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THE NORM OF A SYMMETRIC ELEMENTARY OPERATOR

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ABSTRACT. The norm of the operator $x \mapsto a^*xb + b^*xa$ on $A = B(\mathcal{H})$ (or on any prime C*-algebra A) is computed for all $a, b \in A$ and is shown to be equal to the completely bounded norm.

1. INTRODUCTION

Given a C*-algebra A, an operator on A of the form

$$T : A \to A, \quad Tx = \sum_{j=1}^{n} a_j x b_j \ (x \in A),$$

where $a_j, b_j \in A$ are fixed, is called an *elementary operator* and the smallest n for which T can be expressed in such a form is the *length* of T. Sometimes the norm of such an operator is equal to the completely bounded norm, hence, to the Haagerup norm of the corresponding tensor $\sum a_i \otimes b_i$ (see [10] and [5]), but in general there is no simple formula known for computing the norm of an elementary operator even if $A = B(\mathcal{H})$, the algebra of all bounded operators on a complex Hilbert space $\mathcal{H}$ (see [8] for a survey). Although the case of generalized derivations ($x \mapsto ax + xb$) on $B(\mathcal{H})$ was already settled by Stampfli [13] more than thirty years ago (see [2] for more), a slightly more general operator

$$T_{a,b} : B(\mathcal{H}) \to B(\mathcal{H}), \quad T_{a,b}x = axb + bxa$$

still presents a problem. Contrary to what one might expect from automatic complete positivity of positive elementary operators of length two (see [14] and the references there) by an analogy, the norm of $T_{a,b}$ can be different from the completely bounded norm. It was conjectured by Mathieu [7] that $\|T_{a,b}\| \geq \|a\|\|b\|$ for all $a, b \in B(\mathcal{H})$, and if a and b are selfadjoint, this was confirmed by Stachó and Zalar [12], but for general a, b this is still open and the best estimate known seems to be the one in [11].

In this note we shall deduce a formula for the norm of the operator $T_{a,b}$ when a and b are selfadjoint. (The restriction $T_{a,b}|B(\mathcal{H})_{sa}$ is a special case of the Jacobson-McCrimmon operator.) More generally, we shall prove the following.

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To simplify the notation, put

$$p=\frac{1}{2}t(s+\frac{1}{s}),\ q=\frac{1}{2}(s-\frac{1}{s}),\ r=\frac{1}{2t}(s+\frac{1}{s})$$

and let

$$\Lambda=\{(p,q,r)\in\mathbb{R}^3:\ p>0,\ r>0,\ pr-q^2=1\}.$$

Note that the map $(t,s)\mapsto(p,q,r)$ from $\mathbb{R}^+\times\mathbb{R}^+$ on Λ is surjective (in fact bijective). Furthermore, for each

$$\lambda=(p,q,r)\in\Lambda,\quad \text{let}\quad c_\lambda=pa^*a+rb^*b+2q\mathrm{Im}(a^*b). \tag{2.5}$$

Then from the above computation we have

$$\|S_{a,b}|\mathrm{B}(\mathcal{H})_{sa}\|=\sup_{\|\xi\|=1}\inf_{\lambda\in\Lambda}\langle c_\lambda\xi,\xi\rangle. \tag{2.6}$$

We note that c_λ is positive since

$$c_\lambda=p(a-i\frac{q}{p}b)^*(a-i\frac{q}{p}b)+\frac{1}{p}b^*b. \tag{2.7}$$

Moreover, denoting $m=\inf_{t\in\mathbb{R}}\|a-itb\|$, (2.7) implies that

$$\|c_\lambda\|\geq p\|a-i\frac{q}{p}b\|^2\geq pm^2. \tag{2.8}$$

Hence $\|c_\lambda\|\to\infty$ as p (or, similarly, r) tends to ∞ if $m\neq 0$. Since $pr-q^2=1$ for all $\lambda=(p,q,r)\in\Lambda$, it follows by a standard compactness argument that there exists a $\lambda_0\in\Lambda$ such that $\|c_{\lambda_0}\|=\inf_{\lambda\in\Lambda}\|c_\lambda\|$ if a and ib are linearly independent over $\mathbb{R}$.

We shall use the usual notation $w(a)$ for the numerical radius of an operator $a\in\mathrm{B}(\mathcal{H})$. It is well known that $w(a)=\|a\|$ if a is selfadjoint. The following lemma implies that (at least if $\mathcal{H}$ is finite dimensional) we may interchange "$\sup_\xi$" and "$\inf_\lambda$" in (2.6).

