Electronic Journal of Qualitative Theory of Differential Equations
2004, No. **5**, 1-10; **http://www.math.u-szeged.hu/ejqtde/**

**Complete polynomial vector fields in simplexes
with application to evolutionary dynamics**

by N.M. BEN YOUSIF

ABSTRACT. We describe the complete polynomial vector fields and their fixed points in a finite-dimensional simplex and we apply the results to differential equations of genetical evolution models.

AMS Classification. Primary: 34C07 , secondary: 34C23, 37Cxx.

Key words: polynomial vector field, fixed point, mutation, selection, evolution.

1. Introduction

Well-known genetical models [3],[5],[2] of the time evolution of a closed population consisting of N different species describe the rates $r_1(t), \ldots, r_N(t)$ of the respective species within the whole population at time $t \geq 0$ as the solution of the ordinary system of differential equations $dr_k(t)/dt = F_k(r_1(t), \ldots, r_N(t))$ $(k = 1, \ldots, N)$ where the functions F_k are some polynomials of degree at most 3. During a seminar on such models one has launched the question what are the strange consequences of the assumption that the evolution has no starting point in time, in particular what can be stated on the non-changing distributions in that case.

In this paper we provide the complete algebraic description of all polynomial vector fields (with arbitrary degrees) $V(x) = (F_1(x), \ldots, F_N(x))$ on $\mathbb{R}^N$ which give rise to solutions for the evolution equation defined for all time parameters $t \in \mathbb{R}$ and satisfying the natural rate conditions $r_1(t), \ldots, r_N(t) \geq 0$, $\sum_{k=1}^N r_k(t) = 1$ whenever $r_1(0), \ldots, r_N(0) \geq 0$ and $\sum_{k=1}^N r_k(0) = 1$. On the basis of the explicit formulas obtained, we describe the structure or the set of zeros for such vector fields (which correspond to the non-changing distributions).

2. Results

Throughout this work $\mathbb{R}^N := \{(\xi_1, \ldots, \xi_N) : \ \xi_1, \ldots, \xi_N \in \mathbb{R}\}$. denotes the N-dimensional vector space of all real N-tuples. We reserve the notations $x_1, \ldots, x_N$ for the standard coordinate functions $x_k : (\xi_1, \ldots, \xi_N) \mapsto \xi_k$ on $\mathbb{R}^N$. Our purpose will be to describe the *complete polynomial vector fields on the unit simplex*

$$S := \big(x_1 + \cdots + x_N = 1,\ x_1, \ldots, x_N \geq 0\big)$$

along with their fixed points for the cases of degree ≤ 3.

Recall [1] that by a vector field on S we simply mean a function $S \to \mathbb{R}^N$. A function $\varphi : S \to \mathbb{R}$ is said to be polynomial if it is the restriction of some polynomial of the linear coordinate functions $x_1, \ldots, , x_N$: for some finite system coefficients $\alpha_{k_1 \ldots k_N} \in \mathbb{R}$ with $k_1, \ldots, k_N \in \{0, 1, \ldots\}$) we can write $\varphi(p) = \sum_{k_1,\ldots,k_N} \alpha_{k_1 \ldots k_N} x_1^{k_1} \cdots x_N^{k_N} \quad (p \in S)$. In accordance with this terminology, a vector field V on S is a polynomial vector field if its components $V_k := x_k \circ V$ (that is $V(p) = (V_1(p), \ldots, V_N(p))$ for $p \in S$) are polynomial functions. It is elementary that given two polynomials $P_m = P_m(x_1, \ldots, x_N) : \mathbb{R}^N \to \mathbb{R}$ $(m = 1, 2)$, their restrictions to S coincide if and only if the difference $P_1 - P_2$ vanishes on the affine subspace $A_S := \big(x_1 + \cdots + x_N = 1\big)$ generated by S. We shall see later (Lemma 3.1) that a polynomial $P = P(x_1, \ldots, x_N)$ vanishes on the affine subspace $M := \big(\gamma_1 x_1 + \cdots + \gamma_N x_N = \delta\big)$ iff $P = (\gamma_1 x_1 + \cdots + \gamma_N x_N - \delta) Q(x_1, \ldots, x_N)$ for some polynomial Q. Thus polynomial vector fields on S admit several polynomial extensions to $\mathbb{R}^N$ but any two such extensions differ only by a vector field of the form $(x_1 + \cdots + x_N - 1)W$. A polynomial vector field $V : S \to \mathbb{R}^N$ is said to be complete in S if for any point $p \in S$ there is a (necessarily unique) curve $C_p : \mathbb{R} \to S$ such that $C_p(0) = p$ and $\frac{d}{dt} C_p(t) = V(C_p(t)) \quad (t \in \mathbb{R})$.

