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POINTWISE PERIODIC SELF-MAPS OF SUBSPACES OF 2-DIMENSIONAL MANIFOLDS

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ABSTRACT

Let M be a 2-dimensional closed manifold, orientable or non-orientable. The construction of every compact locally connected subspace X of M without cut-points is analyzed. It is proved that every orientation-preserving (or reversing, or relatively preserving) pointwise periodic continuous self-map of X can be extended to a periodic self-homeomorphism of M (or of a 2-dimensional compact submanifold of M). In addition, every orientation-preserving (or reversing, or relatively preserving) pointwise periodic continuous self-map f of any path-connected subspace of M is proved to be a periodic self-homeomorphism, the number of the shorter-periodic points of f is shown to be finite, and generalization of Weaver's conclusion is given.

Key words: pointwise periodic self-map, pseudo-open disc, left (right) side of a directed arc, orientation-preserving (reversing) map, lift of a homeomorphism.

Periodic motions in spaces are objects of study that are universally existing and draw a widespread attention. The periodic self-homeomorphism is one of the significant forms in periodic motion. It is well known that every pointwise periodic self-homeomorphism on a connected topological manifold must be periodic[1,2]. Kerékjártó pointed out that every periodic self-homeomorphism on the disc is topologically conjugate to a rotation or a reflection[3,4]. Weaver[5] solved a problem posed by Epstein[6]. He proved that a pointwise periodic self-homeomorphism of a compact connected subset of a 2-dimensional orientable manifold is periodic under some stronger conditions, and that the number of its shorter-periodic points is finite. Recently, periodic differentiable homeomorphisms on surfaces and pointwise periodic self-maps of some spaces have been carrying on uninterruptedly[7-9].

In this paper we analyze the orientation-preserving (or reversing, or relatively preserving) pointwise periodic continuous self-maps of any compact, locally connected subspace of a 2-dimensional manifold M without cut points, and find an internal relation between the above self-maps and the periodic self-homeomorphisms of M (or of some 2-dimensional submanifold of M). In addition, we discuss self-maps of path-connected subspaces of M and give generalization of Weaver's conclusion.

I. Characteristics of Compact, Locally Connected Subspaces of 2 Manifolds Without Cut Points

It is well known that every 2-dimensional manifold can be endowed with the differentiable structure and can be imbedded in the Euclidean space R^3 or R^4. Hence we might as well assume that in this paper M is a given 2-dimensional orientable or non-orientable smooth closed manifold in R^3 or R^4. For any $x, y \in M$, let $d(x,y)$ (or $d_M(x,y)$) denote the length of the shortest arc on M joining x to y, and let

$$B_{xr} = \{z \in M : d(z,x) \leqslant r\}, \quad (\forall r > 0).$$

For any $Y \subset M$, we write $\mathring{Y} = Y - \partial Y$, where ∂Y denotes the boundary of Y in M if Y is not an arc, and the two end points of Y if Y is an arc (or an open arc). A point $y \in Y$ is called a cut point[10] of Y if the number of connected components of $Y - \{y\}$ is greater than that of Y. For any path-connected set W in Y, we write

$$G_y(W) = W \cup (\cup \{D : D \text{ is a disc in } Y \text{ and } \partial D \subset W\}).$$

For any arc L and $\{x, y\} \subset L$, we denote by $[x, y]_L$ (or $(x, y)_L$) the subarc (or the open subarc) of L with end points x and y.

Definition 1.1. Let U be an open set in M. If $\overline{U}$ is a disc and $\partial U = \partial \overline{U}$, then we call U an open disc. If there exists a non-injective continuous map h from the unit disc B^2 in the complex plane onto U and an $\varepsilon > 0$ such that

(i) $h|\mathring{B}^2$ is a homeomorphism from $\mathring{B}^2$ onto U,

(ii) $h|A$ is also a homeomorphism from A onto $h(A)$ for each arc A on ∂B^2 with length not greater than ε, then we call U a pseudo-open disc, and h a pseudo-disc projection.

Lemma 1.1. *Let X be a subspace of the sphere S^2 containing more than one point. If X is compact, connected, locally connected and has no cut points, then every connected component of $S^2 - X$ is an open disc, and for any $\varepsilon > 0$, the number of the connected components of $S^2 - X$ with diameters greater than ε is finite.*

Proof. We notice that X has no cut points. By the proposition (2.3) in Chap. 6 of Ref. [10], it is easy to see that the boundary of any connected component U of $S^2 - X$ is a circle, and hence U is an open disc. Furthermore, by Theorem (4.4) in the same chapter mentioned above it follows that X has some properties analogous to the E-continuum. Therefore, the number of the connected components of $S^2 - X$ with diameters greater than ε is finite. The proof is finished.