Lemma 2.3. *Suppose that a and ib are linearly independent over $\mathbb{R}$ and that $\mathcal{H}$ is finite dimensional. Then there exist $\lambda_0=(p_0,q_0,r_0)\in\Lambda$ and a unit vector $\xi\in\mathcal{H}$ such that*

$$w(c_{\lambda_0})=w_0:=\min_{\lambda\in\Lambda}w(c_\lambda),$$

$$\frac{1}{r_0}\|a\xi\|^2=\frac{1}{2}w_0=\frac{1}{p_0}\|b\xi\|^2\quad\textit{and}\quad \mathrm{Im}\langle b\xi,a\xi\rangle=-\frac{1}{2}q_0w_0.$$

Moreover, $\inf_{\lambda\in\Lambda}\langle c_\lambda\xi,\xi\rangle\geq w_0$; hence

$$\sup_{\|\eta\|=1}\inf_{\lambda\in\Lambda}\langle c_\lambda\eta,\eta\rangle=\inf_{\lambda\in\Lambda}\sup_{\|\eta\|=1}\langle c_\lambda\eta,\eta\rangle.$$

Proof. We have already seen in the argument following (2.8) that there exists $\lambda_0=(p_0,q_0,r_0)\in\Lambda$ satisfying $w(c_{\lambda_0})=\inf_{\lambda\in\Lambda}w(\lambda)$. Put $c_0=c_{\lambda_0}$, $s=p_0-p$, $t=q-q_0$ and write $r=\frac{1+q^2}{p}$ as the sum of the Taylor polynomial of degree one (in s, t) plus the remainder

$$r=r_0+\frac{r_0}{p_0}s+2\frac{q_0}{p_0}t+R(s,t),\ \text{where}\ R(s,t)=\frac{p_0t^2+r_0s^2+2q_0ts}{p_0p}.$$

Then by the definition (2.5) of c_λ we have

$$c_\lambda=c_0+sd+te+R(s,t)b^*b, \tag{2.9}$$

where

$$d=\frac{r_0}{p_0}b^*b-a^*a\quad\text{and}\quad e=2\frac{q_0}{p_0}b^*b+2\mathrm{Im}(a^*b). \tag{2.10}$$

By the minimality of w_0 (and since $\dim\mathcal{H}<\infty$), for each $\lambda=(p,q,r)\in\Lambda$ there exists a unit vector $\xi_\lambda\in\mathcal{H}$ such that $\langle c_\lambda\xi_\lambda,\xi_\lambda\rangle\geq w_0$; that is,

$$\langle c_0\xi_\lambda,\xi_\lambda\rangle+s\langle d\xi_\lambda,\xi_\lambda\rangle+t\langle e\xi_\lambda,\xi_\lambda\rangle+R(s,t)\|b\xi_\lambda\|^2\geq w_0\geq\langle c_0\xi_\lambda,\xi_\lambda\rangle, \tag{2.11}$$

where λ depends on (s,t). For a fixed $(u,v)\in\mathbb{R}^2$, we may replace in (2.11) (s,t) by $(u/n,v/n)$ $(n\in\mathbb{N})$ to get a unit vector $\xi_n=\xi_{\lambda(u/n,v/n)}$ such that

$$\langle c_0\xi_n,\xi_n\rangle+\frac{u}{n}\langle d\xi_n,\xi_n\rangle+\frac{v}{n}\langle e\xi_n,\xi_n\rangle+R(\frac{u}{n},\frac{v}{n})\|b\xi_n\|^2\geq w_0\geq\langle c_0\xi_n,\xi_n\rangle. \tag{2.12}$$

By choosing a subsequence, we may assume that the vectors ξ_n converge to some unit vector $\xi_{u,v}\in\mathcal{H}$. From (2.12) we first conclude that $\lim\langle c_0\xi_n,\xi_n\rangle=w_0$, hence

$$\langle c_0\xi_{u,v},\xi_{u,v}\rangle=w_0 \tag{2.13}$$

and then (since $R(s/n,t/n)$ converges to 0 as $1/n^2$) that

$$u\langle d\xi_{u,v},\xi_{u,v}\rangle+v\langle e\xi_{u,v},\xi_{u,v}\rangle\geq 0. \tag{2.14}$$

