mathbb{C})$.

ii) The simple quadratic JB-algebras.*

iii) The JB-algebras of the form* A^+, *where A is a topologically simple C*-algebra.*

iv) The JB-algebras of the form* $H(A,\tau)$, *where A is a topologically simple C*-algebra with *-involution* τ.

Theorem F.6 has been used in [FeGarRod], together with results by M. Mathieu in [Mat2], to show that prime JB*-algebras are ultraprime in the strong sense that *ALL normed ultrapowers of a prime JB*-algebra are prime (JB*-algebras)*, thus obtaining in view of Theorem B.27 a new proof of Proposition D.9. Since geometrically primitive JB*-algebras are prime, Theorem F.6 suggests the possibility of describing geometrically primitive JB*-algebras by selecting carefully, among the prime JB*-algebras there listed, those that are geometrically primitive. This careful selection has been also made in the paper under review (see [FeGarRod; Theorem 4.2]), thus completing the structure theory of noncommutative JB*-algebras in the finer way we have told in Section D. But, if Theorem 4.2 in [FeGarRod] is regarded under the light of the very recent work by J. A. Anquela, F. Montaner, and T. Cortés [AnMoCor] on zelmanovian treatment of Hogben-McCrimmon primitiveness, then it also gives us as a free gift an affirmative answer to an old question (that the reader can easily imagine). Let's therefore formulate this theorem in its definitive version.

Theorem F.9. *For a JB*-algebra J, the following three assertions are equivalent:*

1) J is geometrically primitive (i.e., J has a faithful type I factor representation).

2) J is of one of the following types:

i) $J=M_3^8(\mathbb{C})$,

ii) J is a simple quadratic JB-algebra,*

iii) there exists a primitive C-algebra A such that J is a closed *-invariant Jordan subalgebra of Mult(A) containing A,* *iv) there exists a primitive C*-algebra A generated as a C*-algebra by H(A,τ), for some *-involution τ on A, such that J is a closed *-invariant Jordan subalgebra of Mult(A) contained in H(Mult(A),τ) and containing H(A,τ).*

3) J is primitive (in the sense of Hogben-McCrimmon).

With the well-known result of Dixmier that separable prime C*-algebras are primitive, the above theorem implies easily that *separable prime JB*-algebras are primitive.* Much more interesting is the next corollary, many years expected. Recall that the maximal modular inner ideals of a noncommutative Jordan algebra *J* are defined as the maximal modular inner ideals of the Jordan algebra J^+, and that the core of a maximal modular inner ideal *M* of *J* is defined as the largest ideal of *J* contained in *M*. Recall also the notion of a "primitive M-ideal" of a Banach space (see before D.1).

Corollary F.10. *For an arbitrary noncommutative JB*-algebra J, the cores of maximal modular inner ideals of J agree with the primitive M-ideals of the Banach space of J.*

The JB-versions of the equivalence (1)⇔(2) in Theorem F.9 and of Corollary F.10, although not explicitly given in the literature, can be considered folklore in the classical nonzelmanovian theory of JB-algebras (see [Bu1] and [Bu2]). The points of difficulty in the case of a JB*-algebra *J* are that primitive M-ideals of the Banach space of *J* depend on the complex geometry of *J* and it is not obvious how this dependence reduces to the self-adjoint part of *J* (a fact already overcome in the classical theory [PaPeRod2]), and mainly that modular maximal inner ideals of *J* need not be *-invariant.

The subsequent Zel'manov type theorem for normed Jordan algebras we shall review involves very strong algebraic requirements, namely, the simplicity of the algebra and the existence of a unit element, but no additional assumptions should be made (even completeness is not assumed). This theorem has been proved by M. Cabrera and A. Rodríguez [CRod4], and reads as follows,

Theorem F.11. *Up to bicontinuous isomorphisms, the simple (complete) normed complex Jordan algebras with a unit are the following:*

i) $H_3(\mathbb{O}_{\mathbb{C}})$,

ii) the Jordan algebras $J(X,f)$ *of a continuous nondegenerate symmetric bilinear form* f *on a (complete) normed complex space* X *of dimension* ≥ 2,

iii) the Jordan algebras of the form A^+, *where* A *is a simple (complete) normed associative complex algebra with a unit, and*

iv) the Jordan algebras of the form $H(A,\tau)$, *where* A *is a simple (complete) normed associative complex algebra with a unit and* τ *is an isometric linear algebra involution on* A.

