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Norms of Elementary Operators and Characterization of Norm-Attainable Operators

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Abstract

Let H be a complex Hilbert space and $B(H)$ the algebra of all bounded linear operators on H. In this note, we determine the norm of the Jordan Elementary operator $\mathcal{U}_{A,B} : B(H) \to B(H)$ defined by $\mathcal{U}_{A,B}(X) = AXB + BXA$, $\forall\ X \in B(H)$ and A, B fixed in $B(H)$. In particular, we prove that $\|\mathcal{U}_{A,B}\| \geq \|A\|\|B\|$. We also extend our work by characterizing norm-attainable operators using this norm.

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1 Introduction

Studies on properties of elementary operators have been of great concern to many mathematicians. These properties (Numerical ranges, Spectrum, Compactness, Positivity etc) have been studied with excellent results obtained (see

[1-3, 7-23]). The norm property has also been considered in a large number of papers but still it remains interesting to many mathematicians. This is so because calculating these norms involves finding a formula that describes the norms of elementary operators in terms of their coefficients (see details in [1-23]). Up-to-date, there is no known formula for calculating the norm of an arbitrary elementary operator acting on general Banach algebras.
However, results on norms have been obtained in special cases. The upper estimates of these norms are trivially obtainable but estimating them from below has proved to be difficult. Considering the results in these special cases, Mathieu [9] proved that for prime C*-algebras, $\|AXB + BXA\| \geq \frac{2}{3}\|A\|\|B\|$, Cabrera and Rodriguez [4] obtained $\|AXB+BXA\| \geq \frac{1}{20412}\|A\|\|B\|$ for JB*-algebras while Stacho and Zalar [17] proved that for standard operator algebras $\|AXB+BXA\| \geq 2(\sqrt{2}-1)\|A\|\|B\|$. Timoney [21, 22], Blanco, Boumazgour and Ransford [3] showed that $\|AXB + BXA\| \geq \|A\|\|B\|$, Nyamwala [11], Nyamwala and Agure [12] proved that $\|AXB + BXA\| = 2\|A\|\|B\|$. Recent studies by Hong-Ke Du, Yue-Qing Wang, and Gui-Bao Gao [7] gave a fundamental result on the elementary operator acting on separable complex Hilbert space while Seddik [16] characterized normaloid operators using the injective norm. All the results obtained in all these studies show that the norms lie between 1 and 2 that is, $\|A\|\|B\| \leq \|AXB+BXA\| \leq 2\|A\|\|B\|$. In our work, we shall also consider a special case of the Jordan elementary operator implemented by norm-attainable operators. We therefore, arrange our work in the various sections in the following order:
1. Introduction; 2. Preliminaries ; 3. Isometries(Co-Isometries), Unitaries and norm-attainability; 4. Characterization of Norm-attainable operators; 5. Norms of Elementary operators.

2 Preliminaries

Consider a C*-algebra $\mathcal{A}$ and let $T_{A,B} : \mathcal{A} \to \mathcal{A}$. T is called an elementary operator if it has the following representation:

$$T(X) = \sum_{i=1}^{n} A_i X B_i,\ \forall\ X \in \mathcal{A},$$

where $A_i,\ B_i$ are fixed in $\mathcal{A}$ or $\mathcal{M}(\mathcal{A})$ where $\mathcal{M}(\mathcal{A})$ is the multiplier algebra of $\mathcal{A}$. Let $\mathcal{A} = B(H)$. For $A,\ B \in B(H)$ we define the particular elementary operators:

(i) the left multiplication operator $L_A : B(H) \to B(H)$ by:
$L_A(X) = AX,\ \forall\ X \in B(H)$.

(ii) the right multiplication operator $R_B : B(H) \to B(H)$ by :
$R_B(X) = XB,\ \forall\ X \in B(H)$.

(iii) the generalized derivation (implemented by A, B) by:
$\delta_{A,B} = L_A - R_B$.

(iv) the basic elementary operator (implemented by A, B) by:
$M_{A,\,B}(X) = AXB, \ \forall\ X \in B(H)$.

(v) the Jordan elementary operator (implemented by A, B) by:
$\mathcal{U}_{A,\,B}(X) = AXB + BXA, \ \forall\ X \in B(H)$.

From this stage and in the sequel, we denote a bounded linear operator and a norm-attainable operator in $B(H)$ by A and A_N respectively.