Now we still consider the given smooth closed manifold M in R^3 or R^4. Take a positive constant $\delta_0 = \delta_0(M)$ such that for any $\varepsilon \in (0, 9\delta_0]$ and $x \in M$ it holds that: (i) $B_{x\varepsilon}$ is a disc in M; (ii) if y and $z \in B_{x\varepsilon}$, then the shortest arc A_{yz} in M joining y to z is unique, and $\mathring{A}_{yz} \subset \mathring{B}_{x\varepsilon}$. For any compact set $X \subset M$, if every connected component of $M - X$ is simply connected, then we say that X is genus-full in M. For any connected component U of $M - X$ and any $x \in \partial U$, if x is a boundary point of only one (or more than one) connected component of $U \cap B_{x\delta_0}$, then we call x a unilateral (or multilateral) boundary point.

Theorem 1.1. *Let X be a compact, connected, locally connected and genus-full subspace of M without cut points. Suppose that X contains more than one points. Then*

(1) *every connected component of $M - X$ is an open disc or a psuedo-open disc;*

(2) *there is at most a finite number of connected components of $M - X$ which are pseudo-open discs;*

(3) *for any $\varepsilon > 0$, the number of the connected components of $M - X$ with diameters greater than ε is finite.*

Proof. (1) Let U be a connected component of $M - X$. Choose a finite number of points $x_1, \cdots, x_n$ from ∂U such that $\bigcup_{i=1}^{n} \mathring{B}_{x_i\delta_0}$ covers ∂U. For $i \in Z_n \equiv \{1, \cdots, n\}$, since X is locally path-connected, the connected components of $U \cap \mathring{B}_{x_i,3\delta_0}$ intersecting $B_{x_i\delta_0}$ are of a finite number. Suppose they are $U_{i1}, U_{i2}, \cdots, U_{im_i}(m_i \geqslant 1)$. Since X has no cut point, by Lemma 1.1 we know that ∂U_{ij} is a circle ($\forall j \in Z_{m_i}$), and the connected components (where every component is an open arc) of $\partial U_{ij} - \partial B_{x_i,3\delta_0}$ intersecting $B_{x_i,2\delta_0}$ are of a finite number, as well. Suppose they are $L_{ij1}, L_{ij2}, \cdots, L_{ij\tau_{ij}}(\tau_{ij} \geqslant 1)$. Obviously, $\partial U = \bigcup_{i=1}^{n} \bigcup_{j=1}^{m_i} \bigcup_{k=1}^{\tau_{ij}} L_{ijk}$. Therefore, if U has no multilateral boundary point, then ∂U is a 1-dimensional compact manifold. Since U is simply connected, ∂U is connected. Hence $\partial U = \partial \overline{U}$ is a circle, and U is an open disc.

Now we consider the case that U has multilateral boundary points. Let

$$\delta_{ij} = \min\{\delta_0, d(L_{ijk} \cap B_{x_i,2\delta_0}, L_{ijl} \cap B_{x_i,2\delta_0}): 1 \leqslant k < l \leqslant \tau_{ij}\},$$

$$\delta_1 = \min\{\delta_{ij}/9: j \in Z_{m_i}, i \in Z_n\}.$$

Because U is simply connected, we can choose a disc $D' \subset U$ such that the distance $\varepsilon_w \equiv d(w, D')$ from any point w of U to D' is less than δ_1. Write $B_w = B_{w,\varepsilon_w/2}$. Choose a finite number of points $w_1, \cdots, w_\lambda$ in ∂U such that $\{\mathring{B}_{w_1}, \cdots, \mathring{B}_{w_\lambda}\}$ covers ∂U. Let V be the connected component of $U - \bigcup_{\alpha=1}^{\lambda} B_{w_\alpha}$ containing D', and $D = \overline{V}$.

Then D is a disc with piecewise smooth boundary, and $\partial D = \partial V \subset \bigcup_{\alpha=1}^{\lambda} \partial B_{w_\alpha}$. Take a subset Y_0 of ∂D with a finite number of points such that every connected component of $\partial D \cap \partial B_{w_\alpha}$ contains at least one point of Y_0 if $\partial D \cap \partial B_{w_\alpha} \neq \emptyset$. Suppose that the points of Y_0 are $y_0, y_1, \cdots, y_{\mu-1}, y_\mu = y_0$ in the order of some assigned orientation $\boldsymbol{\partial D}$ of the circle ∂D. For each $\beta \in Z_\mu$, choose $\alpha(\beta) \in Z_\lambda$ such that $y_\beta \in \partial B_{w_{\alpha(\beta)}}$. Let $A^{(\beta)}$ be the shortest arc on M joining y_β to $W_{\alpha(\beta)}$, z_β be the nearest point in $A^{(\beta)} \cap \partial U$ to y, and $A_\beta = [y_\beta, z_\beta]_{A^{(\beta)}}$. Write $\alpha(0) = \alpha(\mu)$, $z_0 = z_\mu$, and $A_0 = A_\mu$. Obviously, $\mathring{A}_1$,