Our main results are as follows.

2.1. Theorem. *A polynomial vector field $V : S \to \mathbb{R}^N$ is complete in S if and only if with the vector fields*

$$Z_k := x_k \sum_{j=1}^{N} x_j (e_j - e_k) \qquad (k = 1, \ldots, N)$$

where e_j is the standard unit vector $e_j := (0, \ldots, 0, \overbrace{1}^{j}, 0, \ldots, 0)$, we have

$$V = \sum_{k=1}^{N} P_k(x_1, \ldots, x_N) Z_k$$

for some polynomial functions $P_1, \ldots, P_N : \mathbb{R}^N \to \mathbb{R}$.

2.2. Theorem. *Given a complete polynomial vector field V of S, there are polynomials $\delta_1, \ldots, \delta_N : \mathbb{R}^{N-1} \to \mathbb{R}$ of degree less than as that of V such that the vector field*

$$\widetilde{V} := \sum_{k=1}^{N-1} x_k \Big[\delta_k(x_1, \ldots, x_{N-1}) - \sum_{\ell=1}^{N-1} x_\ell \delta_\ell(x_1, \ldots, x_{N-1}) \Big] e_k +$$

$$+ (x_1 + \cdots + x_{N-1} - 1) \sum_{\ell=1}^{N-1} x_\ell \delta_\ell(x_1, \ldots, x_{N-1}) e_N$$

coincides with V *on* S *. The points of the zeros of* V *inside the facial subsimplices* $S_K := S\cap(x_1,\ldots,x_K>0=x_{K+1}=\cdots=x_N)$ $(K=1,\ldots,N)$ *can be described as*

$$(*)\qquad \begin{aligned} &S_N\cap(V=0)=S\cap\bigcup_{k=1}^{N-1}\big(\delta_k(x_1,\ldots,x_{N-1})=0\big),\\ &S_K\cap(V=0)=S_K\cap\big(\delta_1(x_1,\ldots,x_{N-1})=\cdots=\delta_K(x_1,\ldots,x_{N-1})\big)\qquad (K<N).\end{aligned}$$

Finally we turn back to our motivativation the genetical time evolution equation for the distribution of species within a closed population. Namely in [2] we have the system

$$(V)\qquad \begin{aligned} \frac{d}{dt}x_k =&\Big(\sum_{i=1}^N g(i)x_i - g(k)\Big)x_k +\\ &+\sum_{i,j=1}^N w(i,j)x_ix_j\Big[\sum_{\ell=1}^N M(i,j,\ell)\varepsilon(i,j,\ell,k)-x_k\Big]\end{aligned}$$

for describing the behaviour of the rates $x_1(t),\ldots,x_N(t)$ at time t of the N species of the population. here the terms $g(k), M(i,j,\ell)$ and $\varepsilon(i,j,\ell,k)$ are non-negative constats with $\sum_{\ell=1}^N M(i,j,\ell)=\sum_{k=1}^N \varepsilon(i,j,\ell,k)=1$. Observe that this can be written as

$$\frac{d}{dt}x=\sum_{k=1}^N g(k)Z_k + W$$

with the vector fields

$$Z_k{:=}x_k\sum_{j=1}^N x_j(e_j-e_k),\quad W:=\sum_{i,j,k=1}^N w(i,j)x_ix_j\Big[\sum_{\ell=1}^N M(i,j,\ell)\varepsilon(i,j,\ell,k)-x_k\Big]e_k,$$

respectively. As a consequence of Theorems 2.1 and 2.2 we obtain the following.