In connection with future developments of zelmanovian techniques in normed Jordan algebras without additional geometric requirements, the proof of this Theorem is more instructive than that of Theorem F.6. In fact, looking at the original version of Zel'manov theorem (Theorem F.1), applying that normed simple complex algebras with a unit are central (use the Gelfand-Mazur theorem, and compare with the much more general result B.14), and taking into account that, under the assumption of simplicity, cases (iii) and (iv) there can be unified by saying that the simple algebra J in such cases is of the form $H(A,\tau)$ for some τ-simple associative algebra A with linear algebra involution τ, the main difficulty to be overcome to arrive to Theorem F.11 is the so called "the norm-extension problem". That is, if the topology of the norm of J can be obtained by restricting to J the topology of a suitable algebra norm on A. Concerning the proof of Theorem F.11, this problem has been solved in [CRod4] by means of the next lemma. Recall that, given an associative algebra A with a linear (algebra) involution τ, (A,τ) is said to be a τ-tight envelope of a Jordan subalgebra $J\subseteq H(A,\tau)$ if A is generated by J and every nonzero τ-invariant ideal of A meets J. Also we refer to [McZe] for the concept of the ideal I_5 (the largest ideal consisting of "imbedded pentad eaters") in the free special Jordan algebra in a countable set of generators, as well as for the meaning of the symbol $I_5(J)$ for a given special Jordan algebra J.

Lemma F.12. *Let* A *be a real or complex associative algebra with a unit an a linear involution* τ, *let* J *denotes the Jordan algebra* $H(A,\tau)$, *and assume $I_5(J)=J$ and that (A,τ) is a τ-tight envelope of J. Then, for any algebra norm $\|.\|$ on J, there exists an algebra norm on A making the involution τ isometric and whose restriction to J is equivalent to $\|.\|$.*

The norm-extension problem has been reconsidered in the paper of A. Rodríguez, A. Slin'ko, and E. Zel'manov [RodSlZe], where the arguments in the proof of Lemma F.12 have been systematically exploited and significantly improved. As a first remarkable result they obtain the following relevant sufficient (and of course necessary) condition for the solution of the norm-extension problem.

Theorem F.13. *Let A be a real or complex associative algebra with a linear involution τ, let J denote the Jordan algebra $H(A,\tau)$, assume that (A,τ) is a τ-tight envelope of J, and let $\|.\|$ be an algebra norm on J. Then there exists an algebra norm on A making the involution τ isometric and whose restriction to J is equivalent to $\|.\|$ if (and only if) the "tetrad mapping" $(h,k,l,m)\longrightarrow hklm+mlkh$ from $J\times J\times J\times J$ into J is continuous for the topology of the norm $\|.\|$.*

It is proved in [RodSlZe] the automatic continuity of the tetrad mapping under reasonable algebraic and topological conditions. Then, using Theorem F.13 and parodying some arguments in the proof of Theorem F.11 for the complete normed case (see [CRod4; Theorem 2]), the following theorem is shown. This theorem answers a never explicitly posed old problem (compare with the result in [Rod2], already commented in Section B, that, if A is an associative real or complex semiprime algebra such that A^+ is a Jordan-Banach algebra, then A itself is a Banach algebra).