$\cdots,\mathring{A}_\mu$ are all contained in U, and they are pairwise disjoint. If $z_\gamma = z_\beta$ for some $\gamma \in Z_\mu - \{\beta\}$, then we can make a small movement of z_γ or z_β such that $z_\gamma \neq z_\beta$ (while z_γ and z_β still are points in ∂U, A_γ and A_β still are the shortest arcs joining y_γ with z_γ and y_β with z_β, respectively, and $\mathring{A}_\gamma \cup \mathring{A}_\beta \subset U$). Hence we can assume that $A_1,\cdots,A_\mu$ are also pairwise disjoint. For $\beta \in Z_\mu$, let L_β be the arc on ∂D from $y_{\beta-1}$ to y_β along the orientation $\boldsymbol{\partial D}$. Write $\Gamma_\beta = A_{\beta-1} \cup L_\beta \cup A_\beta$. Since L_β consists of arcs of at most two circles in $\{\partial B_{w_\alpha}: \alpha \in Z_\lambda\}$, the diameter of $\Gamma_\beta <$ the sum of the diameters of four discs in $\{B_{w_\alpha}: \alpha \in Z_\lambda\} < 4\delta_1 < \delta_0/2$. Choose $i(\beta) \in Z_n$ such that $z_\beta \in \mathring{B}_{x_{i(\beta)}\delta_0}$. By the definition of δ_1 it follows that there exist $j(\beta) \in Z_{m_{i(\beta)}}$ and $k(\beta) \in Z_{r_{i(\beta)},j(\beta)}$ such that the two end points $z_{\beta-1}$ and z_β of the arc Γ_β are in the same connected component $L_{i(\beta),j(\beta),k(\beta)}$ of $\partial U_{i(\beta),j(\beta)} \cap \mathring{B}_{x_{i(\beta)},3\delta_0}$. Denote by Ω_β the subarc of the open arc $L_{i(\beta),j(\beta),k(\beta)}$ from $z_{\beta-1}$ to z_β. Then $\Gamma_\beta \cup \Omega_\beta$ is a circle in $\mathring{B}_{x_{i(\beta)},3\delta_0}$. Let $D_{\beta 1}$ be the closed domain in $B_{x_{i(\beta)},3\delta_0}$ surrounded by this circle. Then $D_{\beta 1}$ is a disc. Choose a point $v_0 \in \mathring{D}$. Make arcs $P_0,P_1,\cdots,P_{\mu-1}$, $P_\mu = P_0$ on D such that $\partial P_\beta = \{v_0, y_\beta\}, \mathring{P}_\beta \subset \mathring{D} (\forall \beta \in Z_\mu)$, and $\mathring{P}_1,\cdots,\mathring{P}_\mu$ are pairwise disjoint. Evidently, $P_{\beta-1} \cup L_\beta \cup P_\beta$ is a circle on D. Suppose the closed domain in D surrounded by this circle is $D_{\beta 2}$. Then $D_{\beta 2} \cap D_{\beta 1} = L_\beta$. Let $D_\beta = D_{\beta 1} \cup D_{\beta 2}$. Then D_β is also a disc. For any $b \in Z$, let

$$\Delta_b = \{re^{t\cdot\sqrt{-1}} \in B^2: r \in [0,1], t \in [2(b-1)\pi/\mu, 2b\pi/\mu]\},$$

$$Q_b = \Delta_b \cap \Delta_{b+1},\ \Omega'_b = \overline{\partial\Delta_b - Q_{b-1} - Q_b}.$$

Clearly, we can define a continuous map $h: B^2 \to \overline{U}$ such that $h|\Delta_\beta$ is a homeomorphism from Δ_β to D_β, and

$$h(Q_\beta) = P_\beta \cup A_\beta,\ h(\Omega'_\beta) = \Omega_\beta, \forall \beta \in Z_\mu.$$

It is easy to check that h satisfies all conditions in Definition 1.1. Hence in this case U is a pseudo-open disc.

(2) and (3). From Lemma 1.1 we can deduce easily that the number of the connected components of $M - X$ with diameters greater than any given positive ε is finite. In addition, since X has no cut-points, by Lemma 1.1 we know that every connected component of $M - X$ with diameter less than $9\delta_0$ is an open disc. Hence $M - X$ at most contains a finite number of connected components which are pseudo-open discs. The proof of Theorem 1.1 is complete.

Remark 1.1. Let X be a compact, locally connected subspace of M without cut point. Even if X is not connected, we can apply Theorem 1.1 to describe the structural characteristics of every connected component of X, and thereby know those of X itself. Moreover, suppose X is connected but not genus-full. Then we can take a non-nulhomotopic circle C in a non-simply-connected connected component U of $M - X$ and do surgery[11] along C. Thus we can obtain a manifold with smaller genus but still containing X. Therefore, the two conditions "X is connected and is genus-full in M" in Theorem 1.1 can be removed in fact (of course, while these two conditions are removed, the conclusions of the theorem need to be revised accordingly).

II. Pointwise Periodic Self-Maps of Compact, Locally Connected Subspaces of 2 Manifolds Without Cut Points

Let the 2-dimensional manifold M, $d=d_M$ and the positive constant δ_0 be the same as in the previous section. When M is orientable, we assume that an orientation of M has been assigned, and that the positive direction of the boundary ∂D of any disc D in M is derived from the orientation of D by the boundary operation. For the sake of imagery, we say the positive direction of ∂D defined above to be in a counter clockwise sense.