2.3. Theorem. *Let* $N\geq 3$ *. Then the time evolution of the population can be retrospected up to any time* $t\leq 0$ *starting with any distribution* $(x(0),\ldots,x_N(0))\in S$ *if and only if the term* W *vanishes on* S *, that is if simply* $d/dt\ x=\sum_{k=1}^N g(k)Z_k(x_1,\ldots,x_N)$ *. In this case the set of the stable distributions has the form*

$$\bigcup_{\gamma\in\{g(1),\ldots,g(N)\}} S\cap(x_m=0\text{ for } m\notin J_\gamma)\qquad\text{where } J_\gamma:=\{m:\ g(m)=\gamma\}\ .$$

2.4. Corollary. *If* $g(1),\ldots,g(N)\geq 0$ *and the vector field* (V) *is complete in* S *then*

$$\frac{d}{dt}\sum_{k=1}^N g(k)x_k(t)\geq 0$$

for any solution $t \mapsto x(t) \in S$ *of the evolution equation* $dx/dt = V(x)$ *.*

3. Proof of Theorem 2.1

As in the previous section, we keep fixed the notations $e_1, \dots, e_N, x_1, \dots, x_N, S$ for the standard unit vectors, coordinate functionals and unit simplex in $\mathbb{R}^N$, and $V : \mathbb{R}^N \to \mathbb{R}^N$ is an arbitrarily fixed polynomial vector field. We write $\langle u, v \rangle := \sum_{k=1}^N x_k(u) x_k(v)$ for the usual scalar product in $\mathbb{R}^N$.

According to [4, (2,2)], V is complete in S if and only if

$$V(p) \in T_p(s) := \{v \in \mathbb{R}^N : \ \exists\, c : \mathbb{R} \to S \ , \ \ c(0) = p \ , \ \ \frac{d}{dt}\Big|_{t=0} c(t) = v\} \qquad \text{for all } p \in S.$$

By writing

$$\overline{e} := \frac{1}{N} \sum_{k=1}^N e_k, \quad u_k := e_k - \overline{e}, \quad S_k := S \cap (x_k = 0) \qquad (k = 1, \dots, N)$$

for the center, the vectors connecting the vertices with the center and the maximal faces of S , it is elementary that

$$T_p(S) = \big\{ v : \ \langle v, \overline{e} \rangle = 0 \big\} \quad \text{if } p \in S \setminus \textstyle\bigcup_{k=1}^N S_k),$$
$$T_p(S) = \big\{ v : \ \langle v, \overline{e} \rangle = \langle v, u_k \rangle = 0 \ \ (k \in K_p) \big\} \quad \text{if } p \in \textstyle\bigcup_{k=1}^N S_k \text{ and } K_p := \{k : \ p \in S_k\}$$

for any non-empty subset K of $\{1, \dots, N\}$. Since the vector field V is polynomial by assumption, it follows that

$$\begin{aligned} &V \text{ is complete in } S \iff \\ &\iff \langle V(p), \overline{e} \rangle = 0 \quad (p \in S) \quad \text{and} \quad \langle V(p), u_m \rangle = 0 \quad (p \in S_m, \ m = 1, \dots, N). \end{aligned}$$

Let us write

$$L_S := (x_1 + \dots + x_N = 1), \quad L_{S_m} := L_S \cap (x_m = 0) \quad (m = 1, \dots, N)$$

for the hyperplane supporting S , and for the affine submanifolds generated by the faces S_m , respectively. Since $e_k = u_k + \overline{e}$ and since polynomials vanishig on a convex set vanish also on its supporting affine submanifold, equivalently we can say

$$\begin{aligned} &V \text{ is complete in } S \iff \\ &\iff \ \langle V(p), \overline{e} \rangle = 0 \text{ for } p \in L_S \text{ and } \langle V(p), e_m \rangle = 0 \text{ for } p \in L_{S_m} \quad (m = 1, \dots, N). \end{aligned}$$

If $P_1,\ldots,P_N : \mathrm{I\!R}^N \to \mathrm{I\!R}$ are polynomial functions then, with the vector fields $Z_k := x_k \sum_{j=1}^N x_j(e_j - e_k)$ $(k=1,\ldots,N)$, we have

$$\begin{aligned}
\left\langle \sum_{k=1}^N P_k(p) Z_k(p), \overline{e} \right\rangle &= \sum_{k=1}^N P_k(p) \langle Z_k(p), \overline{e}\rangle = \\
&= \sum_{k=1}^N P_k(p) \left\langle Z_k(p), \frac{1}{N}\sum_{\ell=1}^N e_\ell \right\rangle = \\
&= \frac{1}{N}\sum_{k=1}^N P_k(p) \sum_{j,\ell=1}^N \langle x_k(p)x_j(p)(e_j - e_k), e_\ell\rangle = \\
&= \frac{1}{N}\sum_{k=1}^N P_k(p)x_k(p) \sum_{j:\ j\neq k}\sum_{\ell=j,k} x_j(p)\langle e_j - e_k, e_\ell\rangle = \\
&= \frac{1}{N}\sum_{k=1}^N P_k(p)x_k(p) \sum_{j:\ j\neq k} x_j(p) \sum_{\ell=j,k} \langle e_j - e_k, e_\ell\rangle = \\
&= \frac{1}{N}\sum_{k=1}^N P_k(p)x_k(p) \sum_{j:\ j\neq k} x_j(p)[1-1] = 0
\end{aligned}$$