Definition 2.1. *For an operator $A \in B(H)$ we define:*

(i) Numerical range by $W(A) = \{\langle Ax, x\rangle :\ x \in H,\ \|x\| = 1\}$,

(ii) Numerical radius by $w(A) = \sup\{|z| :\ z \in W(A)\}$.

We note that for $A,\ Q \in B(H)$, A is said to be positive if $\langle Ax, x\rangle \geq 0, \ \forall\ x \in H$ and Q an isometry(Co-isometry) if $Q^*Q = QQ^* = I$ where I is an identity operator in $B(H)$. A subalgebra $\mathcal{M} \subset B(H)$ is a standard operator algebra on H if it contains $F(H)$ where $F(H)$ is the ideal of finite rank operators. It's clear that $F(H) \subset B(H)$.

Definition 2.2. *An operator $A \in B(H)$ is said to be norm-attainable if for any unit vector $x_0 \in H$, $\|Ax_0\| = \|A\|$.*

At this juncture, we state a theorem by Du [7], which gave the limelight to this present paper.

Theorem 2.3. *(from [7, Theorem 1.1]) Let $\tilde{A} = A_1, \ldots , A_n$ and $\tilde{B} = B_1, \ldots , B_n$ be in $B(H)$ and let $T_{\tilde{A},\tilde{B}} :\ B(H) \to B(H)$ be defined by $T_{\tilde{A},\tilde{B}}(X) = \sum_{i=1}^{n} A_iXB_i, \ \forall\ X \in B(H)$, then*

$$\|T_{\tilde{A},\tilde{B}}\| = \sup_{U \in \mathcal{U}(H)} \|T_{\tilde{A},\tilde{B}}(U)\|.$$

Moreover, there is a contraction $X \in B(H)_1$ such that $\|T_{\tilde{A},\tilde{B}}(X)\| = \|T_{\tilde{A},\tilde{B}}\|$ if and only if there is a unitary $U \in \mathcal{U}(H)$ such that $\|T_{\tilde{A},\tilde{B}}(U)\| = \|T_{\tilde{A},\tilde{B}}\|$.

See [7] for proof.

Remark 2.4. *(i) $\mathcal{U}(H)$ is the algebra of all unitaries while $B(H)_1$ is a unit ball; (ii) An operator $T_{\tilde{A},\tilde{B}}$ is said to be norm-attainable if there is a contraction $X \in B(H)_1$ such that $\|T_{\tilde{A},\tilde{B}}(X)\| = \|T_{\tilde{A},\tilde{B}}\|$. For more details on norm attainability see [6, 7].*

In the next section, we state some Lemmas that form prerequisite to our main results.

3 Isometries(Co-Isometries), Unitaries and Norm-Attainability

We state following Lemma found in [7, Lemma 2.1] and include its proof for completion.

Lemma 3.1. *If $A \in B(H)$ then there exists two isometries or co-isometries $\Gamma_1, \Gamma_2 \in B(H)$ such that $A = \frac{1}{2}(\Gamma_1 + \Gamma_2)$. Furthermore, if $dimN(A) = dimN(A^*)$, $(N(A)$ is a null space$)$ then Γ_1, Γ_2 can be taken as unitaries.*

Proof. Let $A \in B(H)$ and $A = \Gamma Q$ be the polar decomposition of A. Since $A \in B(H)$, Q is a positive contraction in $B(H)$ hence $I - Q^2$ is also positive in $B(H)$. Define operators W_1, W_2 by $W_1 = Q + i(I - Q^2)^{\frac{1}{2}}$ and $W_2 = Q - i(I - Q^2)^{\frac{1}{2}}$, respectively. It's clear that $W_2 = W_1^*$ and therefore, it follows directly that $W_1 W_1^* = W_1^* W_1 = I$ and $W_2 W_2^* = W_2^* W_2 = I$, so W_1, W_2 are unitaries and $Q = \frac{1}{2}(W_1 + W_2)$.
The proof of the second part follows immediately by considering a null space, $N(A)$, and taking into account the results of [7, Lemma 2.1]. □

Theorem 3.2. *If the elementary operator $T_{\tilde{A},\tilde{B}}$ is norm-attainable then there exists an isometry or a co-isometry Q_0 such that $\|T_{\tilde{A},\tilde{B}}\| = \|\Sigma_{i=1}^{n} A_i Q_0 B_i\|$.*