Definition 2.1. Suppose that M is orientable, A is an arc on M with diameter not greater than $8\delta_0, \partial A=\{u_0, v_0\}$, and $\boldsymbol{A}$ is the directed arc formed by A with a direction (take u_0 as the initial point, and v_0 the terminal point). Let $y\in M-A, d(y,A)=r_0\leqslant\delta_0$, and $d(y,\partial A)=r_1>r_0$. If the condition

(i)$_t$ $[x, z]_A\subset\mathring{B}_{yr_1}$, for any $\{x,z\}\subset A\cap\partial B_{yt}$

holds for $t=r_0$, then there is $r\in(r_0, r_1)$ such that the condition (i)$_r$ also holds, and we can choose a connected component L of $A-\partial B_{yr}$ such that $d(y, L)=r_0$. Suppose $\partial L=\{u,v\}$, where $u\in(u_0,v)_A$. Denote by W the connected component of $\mathring{B}_{yr}-[u,v]_A$ containing the point y. If the directed arc $[u,v]_{\boldsymbol{A}}$ is in a counter clockwise (or clockwise) sense in the circle ∂W, then we say that the point y is on the left (or right) side of the directed arc $\boldsymbol{A}$.

We can easily show that, in Definition 2.1, it is independent of the selections of r and L whether the point y is on the left (or right) side. If in the previous definition, the condition

(ii) $r_1\geqslant d(u_0,v_0)/4$, and $r_0\leqslant\delta$ for some $\delta>0$

holds also, then we say that y is on the δ-left (or δ-right) side of the middle of $\boldsymbol{A}$. Obviously, if y is on the left (or right) side of $\boldsymbol{A}$ and $d(y, A)\leqslant\delta$, then y is on the δ-left (or δ-right) side of the middle of some directed subarc of $\boldsymbol{A}$.

Definition 2.2. Suppose M is orientable, $X\subset M$, the number of the path connected components of X is finite, and $f: X\to M$ is a continuous injection. We say that f is orientation-preserving (or reversing) if for any directed arc $\boldsymbol{A}$ in X with diameter not greater than $8\delta_0$, there exists a positive number $\delta=\delta(A)$ such that when a point y of $X-A$ is on the δ-left side of the middle of $\boldsymbol{A}$, $f(y)$ is on the left (or right) side of the directed arc $f(\boldsymbol{A})$.

Obviously, Definition 2.2 is a generalization of the usual definition of orientation-preserving (and reversing) self-homeomorphisms of a 2-dimensional orientable manifold.

Theorem 2.1. *Suppose that M is orientable, and X is a compact, connected, locally connected and genus-full subspace of M without cut points. Then a map $f: X\to X$ can be extended to an orientation-preserving (or reversing) periodic self-homeomorphism of M if and only if f is an orientation-preserving (or reversing) pointwise periodic continuous self-map of X.*

Proof. The necessity of the condition is evident. In the following we consider only the sufficiency.

We see first that the pointwise periodic self-map f is surely injective and surjective. Moreover, X is a compact Hausdoff space. Hence f is a homeomorphism.

If X contains a circle only, then M is a sphere. In this case, the theorem holds obviously. Now we assume that X contains more than one circles. Let the connected components of $M - X$ with a finite or infinite number be $W_1, \cdots, W_n, U_1, U_2, \cdots$, where $W_1, \cdots, W_n$ are pseudo-open discs, and $U_1, U_2, \cdots$ are open discs. We may consider only the case $n > 0$. For any directed arc $\boldsymbol{L}$ on ∂W_i, whose corresponding undirected arc is L, if there exists an open set V in M containing $\mathring{L}$ such that every point in V on the left (or right) side of $\boldsymbol{L}$ is in W_i, then we call $\boldsymbol{L}$ a counter clockwise (or clockwise) regular (directed) arc on ∂W_i. Let $B_i = B^2 \times \{i\}$. For each $i \in Z_n$, by Theorem 1.1 we can choose a pseudo disc projection $h_i: B_i \to \overline{W}_i$ as described in Definition 1.1 and take a positive ε_0 independent of i such that

(i) If A is an arc on ∂B_i with length not greater than ε_0, then $h_i | A$ is a homeomorphism from A onto $h_i(A) \subset \partial W_i$. Furthermore, let $\boldsymbol{A}$ be the directed arc formed by A together with the counter clockwise sence of ∂B_i. Then $h_i(\boldsymbol{A})$ is a counter clockwise regular arc on ∂W_i.

(ii) For any counter clockwise (or clockwise) regular arc $\boldsymbol{L}$ on ∂W_i, there exists a unique counter clockwise (or clockwise) directed arc $\boldsymbol{A}$ on ∂B_i satisfying $h_i(\boldsymbol{A}) = \boldsymbol{L}$.