Theorem F.14. *Let A be a real or complex associative algebra with a linear involution τ, let J denote the Jordan algebra $H(A,\tau)$,* and *assume that (A,τ) is a τ-tight envelope of J and that J is semiprime. Then, for any complete algebra norm $\|.\|$ on J, there exists an algebra norm $|\!|\!|.|\!|\!|$ on A making τ isometric and satisfying the following properties:*

i) The restriction of $|\!|\!|.|\!|\!|$ to J is equivalent to $\|.\|$.

ii) If $A^\wedge$ denotes the completion of $(A,|\!|\!|.|\!|\!|)$, and if $\tau^\wedge$ stands for the only continuous (automatically isometric) involution on $A^\wedge$ that extends τ, then $J=H(A^\wedge,\tau^\wedge)$.

iii) Every nonzero $\tau^{\wedge}$-invariant ideal of $A^{\wedge}$ meets J.

Let's note that the assumption of completeness for $(J, \|.\|)$ in the theorem cannot be dropped in general because, without completeness of $\|.\|$, counter-examples to the norm-extension problem are given in [RodSlZe] even if (A,τ) is a τ-tight envelope of J $(=H(A,\tau))$ and J is semisimple. The paper we are reviewing contains also other interesting positive results about the norm-extension problem, as the following.

Proposition F.15. *Let A be a real or complex associative algebra with a linear involution τ, let J denote the Jordan algebra $H(A,\tau)$, and assume that (A,τ) is a τ-tight envelope of J, and that J has been endowed with an algebra norm $\|.\|$ in such a way that there is a positive number K satisfying $\|s^2\|\leq K\|D_s\|^2$ for those s in $S(A,\tau)$ for which the mapping $D_s : h \longrightarrow [s,h]$ from J into J is continuous. Then D_s is continuous for all s in $S(A,\tau)$, and there exists a positive number ρ (depending only on K) such that, by defining for h in J and s in $S(A,\tau)$ $|\!|\!| h+s |\!|\!| := \rho(\|h\|+\|D_s\|)$, $|\!|\!| . |\!|\!|$ is an algebra norm on A. Clearly $|\!|\!| a^* |\!|\!| = |\!|\!| a |\!|\!|$ for all a in A, and the restriction of $|\!|\!| . |\!|\!|$ to J is $\rho\|.\|$.*

To conclude the review on Zel'manov's type theorems for normed Jordan algebras, let's finally refer to the work of M. Cabrera and A. Rodríguez [CRod6] on Zel'manov's treatment of nondegenerately ultraprime Jordan-Banach algebras. A normed Jordan algebra J is said to be nondegenerately ultraprime if there exists a countably incomplete ultrafilter $\mathcal{U}$ on a suitable set such that the corresponding normed ultrapower $J_{\mathcal{U}}$ is prime and nondegenerate. Examples of nondegenerately ultraprime Jordan Banach algebras are all prime JB*-algebras. With the Beidar-Mikhalev-Slin'ko characterization of prime nondegenerate Jordan algebras [BeiMikSl], it can be proved easily that *a normed Jordan algebra J is nondegenerately ultraprime if and only if there exists $k>0$ such that $\|U_{x,y}\|\geq k\|x\|\|y\|$ for all x,y in J* (hence *all normed ultrapowers of a nondegenerately ultraprime normed Jordan algebra are prime and nondegenerate*). Following ideas by M. Mathieu ([Mat1] and [Mat3]), Cabrera and Rodríguez introduce "ultra-τ-prime" normed associative algebras with continuous linear involution τ, that can be characterized without any reference to ultrapowers as those normed associative algebras

A with continuous linear involution τ satisfying

$$Max\{\|M_{a,b}\|, \|M_{\tau(a),b}\|\}\geq k\|a\|\|b\|$$

for some fixed $k>o$ and all a,b in A, where $M_{a,b}(c):=acb$ (so that the completion $(A^\wedge,\tau)$ of an ultra-τ-prime normed associative algebra (A,τ) is an ultra-τ-prime Banach algebra). For such an ultra-τ-prime normed associative algebra (A,τ), a large τ-invariant subalgebra $Q_b(A)$ of its symmetric Martindale ring of quotients (the one of "bounded maximal essentially defined double centralizers") can be converted in an ultra-τ-prime normed algebra in such a way that the natural embedding $A\hookrightarrow Q_b(A)$ becomes a topological embedding. Then Jordan subalgebras of $Q_b(A)$ contained in $H(Q_b(A),\tau)$ and containing $H(A,\tau)$ as an ideal are examples of nondegenerately ultraprime normed Jordan algebras. The main result in [CRod6] is the following.