for any point $p \in \mathrm{I\!R}^N$ (not only for $p \in S$). On the other hand, if $p \in S_m$ then $x_m(p) = 0$ and

$$\begin{aligned}
\left\langle \sum_{k=1}^N P_k(p) Z_k(p), e_m \right\rangle &= \sum_{k=1}^N P_k(p) \left\langle Z_k(p), e_m\right\rangle = \\
&= \sum_{k=1}^N P_k(p)x_k(p) \sum_{j:\ j\neq k} x_j(p)\langle e_j - e_k, e_m\rangle = \\
&= \sum_{k:\ k\neq m} P_k(p)x_k(p) \sum_{j:\ j\neq k,m} x_j(p)\langle e_j - e_k, e_m\rangle = 0.
\end{aligned}$$

This means that the vector fields of the form $V := \sum_{k=1}^N P_k Z_k$ with arbitrary polynomials $P_1,\ldots,P_N$ are complete in S , morover $\langle V(p), \overline{e}\rangle = 0$ for all $p \in \mathrm{I\!R}^N$.

To prove the remaining part of the theorem, we need the following lemma.

3.1. Lemma. *If $P : \mathrm{I\!R}^N \to \mathrm{I\!R}$ is a polynomial function and $0 \neq \phi : \mathrm{I\!R}^K \to \mathrm{I\!R}$ is an affine function* * *such that $P(q) = 0$ for the points q of the hyperplane $\{q \in \mathrm{I\!R}^N :\ \phi(q) = 0\}$ then ϕ a divisor of P in the sense that $P = \phi Q$ with some (unique) polynomial $Q : \mathrm{I\!R}^N \to \mathrm{I\!R}$.*

---

* That is ϕ is the sum of a linear functional with a constant.

Proof. Trivially, any two hyperplanes are affine images of each other. In particular there is a one to one affine (i.e linear + constant) mapping $A : \mathbb{R}^N \leftrightarrow \mathbb{R}^N$ such that $\{q \in \mathbb{R}^N : \ \phi(p) = 0\} = A\big(\{q \in \mathbb{R}^N : \ x_1(q) = 0\}\big)$. Then $R := P \circ A$ is a polynomial function such that $R(q) = 0$ for the points of the hyperplane $\{q \in \mathbb{R}^N : \ x_1(q) = 0\}$. We can write $R = \sum_{k_1,\ldots,k_N=0}^d \alpha_{k_1,\ldots,k_N} x_1^{k_1} \cdots x_N^{k_N}$ with a suitable finite family of coefficients $\alpha_{k_1,\ldots,k_N}$. By the Taylor formula, $\alpha_{k_1,\ldots,k_N} = \frac{\partial^{k_1+\cdots+k_N}}{\partial x_1^{k_1}\cdots\partial x_N^{k_N}}\Big|_{x_1=\cdots=x_N=0} R$. It follows $\alpha_{k_1,\ldots,k_N} = 0$ for $k_1 > 0$, since R vanishes for $x_1 = 0$. This means that $R = x_1 R_0$ with the polynomial $R_0 := \sum_{k_1=1}^d \sum_{k_2,\ldots,k_N=0}^d x_1^{k_1-1} x_2^{k_2} \cdots x_N^{k_N}$. By the same argument applied for the polynomial function ϕ of degree $d = 1$ in place of R, we see that $\phi \circ A = \alpha x_1$ for some constant (polynomial of degree 0) $\alpha \neq 0$. That is $\phi = \alpha x_1 \circ A^{-1}$. Therefore

$$P = R \circ A^{-1} = [x_1 R_0] \circ A^{-1} = (x_1 \circ A^{-1})(R_0 \circ A^{-1}) = \phi \cdot (\frac{1}{\alpha} R_0 \circ A^{-1}).$$

Since the inverse of an affine mapping is affine as well, the function $Q := \frac{1}{\alpha} R_0 \circ A^{-1}$ is a polynomial which suits the statement of the lemma. $\square$

3.2. Corollary. *A polynomial vector field* $\widetilde{V} : \mathbb{R}^N \to \mathbb{R}^N$ *coincides with* V *on* S *iff it has the form* $\widetilde{V} = V + (x_1 + \cdots + x_N - 1)W$ *with some polynomial vector field* $W : \mathbb{R}^N \to \mathbb{R}^N$.