We write the directed arc $\boldsymbol{A}$ mentioned in the property (ii) by $\boldsymbol{A} = h_i^{-1}(\boldsymbol{L})$. It should be noticed that the inverse image $h_i^{-1}(L)$ of the arc L corresponding to $\boldsymbol{L}$ may contain not only the arc A corresponding to $\boldsymbol{A} = h_i^{-1}(\boldsymbol{L})$ since $h_i | \partial B_i$ is not injective. Because f is orientation-preserving (or reversing), for any counter clockwise regular are $\boldsymbol{L}$ on ∂W_i, $f(\boldsymbol{L})$ is a counter clockwise (or clockwise) regular arc on the boundary $\partial W_{j(i)}$ of some pseudo-open disc $W_{j(i)}$. Thus the homeomorphism $f|\partial W_i$: $\partial W_i \to \partial W_{j(i)}$ can be lifted to $\tilde{f}_i: \partial B_i \to \partial B_{j(i)}$, where the definition of $\tilde{f}_i$ is given as follows: For any $z_0 \in \partial B_i$, take arbitrarily a counter clockwise directed arc $\boldsymbol{A}$ on ∂B_i with the initial point z_0 and with length not greater than ε_0, and let $\tilde{f}_i(z_0)$ be the initial point of the directed arc $h_{j(i)}^{-1} f h_i(\boldsymbol{A})$. Obviously, if $\boldsymbol{A}_1$ is a directed subarc of $\boldsymbol{A}$ and the initial point of $\boldsymbol{A}_1$ is also z_0, then $h_{j(i)}^{-1} f h_i(\boldsymbol{A}_1)$ and $h_{j(i)}^{-1} f h_i(\boldsymbol{A})$ have the same initial point. Hence the definition of $\tilde{f}_i(x)$ is independent of the length of the taken directed arc $\boldsymbol{A}$. Since $\tilde{f}_i(\boldsymbol{A})$ $(\equiv \{\tilde{f}_i(z): z \in A\}$, with the initial point $\tilde{f}_i(z_0)) = h_{j(i)}^{-1} f h_i(\boldsymbol{A})$, $\tilde{f}_i$ is continuous. By the 1-1 correspondence between the set $\{\boldsymbol{L}\}$ of counter clockwies regular arcs on ∂W_i and the set $\{f(\boldsymbol{L})\}$ of counter clockwise (or clockwise) regular arcs on $\partial W_{j(i)}$, and by the above properties (i) and (ii), we see that $\tilde{f}_i$ is both injective and surjective. Hence $\tilde{f}_i$ is a homeomorphism, satisfying $h_{j(i)} \tilde{f}_i = f h_i | \partial B_i$.

Let the extension $\tilde{F}_i: B_i \to B_{j(i)}$ of $\tilde{f}_i$ be defined by

$$\tilde{F}_i(re^{t\cdot\sqrt{-1}}) = r \cdot \tilde{f}(e^{t\cdot\sqrt{-1}}), \ \forall r \in [0,1], t \in R,$$

where for simplicity, we denote points of B_i and $B_{j(i)}$ by complex numbers in B^2.

Clearly, $\tilde{F}_i$ is a homeomorphism. Define an extension F_i: $\overline{W}_i \to \overline{W}_{j(i)}$ of $f|\partial W_i = h_{j(i)}\tilde{f}_i h_i^{-1}|\partial W_i$ by $F_i = h_{j(i)}\tilde{F}_i h_i^{-1}$. From the properties of pseudo disc projection described in Definition 1.1, we know that F_i is a homeomorphism.

Let $B' = \bigcup_{i=1}^{n} B_i$, $\partial B' = \bigcup_{i=1}^{n} \partial B_i$, and $W' = \bigcup_{i=1}^{n} \overline{W}_i$. Define the self-homeomorphisms $\tilde{\varphi}$, $\tilde{\Phi}$ and Φ of $\partial B', B'$ and W' by $\tilde{\varphi}|\partial B_i = \tilde{f}_i$, $\tilde{\Phi}|B_i = \tilde{F}_i$ and $\Phi|\overline{W}_i = F_i$ $(\forall i \in Z_n)$, respectively. Then Φ is an extension of $\varphi \equiv f|\left(\bigcup_{i=1}^{n} \partial W_i\right)$. By the pointwise periodicity of φ, it is easy to check that the lift $\tilde{\varphi}$ of φ is also pointwise periodic, and hence we can further derive that $\tilde{\varphi}$ is periodic. Thus $\tilde{\Phi}$ and Φ are periodic.

Similarly (in fact it is simplier), let $U' = U_1 \cup U_2 \cup \cdots$. We can extend $\psi \equiv f|\partial U'$ to $\Psi: \overline{U}' \to \overline{U}'$ and make Ψ be pointwise periodic. Let $F: M \to M$ be $F|X = f$, $F|W' = \Phi$ and $F|\overline{U}' = \Psi$. Then F is a pointwise periodic self-homeomorphism of M. By the Montgomery theorem[2] it follows that F is periodic. This completes the proof of Theorem 2.1.