Theorem F.16. *Up to bicontinuous isomorphisms, the nondegenerately ultraprime Jordan-Banach complex algebras are the following:*

i) $H_3(\mathbb{O}_{\mathbb{C}})$.

ii) The Jordan-Banach algebras of the form $J(X,\langle .,.\rangle)$, *where* $(X,\langle .,.\rangle)$ *is a regular symmetric self-dual complex Banach space with* $dim(X)\geq 2$ (see definitions before Theorem B.9).

iii) *The closed Jordan subalgebras of* $Q_b(A)^\wedge$ *contained in* $H(Q_b(A)^\wedge,\tau)$ *and containing* $H(A,\tau)$ *as an ideal, where* A *is an ultra-τ-prime complex Banach algebra with continuous linear involution* τ.

The proof of this theorem is very long and difficult. Among the results already referred, it uses Theorems B.17 and F.1, Proposition F.15, and also in its concluding steps, either the proof of Theorem F.11 in the complete normed case, or better Theorem F.14. It uses also the result in [CRod3] asserting that *the centre of a nondegenerate Jordan algebra* J *coincides with the set* $\{x\in J : 2U_{x.y}=U_xU_y+U_yU_x$, *for all* y *in* $J\}$. (This was proved in [CRod3] by strong structural methods but, accordingly to a private communication of Y. A. Medvedev, the proof can be considerably liberated of its structural nature). The proof of theorem F.16 involves also new results which have their own interest, and that we state here.

Proposition F.17. *Let X, Y, and Z be real or complex normed spaces,* *and $F:X\longrightarrow Y$, $G:X\longrightarrow Z$ be bounded linear operators. Then*

$$Sup\{\|F(x)\|\|G(x)\| \ : \ \|x\|\leq 1\}\geq\frac{4\sqrt{2}-5}{7}\|F\|\|G\| \ .$$

Proposition F.18. *Let A be a prime (associative) Banach complex algebra with a unit* 1 *and a (not necessarily continuous) linear involution τ. Then for every s in $S(A,\tau)$ satisfying $\|1-s^2\|<1$ we have*

$$\|M_{1+s,1-s}\|\geq[1+(1-\|1-s^2\|)^{1/2}]^2.$$

Some other minor results in the paper [CRod6] we are reviewing are that *nondegenerately ultraprime Jordan-Banach complex algebras with nonzero socle are nothing but prime nondegenerate Jordan-Banach complex algebras with nonzero socle and minimality of norm topology* (see Theorem B.9), thus extending the corresponding associative theorem in [ArMat] and [PeRiRodVi], and that *the normed ultraproduct of any ultrafiltered family of prime JB*-algebras is a prime JB*-algebra.* Equivalently reformulated: *there exists a universal constant G such that, for every prime JB*-algebra J and for every x,y in J, we have $\|U_{x,y}\|\geq G\|x\|\|y\|$*

Problems and future directions.

We have the conviction that the above reviewed results on zelmanovian treatment of prime nondegenerate normed Jordan algebras, although relevant, are only the first steps of a flourishing theory that should be developed in the next years. Several reasons plead on behalf of this conviction. The first one is that prime nondegenerate normed Jordan algebras subjected until now to a zelmanovian treatment (see Theorems F.6, F.11, and F.16) have the common very restrictive property of being centrally closed (see D.9, B.14, and B.17, respectively), a fact that makes easy the study of cases (i) and (ii) in Zel'manov's theorem, and that perhaps should be taken as an additional natural assumption for future more general results in the line we are reviewing. In a positive direction, since primitive Jordan-Banach complex algebras are centrally closed (Theorem B.18), the observation above encourages to provide a Zel'manov theorem for such algebras, the formulation of which should be close to the following conjecture.