Proof. Observe that $\widetilde{D}$ and V coincide on S iff they coincide on the hyperplane L_S supporting S. We can write $\widetilde{V} = \sum_{k=1}^N \widetilde{P}_k e_k$ resp. $V = \sum_{k=1}^N P_k e_k$ with some scalar valued polynomials $\widetilde{P}_k$ resp. P_k and, by the lemma, we have $\widetilde{P}_k - P_k = 0$ on L_S iff $\widetilde{P}_k - P_k = (x_1 + \cdots + x_N - 1)Q_k$ with some polynomial $Q_k : \mathbb{R}^N \to \mathbb{R}$ $(k = 1, \ldots, N)$, that is if $\widetilde{V} - V = (x_1 + \cdots + x_N - 1)W$ with the vector field $W := \sum_{k=1}^N Q_k e_k$. $\square$

Instead of the generic polynomial vector field V complete in S, it is more convenient to study another $\widetilde{V}$ coinciding with V on S but having additional properties. As in the proof of the previous corollary, we decompose V as $V = \sum_k P_k e_k$. Recall that $V(p) \in T_p(S) \subset \{v : \ \langle v, \overline{e}\rangle = 0\}$ for the points $p \in S$. In terms of the component functions P_k, this means that $\frac{1}{N}\sum_{k=1}^N P_k = 0$ that is $P_N = -\sum_{k:\ k\neq N} P_k$ on S. On the other hand, $x_1 + \cdots + x_N = 0$ that is $x_N = -\sum_{k:\ k\neq N}$ on S. Introduce the vector field

$$\widetilde{V} := \sum_{k=1}^N \widetilde{P}_k e_k$$

where

$$\widetilde{P_k} := \pi_k(x_1, \ldots, x_{N-1}) := P_k(x_1, x_2, \ldots, x_{N-1}, 1 - x_1 - \cdots - x_{N-1}) \quad (k < N),$$

$$\widetilde{P}_N := \pi_N(x_1, \ldots, x_{N-1}) := -\sum_{k=1}^{N-1} \widetilde{P}_k = -\sum_{k=1}^{N-1} \pi_k(x_1, \ldots, x_{N-1}).$$

By its construction, $\widetilde{V}$ coincides with V on S, it is a polynomial of the same degree as V but only in the variables $x_1, \ldots, x_{N-1}$ and it has the property $\sum_{k=1}^N \widetilde{P}_k = 0$ on the whole space $\mathbb{R}^N$. The relations $\widetilde{V}(p) = V(p) \in T_p(S) \subset \{v : \langle v, e_k \rangle = 0\}$ for $p \in S_k$ $(k = 1, \ldots, N)$ mean

$$\widetilde{P}_k(p) = \langle \widetilde{V}(p), e_k \rangle = 0 \qquad \text{for} \quad p \in S_k = (x_k = 0,\ x_1 + \cdots + x_N = 1,\ x_1, \ldots, x_N \geq 0).$$

In terms of the polynomials π_k of $N-1$ variables this can be stated as

$$(**) \qquad \begin{aligned} &\pi_k(\xi_1, \ldots, \xi_{N-1}) = 0 \quad \text{whenever} \quad \xi_k = 0 \qquad (k = 1, \ldots, N-1) \quad \text{and} \\ &-\sum_{k=1}^{N-1} \pi_k(\xi_1, \ldots, \xi_{N-1}) \big[= \pi_N(\xi_1, \ldots, \xi_{N-1})\big] = 0 \quad \text{whenever} \quad \xi_1 + \cdots + \xi_{N-1} = 1. \end{aligned}$$