Now we consider a 2-dimensional non-orientable closed manifold M_q of genus q (for any $q \in Z_+$). It is well known that M_q has a 2-sheeted orientable covering space H_{q-1} of genus $q-1$ (see [11, p. 234]). Concretely speaking, if in R^3 we center-symmetrically add an even number of handles $(2 \cdot [(q-1)/2]$ handles) to the sphere or the torus with the origin O as the center, then the resulting 2-dimensional orientable closed manifold can be taken as H_{q-1}. Identify every pair of symmetric points x and $-x$ in H_{q-1}. Then the resulting identification space can be taken as M_q, and the relevant identification map π: $H_{q-1} \to M_q$ is a 2-sheeted covering map.

Definition 2.3. Let $\pi: H_{q-1} \to M_q$ be as stated above, $X \subset M_q$, $\tilde{X} = \pi^{-1}(X)$, and let f be a continuous self-map of X. If f can be lifted to an injective continuous self-map $\tilde{f}$ of $\tilde{X}$ such that $\pi\tilde{f} = f\pi|\tilde{X}$ and $\tilde{f}$ is orientation-preserving, then we say that f is relatively (to the covering map π) orientation-preserving (or orientation-relatively-preserving).

In the definition, f has an orientation-preserving lift $\tilde{f}$ if and only if f has an orientation-reserving lift $\tilde{g}$ ($\tilde{g}$ and $\tilde{f}$ can be determined each other by $\tilde{g}(x) = -\tilde{f}(x)$ $(\forall x \in \tilde{X})$). Hence we need not define the conception "relatively orientation-reversing".

Theorem 2.2. *Let X be a compact, connected, locally connected and genus-full subspace of M_q without cut points, and f a continuous self-map of X. Then f can be extended to a periodic self-homeomorphism of M_q if and only if f is a relatively orientation-preserving pointwise periodic self-map.*

Proof. Since any self-homeomorphism of M_q can be lifted to an orientation-preserving self-homeomorphism of H_{q-1}, the necessity of the condition in the theorem is clear. Now we prove the sufficiency of the condition. Let $\tilde{X} = \pi^{-1}(X)$. From the properties of X it is easy to check that $\tilde{X}$ is also a compact, connected, locally connected and genus-full subspace of H_{q-1} without cut points. By Definition 2.3, f can be lifted to an orientation-preserving continuous self-map $\tilde{f}$ of $\tilde{X}$. Evidently, $\tilde{f}$ has symmetry, that is, $\tilde{f}(-x) = -\tilde{f}(x)$ for any $x \in X$. From the pointwise periodicity of f it can be deduced that $\tilde{f}$ is also pointwise periodic. Hence, using Theorem 2.1, we can extend $\tilde{f}$ to a periodic self-homeomorphism $\tilde{F}$ of H_{q-1}, and by the symmetry of $\tilde{f}$ we can make $\tilde{F}$ also symmetric. Let $F = \pi\tilde{F}\pi^{-1}$. Then F is an extension of f and a periodic self-homeomorphism of M_q. The proof of Theorem 2.2 is complete.

Now we consider a pointwise periodic continuous self-map φ on a space Y with n connected components $Y_1,\cdots,Y_n$. For any $i \in Z_n$, choose $m(i) \in Z_n$ such that $\varphi^{m(i)}(Y_i) = Y_i$ and $\varphi^k(Y_i) \neq Y_i$ for $1 \leqslant k < m(i)$. Clearly, in the sense of topologcal equivalence, $\varphi\left|\left(\bigcup_{k=1}^{m(i)} \varphi^k(Y_i)\right)\right.$ is determined by $\varphi^{m(i)}|Y_i$. Thus, by Remark 1.1, from Theorems 2.1 and 2.2 we can immediately derive the following

Theorem 2.3. *Let Y be a compact, locally connected subspace of a 2-dimensional closed manifold M without cut points, and φ a continuous self-map of Y. Then the following conditions (1) and (2) are equivalent (in which the word "orientation-preserving (or reversing)" is used in the case that M is orientable, and the word "orientation-relatively-preserving" is used in the case that M is non-orientable).*

(1) *φ is an orientation-preserving (or reversing or relatively-preserving) pointwise periodic self-map.*

(2) *φ can be extended to an orientation-preserving (or reversing or relatively-preserving) periodic self-homeomorphism of some 2-dimensional compact submanifold M_0 of M containing Y (where M_0 may contain some boundary circles and may contain more than one connected components).*

Remark 2.1. For clarifying the construction of every orientation-preserving (or reversing or relatively preserving) pointwise periodic continuous self-map of any compact, locally connected subspace X without cut points of a 2-dimensional closed manifold M, we only need to make a further research about the corresponding periodic self-homeomorphism of M (or of a compact submanifold M_0 of M) by the above three theorems.

III. Pointwise Periodic Self-Maps of Path-Connected Subspaces of 2 Manifolds

In this section we still discuss the pointwise periodic self-map f on a subspace X of a 2-dimensional manifold M. But here we only require X to be path-connected, and do not require X to be compact, locally connected and without cut points again. For any $n \in Z_+$, in the following we write $P_n(f) = \{x \in X$: the (minimum) period of x under f is $n\}$, and $P_n^\sigma(f) = \bigcup_{i=1}^{n} P_i(f)$. Let $P_0(f) = P_0^\sigma(f) = \varnothing$. If $P_n(f) \neq \varnothing$, then we call any point in $P_{n-1}^\sigma(f)$ a shorter-periodic point.