Conjecture F.19. *A Jordan-Banach complex algebra J is primitive (if* *and) only if one of the following statements holds:*

i) $J=H_3(\mathbb{O}_{\mathbb{C}})$.

ii) J is the Jordan algebra J(X,f) of a continuous nondegenerate symmetric bilinear form f on a complex Banach space X of dimension ≥ 2.

iii) There are a primitive associative Banach complex algebra A, and a one-to-one continuous Jordan-homomorphism ϕ *from J into some suitably normed subalgebra of Q(A), such that the range of* ϕ *contains A as an ideal.*

iv) There are a primitive associative Banach complex algebra A with a continuous linear involution τ, *and a one-to-one continuous Jordan-homomorphism* ϕ *from J into some suitably normed* τ*-invariant subalgebra of Q(A), such that the range of* ϕ *is contained in* $H(Q(A),\tau)$ *and contains* $H(A,\tau)$ *as an ideal.*

The reader may have told that the conjecture above is much less adventurous than what would be expected from the positive results referred below. But, even under the additional assumption of the existence of a nonzero socle, not much more is known (see Theorem B.8, and note that, with nonzero socle, primeness is equivalent to primitiveness [FeRod1]). Another reason why we should accept "light" normed versions of Zel'manov prime theorem, like the one in Conjecture F.19, is that, concerning positive results, JB*-algebras have minimality of norm topology (see Theorem D.10), and it is strongly conjectured that ultraprime associative Banach complex algebras, as well as nondegenerately ultraprime Jordan-Banach complex algebras, must also have minimality of norm topology. Then, looking at the comparison of Theorem B.8 with Theorem B.9, it can be expected that, unless for the case of simple algebras with a unit that has a peculiar treatment, "strong" normed versions of Zel'manov prime theorem may exist only for centrally closed prime nondegenerate Jordan-Banach complex algebras with minimality of norm topology. Even, in this case, when we have the Jordan-Banach algebra J sandwiched in the form

$$H(A,\tau)<|J\subseteq H(Q(A),\tau)$$

for a τ-prime associative algebra A with linear involution τ, the following two problems arise:

i) to extend the norm from $H(A,\tau)$ to A (note that Theorem F.14 is not relevant in this situation because the ideal $H(A,\tau)$ needs not be closed in J), and

ii) to extend the norm from A to a large enough τ-invariant subalgebra of $Q(A)$ in such a way that the topology of this extended norm agrees on J with the topology of the given norm on J.

We think it is more promising for the moment to look for "light" normed versions of Zel'manov's theorem, by finding reasonable conditions on associative normed algebras implying the existence of large enough "symmetric algebras of bounded quotients", and by weakening the norm-extension problem in the following way.

Conjecture F.20. *Let A be a τ-prime complex algebra with a linear involution τ, let J denote the Jordan algebra $H(A,\tau)$, and assume $\mathcal{Z}(J)=J$ ($\mathcal{Z}$ denoting the tetrad-eater zelmanovian ideal), and that (A,τ) is a τ-tight envelope of J. Then, for every algebra norm $\|.\|$ on J such that J is an ideal in the completion of $(J,\|.\|)$, there exists an algebra norm $|\!|\!|.|\!|\!|$ on A making continuous the inclusion $(J,\|.\|)\hookrightarrow(A,|\!|\!|.|\!|\!|)$.*

As a hint for an eventual proof of this conjecture, we note that, under the above assumptions, the tetrad mapping of J enjoys a strong separate continuity with respect to the topology of any algebra norm on J (compare with Theorem F.13).

Thinking in "light" normed versions of Zel'manov prime theorem, a problem to be not forgotten is that of the description of Jordan-Banach complex algebras in cases (i) or (ii) of the original theorem. The tensor product $\mathcal{D}\otimes M_3^8(\mathbb{C})$, where $\mathcal{D}$ denotes the disk algebra, becomes an example of an algebra in such a situation.