By the lemma (applied with $N-1$ instead of N), the first $N-1$ equations are equivalent to

$$\pi_k(\xi_1, \ldots, \xi_{N-1}) = \xi_k \varrho_k(\xi_1, \ldots, \xi_{N-1}) \qquad (k = 1, \ldots, N-1)$$

with some polynomials $\varrho_k : \mathbb{R}^{N-1} \to \mathbb{R}$ with degree less than the degree of π_k and V. Also by the lemma (with $N-1$ instead of N), the last equation can be interpreted as

$$-\sum_{k=1}^{N-1} \pi_k(\xi_1, \ldots, \xi_{N-1}) = \pi_N(\xi_1, \ldots, \xi_{N-1}) = [1 - (\xi_1 + \cdots + \xi_{N-1})] \varrho_N(\xi_1, \ldots, \xi_{N-1})$$

with some polynomial $\varrho_N : \mathbb{R}^{N-1} \to \mathbb{R}$ of degree less than that of V. Thus

$$\begin{aligned} &-\sum_{k=1}^{N-1} \xi_k \varrho_k(\xi_1, \ldots, \xi_{N-1}) = [1 - (\xi_1 + \cdots + \xi_{N-1})] \varrho_N(\xi_1, \ldots, \xi_{N-1}), \\ &\sum_{k=1}^{N-1} \xi_k [\varrho_N - \varrho_k](\xi_1, \ldots, \xi_{N-1}) = \varrho_N(\xi_1, \ldots, \xi_{N-1}). \end{aligned}$$

By introducing the polynomials $\delta_k := \varrho_k - \varrho_N$ $(k = 1, \ldots, N-1)$ of $N-1$ variables, we can reformulate the relationships $(**)$ as

$$\begin{aligned} \pi_k &= \xi_k \varrho_k = \xi_k(\delta_k + \varrho_N) \qquad (k = 1, \ldots, N-1), \\ \pi_N &= (1 - \xi_1 - \cdots - \xi_N) \varrho_N, \\ \varrho_N &= -\xi_1 \delta_1 - \cdots - \xi_{N-1} \delta_{N-1} \end{aligned}$$

which is the same as

$$(***) \qquad \begin{aligned} &\pi_k(\xi_1, \ldots, \xi_{N-1}) = \xi_k \Big[\delta_k(\xi_1, \ldots, \xi_{N-1}) - \sum_{\ell=1}^{N-1} \xi_\ell \delta_\ell(\xi_1, \ldots, \xi_{N-1}) \Big] \quad \text{for } k \neq N, \\ &\pi_N = (\xi_1 + \cdots + \xi_N - 1) \sum_{\ell=1}^{N-1} \xi_\ell \delta_\ell(\xi_1, \ldots, \xi_{N-1}) \end{aligned}$$

where $\delta_1, \dots, \delta_{N-1}$ are arbitrary polynomials of the variables $\xi_1, \dots, \xi_{N-1}$.

Summarizing the arguments, we have obtained the followig result.

3.3. Proposition. *Let* $V = \sum_{k=1}^N P_k e_k$ *be a vector field where* $P_1, \dots, P_N : \mathbb{R}^N \to \mathbb{R}$ *are polynomials of the coordinate functions* $x_1, \dots, x_N$ *. Then* V *is complete in the simplex* $S := (x_1 + \cdots + x_N = 1, x_1, \dots, x_N \geq 0)$ *if and only if there exist polynomials* $\delta_1, \dots, \delta_{N-1}$ *of* $N-1$ *variables and degree less than that of* V *such that the vector field* $\widetilde{V} := \sum_{k=1}^N \pi_k(x_1, \dots, x_{N-1}) e_k$ *, where the polynomials* π_k *are given by* $(***)$ *in terms of* $\delta_1, \dots, \delta_{N-1}$ *, coincides with* V *on the hyperplane* $L_S := (x_1 + \cdots + x_N = 1)$ *.* $\square$

On the basis of the proposition we can finish the proof of Theorem 2.1 as follows. Let V be a polynomial vector field complete in S. By the proposition, we can find a vector field $\widetilde{V}$ of the form (*) coinciding with V on S such that where $\delta_1, \dots, \delta_{N-1} : \mathbb{R}^{N-1} \to \mathbb{R}$ are polynomials. It suffices to show that the vector field

$$\widehat{V} := -\sum_{k=1}^{N-1} \delta(x_1, \dots, x_{N-1}) Z_k(x_1, \dots, x_N) = \sum_{k=1}^{N-1} \delta(x_1, \dots, x_{N-1}) \sum_{\ell=1}^{N} x_k x_\ell (e_k - e_\ell)$$

coincides with $\widetilde{V}$ on S. Consider any point $p \in S$ and let $\xi_k := x_k(p) \quad (k = 1, \dots, N)$. Since $\xi_N = 1 - \xi_1 - \cdots - \xi_{N-1}$, it is straightforward to check that indeed