Proof. From Remark (2.4), if the elementary operator $T_{\tilde{A},\tilde{B}}$ is norm-attainable then there exists $X_0 \in B(H)$ such that $\|T_{\tilde{A},\tilde{B}}\| = \|\Sigma_{i=1}^{n} A_i X_0 B_i\|$. Lastly, it's sufficient enough to show that $X_0 = \frac{1}{2}(\Gamma_1 + \Gamma_2)$ for two isometries or co-isometries $\Gamma_1, \Gamma_2 \in B(H)$. This follows immediately from Lemma (3.1). □

Theorem 3.3. *$A \in B(H)$ is norm-attainable if and only if there exists a finite sequence x_i $(i = 1, ..., n)$ of orthonormal vectors such that $\lim_i \|Ax_i\| = \|A\|$.*

Proof. Let Γ be a partial isometry from the range closure of A^* onto the range closure of A. If $A = \Gamma Q$, then $\|Ax\| = \|Qx\|$ for all $x \in H$. Hence, this shows that A is norm-attainable if and only if Q is norm-attainable. Conversely, taking the spectral decomposition of Q as $Q = \int_0^{\|Q\|} \alpha dE_\alpha$ and invoking the analogous argument on the $\Leftarrow$ part of Lemma (2.4) in [7] we obtain the required result. □

Theorem 3.4. *For $A \in B(H)$, A is norm-attainable if and only if its adjoint is norm-attainable.*

Proof. See proof in [7]. □

Corollary 3.5. *If $A, B \in B(H)$ are norm-attainable then $A + B$, $A - B$, λA, AI are norm-attainable where λ is a scalar.*

Proof. The proofs follow explicitly by use of limits, triangle and Cauchy-Schwarz inequalities. □

4 Characterization of Norm-Attainable operators

Theorem 4.1. *Let $\tilde{A} = A_1, \ldots, A_n$ be n-tuples of operators in $B(H)$ then the following properties hold.*

(i) *$\tilde{A}$ is norm-attainable in $B(H^n)$, where $B(H^n)$ is the algebra of bounded operators on H^n.*

(ii) *There exists a finite number of unit functionals $\phi_i \in H^*$ and unit vectors $x_1, \ldots, x_n \in H$ such that $\Sigma_{i=1}^n \phi_i \otimes (\tilde{A}(x_i)) = \lambda_i \|\tilde{A}\|$.*

Proof. (i) is trivial. For part (ii), it's clear that ϕ is linear and bounded so $\|\Sigma_{i=1}^n \phi_i \otimes (\tilde{A}(x_i))\| \leq \Sigma_{i=1}^n \|\phi_i \otimes (\tilde{A}(x_i))\| = \Sigma_{i=1}^n \|\phi_i(x_i)\tilde{A}\| = \Sigma_{i=1}^n |\phi_i(x_i)| \|\tilde{A}\| = \lambda_i \|\tilde{A}\|$.
On the other hand, pick any arbitrary $\epsilon > 0$ and let ϕ_i be unit functionals dependent on ϵ, then $\|\Sigma_{i=1}^n \phi_i \otimes (\tilde{A}(x_i))\| \geq \Sigma_{i=1}^n \|\phi_i \otimes (\tilde{A}(x_i))\| - \epsilon = \Sigma_{i=1}^n \|\phi_i(x_i)\tilde{A}\| = \Sigma_{i=1}^n |\phi_i(x_i)| \|\tilde{A}\| = \lambda_i \|\tilde{A}\|$. This completes the proof. □

Theorem 4.2. *Let $\tilde{A} = A_1, \ldots, A_n$ and $\tilde{B} = B_1, \ldots, B_n$ be in $B(H)$ such that $w(\tilde{B}) = \sup_i\{|z_i| \leq \|\tilde{B}\| : 1 \leq i \leq n\}$. The following properties are equivalent.*

(i) *$\tilde{A}$ is norm attainable in $B(H^n)$,*

(ii) *$\tilde{A} \| I_n$,*

(iii) *$\tilde{A}^* \| \tilde{A}$,*

(iv) *$\tilde{A}^*, \tilde{A}$ is 2-norm-attainable in $B(H^2)$,*

(v) *$\|I + M_{\tilde{A}^*, \tilde{A}}\| = 1 + \|\tilde{A}\|^2$,*

(vi) *$\|I + \lambda_i M_{\tilde{A}, \tilde{A}}\| = 1 + \|\tilde{A}\|^2$, for some unit scalars λ_i,*