Now, let center the attention on a problem concerning Theorem F.16, namely, if the algebras arising in case (iii) of this theorem fall in one of the following two cases:

(1) The closed Jordan subalgebras of $Q_b(A)^\wedge$ containing A as an ideal, where A is an ultraprime complex Banach algebra (and $Q_b(A)$ denotes the Mathieu's symmetric algebra of bounded quotients [Mat3]).

(2) The closed Jordan subalgebras of $Q_b(A)^\wedge$ contained in $H(Q_b(A)^\wedge,\tau)$ and containing $H(A,\tau)$ as an ideal, where A is an ultraprime complex Banach algebra with continuous linear involution τ.

This problem seems to be an essentially associative problem, namely, if every prime ultra-τ-prime complex Banach algebra A, with continuous linear involution τ, is ultraprime. More precisely we pose the following question.

F.21. *Is there a simple ultra-τ-prime complex Banach algebra, with* *continuous linear involution τ, that is not ultraprime?*

Let's finally comment about the possibility of working on normed versions of Zel'manov prime theorem for Jordan triples [Ze2]. Because of the analogy with the binary case, no further considerations seem to be convenient, and we conclude with the natural conjecture about its JB*-version.

Conjecture F.22. *The prime JB*-triples are the following:*

i) The Cartan factors of types C^4, C^5, and C^6.

ii) The JB-subtriples of Mult(A) containing A, where A is a prime C*-algebra.*

iii) The JB-subtriples of Mult(A) contained in $H(Mult(A),\tau)$ and containing $H(A,\tau)$, where A is a prime C*-algebra with *-involution τ.*

iv) The JB-subtriples of Mult(A) contained in $S(Mult(A),\tau)$ and containing $S(A,\tau)$, where A is a prime C*-algebra with *-involution τ.*

G. Addendum after the 1992 Oberwolfach Conference on Jordan algebras.

This survey was basically written immediately before the 1992 Oberwolfach Conference on Jordan algebras, and a summary of it was presented in an address at that Conference. After these events, O. Loos [Lo5] have provided answers to some questions early raised in this paper, and we become pleasingly obliged to review them here in some detail.

In analogy with the notion of a "properly finite spectrum" element of a Jordan triple considered in Section C (which was actually suggested to us by A. Fernández), O. Loos have introduced the concept of a "properly algebraic" element. Properly algebraic elements of a Jordan triple are defined as those elements that are algebraic in every Jordan algebra homotope. Then he has proved the following theorem.

Theorem G.1. *For a semiprimitive complex Jordan-Banach triple, the socle, the set of properly algebraic elements, the largest properly spectrum-finite ideal, and the largest von Neumann regular ideal all coincide.*

In fact Theorem G.1 is a direct consequence of an analogous result also proved in [Lo5] for Jordan-Banach pairs (see [Hes] for a systematic study of Jordan-Banach pairs). The method of proof makes use of Jordan pairs which are only "half-Banach", only one of the spaces is complete but both are normed. The main tools are structural transformations [Lo2] and subquotients [LoN].

Theorem G.1, together with Theorem C.2, provides affirmative answers to question C.4 and to natural variants of Questions C.3 and B.24. In fact it is an almost direct consequence of Theorem G.1 and the Benslimane-Jaa-Kaidí theorem (Theorem B.7) that Question B.24 (in its original formulation) also has an affirmative answer. Indeed, as we have commented after B.24, Theorem B.7 reduces the problem to prove that the socle of a semiprimitive Jordan-Banach complex algebra J is a semiprime ideal of J. But it is easily seen that the largest properly algebraic ideal of an arbitrary Jordan algebra J (the existence of which is guaranteed by [Lo5; Theorem 2.9]) is a semiprime ideal of J. Hence

Theorem G.1 leads to the affirmative answer of B.24, that we precisely state here.

Corollary G.2. *The socle of a semiprimitive Jordan-Banach complex algebra agrees with the largest algebraic (equivalently, spectrum-finite) ideal.*

However, Questions such as C.3 in its original formulation, or the more general one if the socle of a complex semiprimitive Jordan-Banach triple is the largest algebraic ideal, remain unanswered. For other related problems we refer to [Lo5].

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