$$\begin{aligned}\widetilde{V}(p) - \widehat{V}(p) = \sum_{k=1}^{N-1} \xi_k \Big[\delta_k(\xi_1, \dots, \xi_{N-1}) - \sum_{\ell=1}^{N-1} \xi_\ell \delta_\ell(\xi_1, \dots, \xi_{N-1})\Big] e_k + \\ + (\xi_1 + \cdots + \xi_{N-1} - 1) \sum_{\ell=1}^{N-1} \xi_\ell \delta_\ell(\xi_1, \dots, \xi_{N-1}) e_N + \\ + \sum_{k=1}^{N-1} \delta(\xi_1, \dots, \xi_{N-1}) \sum_{\ell=1}^{N} \xi_k \xi_\ell (e_k - e_\ell) = 0 \ .\end{aligned}$$

4. Proof of Theorem 2.2

According to Proposition 3.3, we can take a vector field $\widetilde{V}$ of the form $(*)$ coinciding with V on S where $\delta_1, \dots, \delta_N : \mathbb{R}^{N-1} \to \mathbb{R}$ are polynomials of degree less than that of V. Consider a point $p := (\xi_1, \dots, \xi_N) \in S$. Necessarily $\xi_N = 1 - \xi_1 - \cdots - \xi_{N-1} \geq 0$ and $\xi_1, \dots, \xi_{N-1} \geq 0$. We have $V(p) = 0$ iff

$$\xi_k \Big[\delta_k(\xi_1, \dots, \xi_{N-1}) - \sum_{\ell=1}^{N-1} \xi_\ell \delta_\ell(\xi_1, \dots, \xi_{N-1})\Big] = 0 \quad (k = 1, \dots, N-1).$$

Assume these equations hold with $\xi_1,\ldots,\xi_N>0$, that is $p\in S_N$. Then $\delta_1(\xi_1,\ldots,\xi_{N-1})=\cdots=\delta_{N-1}(\xi_1,\ldots,\xi_{N-1})=\sum_{\ell=1}^{N-1}\xi_\ell\delta_\ell(\xi_1,\ldots,\xi_{N-1})$. However, by writing δ for the common value of the $\delta_k(\xi_1,\ldots,\xi_{N-1})$, we have $\delta=\sum_{\ell=1}^{N-1}\xi_\ell\delta$, that is $\xi_N\delta=\bigl(1-\sum_{\ell=1}^{N-1}\xi_\ell\bigr)\delta=0$ and $\delta=0$.

Assume finally that $K<N$ and $\xi_1,\ldots,\xi_K>0=\xi_{K+1}=\cdots=\xi_N$, that is $p\in S_K$. Then $V(p)=0$ iff

$$\delta_k(\xi_1,\ldots,\xi_K,0,\ldots,0)=\sum_{\ell=1}^{K}\xi_\ell\delta_\ell(\xi_1,\ldots,\xi_K,0,\ldots,0)\qquad(k=1,\ldots,K).$$

Again the $\delta_k(\xi_1,\ldots,\xi_K,0,\ldots,0)$ assume a common value δ. However, in this case $\sum_{\ell=1}^{K}\xi_\ell=1$ and hence δ may be arbitrary for $V(p)=0$. $\square$

5. Proof of Theorem 2.3

First we check that $\sum_{k=1}^N\bigl(\sum_{i=1}^N g(i)x_i-g(k)\bigr)x_ke_k=\sum_{k=1}^N g(k)Z_k$ on S. Indeed, given any index m, from the fact that $\sum_{i=1}^N x_i=1$ on S, it follows

$$\begin{aligned}
&\Bigl\langle\sum_{k=1}^N g(k)Z_k\ ,\ e_m\Bigr\rangle=\sum_{k=1}^N g(k)\langle Z_k,e_m\rangle=\\
&=\sum_{k=1}^N g(k)\Bigl\langle x_k\sum_{i=1}^N x_i(e_i-e_k),e_m\Bigr\rangle=\sum_{i,k=1}^N g(k)x_kx_i\langle e_i,e_m\rangle-\sum_{i,k=1}^N g(k)x_kx_i\langle e_k,e_m\rangle=\\
&=\sum_{k=1}^N g(k)x_kx_m-g(m)x_m\sum_{i=1}^N x_i=\\
&=\Bigl(\sum_{i=1}^N g(i)x_i-g(m)\Bigr)x_m=\Bigl\langle\sum_{k=1}^N\Bigl(\sum_{i=1}^N g(i)x_i-g(k)\Bigr)x_ke_k\ ,\ e_m\Bigr\rangle\,.
\end{aligned}$$