Lemma 3.1. *Suppose M is a 2-dimensional, orientable, smooth, closed manifold, δ_0 is the same as mentioned in Sec. 1, X is a subset of M with diameter ρ, f is an orientation-preserving pointwise periodic self-map of X, and A is an arc in X with end points u and v. If $u\in P_\lambda(f)$, $\mathring{A}\subset P_m(f)$, and $m>\lambda\geqslant 1$, then*

(1) *there exists some $\delta>0$ such that $B_{u\delta}\cap P^\sigma_{m-1}(f)=\{u\}$;*

(2) *let positive numbers $\varepsilon_1<\varepsilon_2<\varepsilon_3<\varepsilon_4<\min\{\delta_0,\rho\}/2$ be taken such that $d(f^i(x),f^i(y))<\varepsilon_{\alpha+1}/2$ for any $\alpha\in Z_3$, $i\in Z_m$ and $\{x,y\}\subset B_{u\varepsilon_\alpha}\cap X$. Then $v\in P_m(f)$ if $A\subset B_{u\varepsilon_1}$.*

Proof. (1) If $\lambda>1$, then we can consider another map $f'=f^\lambda$. Thus we might as well assume that $\lambda=1$. It is easy to show that there exists an arc A_0 in

$$\bigcup_{i=1}^{m} f^i(A-\{v\})$$

such that $u\in\partial A_0$ and $A_1-\{u\},\cdots,A_m-\{u\}$ are pairwise disjoint, where we write $A_j=f^j(A_0)$ for any $j\in Z$. Since f is orientation-preserving, there exist sufficiently small positive numbers $\varepsilon>\delta$ such that for any $i,j\in Z_m$, $k\in Z_{m-1}$ and $y\in B_{u\delta}\cap X$, when y is on the left side of the directed arc $\overrightarrow{A_i\cup A_{i+k}}$ (the direction is from A_i to A_{i+k}), $f^j(y)$ is a point in $B_{u\varepsilon}$ on the left side of $\overrightarrow{A_{i+j}\cup A_{i+k+j}}$. It follows that $B_{u\delta}\cap P^\sigma_{m-1}(f)=\{u\}$.

(2) Suppose $v\in P_\mu(f)$. Let τ be the least common multiple of λ and μ. Then λ, μ and τ are all factors of m. If $\mu<m$, then there are two cases:

a) $\tau<m$. In this case we put

$$L=A,\ m'=m/\tau,\ g=f^\tau,\ v'=v.$$

b) $\tau=m$ and $\mu<m$. In this case, let L be an arc joining u to $f^\mu(u)$ in $A\cup f^\mu(A)$, and let

$$m'=m/\lambda,\ g=f^\lambda,\ v'=f^\mu(u).$$

Then, in both cases a) and b), we have $m'>1$, $\mathring{L}\subset P_{m'}(g)$, $\partial L=\{u,v'\}\subset P_1(g)$, and $L\subset B_{u\varepsilon_2}$ (derived by $A\subset B_{u\varepsilon_1}$). Similarly to the proof of the conclusion (1) mentioned above, we can choose an arc A_0 in $\bigcup_{i=1}^{m'} g^i(L)$ such that $\partial A_0=\{u,v'\}$, and $\mathring{A}_1\cdots$, $\mathring{A}_{m'}$ are pairwise disjoint, where $A_j\equiv g^j(A_0)$ for any $j\in Z$. Since $A_1\cup\cdots\cup A_{m'}\subset B_{u\varepsilon_3}\subset B_{u\delta_0}$, we see that $D_1\equiv G_{B_{u\delta_0}}(A_1\cup\cdots\cup A_{m'})$ is a disc, and $\partial D_1=A_\beta\cup A_{\beta+\gamma}$ for some β and $\gamma\in Z_{m'-1}$. Since g is orientation-preserving, there exists $\alpha\in Z_{m'}-\{\beta\}$ such that $D_2\equiv G_{B_{u\delta_0}}(A_\alpha\cup A_{\alpha+\gamma})\subset D_1$ and $g^{\beta-\alpha}(X\cap D_2)=X-\mathring{D}_1$. Thus, from $X\cap D_2\subset B_{u\varepsilon_3}$ it follows that $X-\mathring{D}_1\subset B_{u\varepsilon_4}$, and hence $X\subset B_{u\varepsilon_4}$. But this contradicts the given condition $2\varepsilon_4<\rho$. Therefore it cannot hold that $\mu<m$. The proof of Lemma 3.1 is complete.