(vii) *$\|I + M_{\tilde{A}, \tilde{B}}\| = 1 + \|\tilde{A}\| \|\tilde{B}\|$.*

Proof. Clearly,$(i) \Leftrightarrow (ii)$ is obvious.
$(i) \Rightarrow (iii)$. There exists a sequence of functionals ϕ_i on $B(H^n)$ and a set of unit scalars λ_i such that $\Sigma_{i=1}^n \phi_i \otimes (\tilde{A}(x_i)) = \lambda_i \|\tilde{A}\|$. Let $\|\tilde{A}\|$ be normal, then by Theorem (3.4) and a proof of Theorem (9) in [16], the result follows.
$(iii) \Rightarrow (i)$ follows from Theorem (4.1).
$(i) \Leftrightarrow (iv)$ is clear.
$(i) \Leftrightarrow (v)$. By letting $i = 1$ in The above theorem the proof follows through.
$(iv) \Rightarrow (i)$ is immediate from the analogous proof of Corollary 1 in [16].
$(i) \Rightarrow (ii)$ follows from [15, Corollary 2.5 and 2.6].
$(i) \Leftrightarrow (vii)$ follows directly from [15, Corollary 2.5] by taking limits over n. □

5 Norms of elementary operators

Theorem 5.1. *Consider a norm-attainable Jordan elementary operator* $\mathcal{U}_{N,A,B}$: $B(H) \to B(H)$, $X \to AXB + BXA$, $\forall X \in B(H.)$ *Assume* $A, B \in B(H)$ *are norm-attainable such that* $A = \Gamma Q$ *and* $B = \Gamma R$ *where* $Q = |A|$, $R = |B|$ *and* Γ *a unitary in* $B(H)$ *then* $\|\mathcal{U}_{N,A,B}|B(H)\| \geq \|A\|\|B\|$.

Proof. Let $A = \Gamma Q$ be apolar decomposition of A. Since Γ is a unitary in $B(H)$, then from Theorem (3.3) Γ is a partial isometry from the range of A^* onto the range of A. This implies that $\|Ax\| = \|Qx\|$, for all unit vectors $x \in H$ hence A is norm-attainable. Similarly, B is norm-attainable. Taking the the spectral decomposition of Q as $Q = \int_0^{\|Q\|} \alpha dE_\alpha$ for some spectral projections E_α, clearly $\|Ax\| = \|A\|$ and $\|Bx\| = \|B\|$. From the definition of operator norm,

$$\begin{aligned}
\|\mathcal{U}_{N,A,B}\| &= \sup\{\|\langle Bx, y\rangle A + \langle Ax, y\rangle B\| : \|x\|\| = \|y\| = 1\} \\
&= \sup\{\|\langle Rx, \Gamma^* y\rangle \Gamma Q + \langle Qx, \Gamma^* y\rangle \Gamma R\| : \|x\|\| = \|y\| = 1\} \\
&= \sup\{\|\langle Rx, \Gamma^* y\rangle Q + \langle Qx, \Gamma^* y\rangle R\| : \|x\|\| = \|y\| = 1\}.
\end{aligned}$$

Substituting the spectral decompositions of Q and R and taking the supremum over the unit vectors x and y we obtain

$$\begin{aligned}
\|\mathcal{U}_{N,A,B}\| &= \|\mathcal{U}_{N,\int_0^{\|Q\|} \alpha_1 dE_{\alpha_1}, \int_0^{\|R\|} \alpha_2 dF_{\alpha_2}}\| \\
&= \|\Gamma_{N,Q,R}\| \\
&\geq \|Q\|\|R\| \\
&= \|A\|\|B\|.
\end{aligned}$$

□

Corollary 5.2. *Let* A *and* B *be as above. If there exists a sequence* x_n *of unit vectors in* H *such that* $\|Ax_n\| = \|A\|$ *and* $\|Bx_n\| = \|B\|$ *then* $\|\mathcal{U}_{N,A,B}|B(H)\| \geq \|A\|\|B\|$ *holds.*

Proof. Follows immediately from a quick check at Theorem (3.3) and the result obtained in [15, Theorem 13]. □

Open Question: What are the necessary and sufficient conditions for Theorem (5.1) above to hold if H is taken to be infinite dimensional and a non-separable Hilbert space?

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