Since, in general, (real-)linear combinations of complete vector fields are complete vector fields (see e.g. [1]), and since the Z_k are complete in S, the field $V=\sum_{k=1}^N g(k)Z_k-W$ is complete in S iff W is complete in S. As we have seen, the polynomial vector field W is complete in S iff $\bigl\langle W,\sum_{k=1}^N e_k\bigr\rangle=0$ and $\langle W(x_1,\ldots,x_N),e_k\rangle=0$ whenever $x_k=0$ for some index k and $\sum_{i=1}^N x_i=1$. It is well known that, by its construction, $\bigl\langle V(x_1,\ldots,x_N),\sum_{i=1}^N e_i\bigr\rangle=0$ and hence $\bigl\langle W(x_1,\ldots,x_N),\sum_{i=1}^N e_i\bigr\rangle=0$ if $\sum_{i=1}^N x_i$ even in the case if V is not complete in S. Fix any index k. By the definition $W:=\sum_{m=1}^N\sum_{i,j=1}^N w(i,j)x_ix_j\bigl[\sum_{\ell=1}^N M(i,j,\ell)\varepsilon(i,j,\ell,m)-x_m\bigr]e_m$ we have $\langle W(p),e_k\rangle=0$ for all points $p:=(\xi_1,\ldots,\xi_N)$ with $\xi_k=0$ and $\sum_{i=1}^N\xi_i=1$ if and only if

$$\sum_{\substack{i,j=1\\ i,j\neq k}}^N w(i,j)\xi_i\xi_j\sum_{\ell=1}^N M(i,j,\ell)\varepsilon(i,j,\ell,k)=0\qquad\text{if }\sum_{\substack{i=1\\ i\neq k}}^N\xi_i=1\,.$$

By elementary properties of bilinear forms, this latter relation holds iff

$$w(i,j)\sum_{\ell=1}^{N} M(i,j,\ell)\varepsilon(i,j,\ell,k)=0 \qquad \text{if } i,j\neq k\,.$$

Since $N\geq 3$, the field W has this property for all indices $k=1,\ldots,N$ iff all these terms vanish and hence $W=0$. Thus V is complete in S iff $W=0$ that is $V=\sum_{k=1}^{N} g(k)Z_k$ on S. In this case, the equation $V(\xi_1,\ldots,\xi_N)=0$ with $(\xi_1,\ldots,\xi_N)\in S$ means

$$\xi_k\big(\sum_{i=1}^{N} g(i)\xi_i-g(k)\big)=0 \qquad (k=1,\ldots,N)$$

along with the conditions $\xi_1+\cdots+\xi_N=1$ and $\xi_1,\ldots,\xi_N\geq 0$. Consider a point $(\xi_1,\ldots,\xi_N)\in S$ and write $J:=\{j:\ \xi_j>0\}$. Then $\xi_k=0$ for $k\notin J$ and hence $\sum_{j\in J}\xi_j=1$ and $V(\xi_1,\ldots,\xi_N)=0$ iff $g(k)=\sum_{i=1}^{N}\xi_i g(i)=\sum_{j\in J}\xi_j g(j)$ for the indices $k\in J$. By writing $\gamma:=\sum_{j\in J}\xi_j g(j)$ for the common value of the $g(k)$ with $(k\in J)$, we see that $V(\xi_1,\ldots,\xi_N)=0$ for any $\xi_1,\ldots,\xi_N)\in S\cap\bigcap_{j\in J}(x_j>0)\cap\bigcap_{i\notin J}(x_i=0)$. This completes the proof. $\square$

REFERENCES

[1] F. Brickell - R.S. Clark, Differentiable Manifolds, Van Nostrand Reinhold Co., London, 1970.

[2] L. Hatvani, *A modification of Tusnády's modell for genenetical evolution,* preprint, 2001.

[3] J. Hofbauer – K. Sigmund, Evolutionary Games and Replicator Dynamics, Cambridge Univ. Press, Cambridge, 1998.

[4] L.L. Stachó, *On nonlinear projections of vector fields,* NLA98: Convex analysis and chaos (Sakado, 1998), 47–54, Josai Math. Monogr., 1, Josai Univ., Sakado, 1999.

[5] G. Tusnády, *Mutation and selection* (Hungarian), Magyar Tudomány, **7** (1997), 792-805.

Address: Department of Mathematics, Alfatex University, Tripoli, Libya
Email: nuri_mofideh@math.u-szeged.hu