Lemma 3.2. *Let $X\subset M\subset R^3$ and $f:X\to X$ be the same as in the above lemma. Suppose that A is an arc in X. Then there exists $m\in Z_+$ such that $A\subset P^\sigma_m(f)$ and $A-P_m(f)$ at most contains a finite number of points.*

Proof. Let $V_n = \mathring{A} - \overline{(A - P_n(f))}$ for $n \geqslant 0$. By Lemma 3.1, we know that $T_0 - V_n$ is a finite set of points for any connected component T_0 of $\overline{V}_n$.

If $V_n = \emptyset$ for any $n \in Z_+$, then each $P_n^\sigma(f)$ does not contains any subarc of A yet. Therefore, there exists a subarc L_n of A in $A - P_n^\sigma(f)$ such that $L_1 \supset L_2 \supset L_3 \supset \cdots$, and hence we obtain a nonempty set $\sum_{n=1}^{\infty} L_n$ of non-periodic points of f. But this contradicts the condition of the lemma. Thus there exists $m = m_0 \in Z_+$ such that $V_m \neq \emptyset$ and $V_i = \emptyset$ for $i < m$.

Let A_0 be a non-trivial connected component of $\overline{V}_m$. If $A_0 \neq A$, then by the same reason we know that there exists $m_1 > m_0$ such that $V_{m_1} \neq \emptyset$. Let A_1 be a non-trivial connected component of $\overline{V}_{m_1}$. By (1) of Lemma 3.1 we see that $A_1 \cap A_0 = \emptyset$, and that there exists an open subarc K_1 of A between A_0 and A_1 satisfying

(i) $K_1 \cap P_{m_0}^\sigma(f) = \emptyset$;

(ii) $K_1 \not\subset P_{m_1}^\sigma(f)$;

(iii) K_1 and A_1 have a common end point.

From the three properties we can deduce again and again that there exist a strictly increasing sequence $m_1, m_2, m_3, \cdots$ of integers, a non-trivial connected component A_n of each $\overline{V}_{m_n}$, and an open subarc K_n of A between A_{n-1} and A_n such that

$$A_n \cap K_{n-1} \neq \emptyset, \quad K_n \cap P_{m_{n-1}}^\sigma(f) = \emptyset, \quad K_n \not\subset P_{m_n}^\sigma(f),$$

and K_n and A_n have exactly a common end point for $n \geqslant 2$. But in this case we also obtain a nonempty set $\bigcap_{n=1}^{\infty} K_n \left(= \bigcap_{n=1}^{\infty} \overline{K}_n!\right)$ of non-periodic points of f, and it leads to a contradiction. Hence we have $A_0 = A$.

Finally, as indicated above, from $A_0 = A$ it follows at once that $A - P_m(f) \subset A_0 - V_m$ is a finite set of points. Lemma 3.2 is proved.

Theorem 3.1. *Suppose M is a 2-dimensional orientable manifold with genus $q \geqslant 0$, X is a path-connected subspace of M, and f is an orientation-preserving pointwise periodic self-map of X. Then f is a periodic self-homeomorphism, and all shorter-periodic points are isolated.*

Proof. Take an arc $A \subset X$ and let $m \in Z_+$ be as mentioned in Lemma 3.2. For any $x \in X$, take another arc A' in X containing both x and a subarc of A. Applying Lemma 3.2 to A', we know that $A' \subset \overline{P_m(f)}$. Thus f is a periodic self-homeomorphism with periodic m. In addition, by Lemma 3.1 we see that all shorter-periodic points are isolated. Theorem 3.1 follows.

Theorem 3.2. *Let X be a path-connected subspace of a 2-dimensional orientable manifold M. Then every orientation-reversing pointwise periodic self-map g of X is a periodic self-homeomorphism. Furthermore, suppose that the period of g is $2m$. Then all shorter-periodic points of g are isolated for even m, and all shorter-periodic points of g except those with periods m are also isolated for odd m.*

Proof. Applying Theorem 3.1 to $f=g^2$, we obtain Theorem 3.2 at once.

Theorem 3.3. *Let X be a path-connected subspace of the 2-dimensional non-orientable manifold M_q. Then every relatively orientation-preserving pointwise periodic self-map f of X is periodic. Furthermore, suppose the period of f is m. Then all shorter-periodic points of f are isolated if m is odd, and all shorter-periodic ponits of f except those with periods $m/2$ are also isolated if m is even.*

Proof. Let $\pi: H_{q-1}\to M_q$ and $\widetilde{X}=\pi^{-1}(X)$ be as described in Sec. II. Lift f to an orientation-preserving continuous map $\tilde{f}:\widetilde{X}\to\widetilde{X}$. Then $\tilde{f}$ is also pointwise periodic. By Theorem 3.1, $\tilde{f}$ is periodic, and shorter-periodic points of $\tilde{f}$ are isolated. Thus, from $f=\pi\tilde{f}\pi^{-1}|X$ we see that Theorem 3.3 holds.

Remark 3.1. It will be proved[1)] that, even if X is not compact, the numbers of the shorter-periodic points mentioned in the above three theorems are all still finite. For example, in Theorem 3.1, we can further obtain that the number of points of P^{σ}_{m-1} is not greater than $m+12q$.

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