Denote by $\mathcal{K}$ the eigenspace of c_0 corresponding to the eigenvalue w_0. Then it follows from (2.13) that $\xi_{u,v}\in\mathcal{K}$. Moreover, since the spatial numerical range W of the compression of $d+ie$ onto $\mathcal{K}$ is convex (see [3]) and compact (since $\dim\mathcal{K}<\infty$), it follows from (2.14) that there exists a unit vector $\xi\in\mathcal{K}$ such that

$$\langle d\xi,\xi\rangle=0\quad\text{and}\quad\langle e\xi,\xi\rangle=0. \tag{2.15}$$

(Otherwise, for some linear functional ω on the plane containing W we would have that $\omega(x,y)<0$ for all $x+iy\in W$; but since ω is necessarily of the form $\omega(x,y)=ux+vy$ for some $(u,v)\in\mathbb{R}^2$, this would contradict (2.14).) Now it follows from (2.15) and the definition (2.10) of d and e that

$$p_0\|a\xi\|^2=r_0\|b\xi\|^2\quad\text{and}\quad p_0\mathrm{Im}\langle b\xi,a\xi\rangle=-q_0\|b\xi\|^2. \tag{2.16}$$

Since $\langle c_0\xi,\xi\rangle=w_0$ and $c_0=p_0a^*a+r_0b^*b+2q_0\mathrm{Im}(a^*b)$ by (2.5), we have that

$$p_0\|a\xi\|^2+r_0\|b\xi\|^2+2q_0\mathrm{Im}\langle b\xi,a\xi\rangle=w_0,$$

from which we compute by using (2.16) and the identity $p_0r_0-q_0^2=1$ that

$$\|a\xi\|^2=\frac{r_0}{2}w_0,\ \|b\xi\|^2=\frac{p_0}{2}w_0\ \text{and}\ \mathrm{Im}\langle b\xi,a\xi\rangle=-\frac{q_0}{2}w_0. \tag{2.17}$$

Finally, for each $\lambda=(p,q,r)\in\Lambda$ we compute, using (2.9), (2.15) and $c_0\xi=w_0\xi$ that

$$\langle c_\lambda\xi,\xi\rangle=w_0+R(s,t)\|b\xi\|^2\geq w_0, \tag{2.18}$$

since $R(s,t)\geq 0$. The inequality $\sup_{\|\eta\|=1}\inf_{\lambda\in\Lambda}\langle c_\lambda\eta,\eta\rangle\leq\inf_{\lambda\in\Lambda}\sup_{\|\eta\|=1}\langle c_\lambda\eta,\eta\rangle$ is a tautology, while the reverse inequality follows from (2.18) since $w_0=\inf_{\lambda\in\Lambda}w(c_\lambda)$. □

Proof of Theorem 1.1. First assume that a and ib are linearly independent over $\mathbb{R}$. Since $\|S_{a,b}|\mathrm{B}(\mathcal{H})_{sa}\|\leq\|S_{a,b}\|\leq\|S_{a,b}\|_{\mathrm{cb}}$ and $\|S_{a,b}\|_{\mathrm{cb}}$ is equal to the Haagerup norm of the tensor $\tau:=a^*\otimes b+b^*\otimes a$ (see [10]), it suffices to prove that $\|\tau\|_h\leq\|S_{a,b}|\mathrm{B}(\mathcal{H})_{sa}\|$. Put $E=S_{a,b}|\mathrm{B}(\mathcal{H})_{sa}$. Observe that for all $\alpha,\beta,\gamma\in\mathbb{R}$ satisfying $\alpha\gamma-\beta^2=1$ we can write τ as

$$\tau=(-i\beta a^*+\gamma b^*)\otimes(\alpha a-i\beta b)+(\alpha a^*+i\beta b^*)\otimes(i\beta a+\gamma b).$$

Hence (after a short computation),

$$\|\tau\|_h \leq \|(\alpha^2+\beta^2)a^*a + (\gamma^2+\beta^2)b^*b + 2\beta(\alpha+\gamma)\mathrm{Im}(a^*b)\|.$$

Observe that for all (p,q,r) as in Theorem 1.1 (that is, $(p,q,r) \in \Lambda$) we can find α,β,γ so that $\alpha^2+\beta^2=p$, $\beta^2+\gamma^2=r$ and $\beta(\alpha+\gamma)=q$. (To show this, the reader may assume, by replacing a and b with ta and $\frac{1}{t}b$ for a suitable t, that $r=p$, which simplifies the computation.) It follows that

$$\|\tau\|_h \leq \inf_{(p,q,r)\in\Lambda} \sup_{\|\xi\|=1} \langle pa^*a + rb^*b + 2q\mathrm{Im}(a^*b)\xi, \xi\rangle = \inf_{\lambda\in\Lambda} \sup_{\|\xi\|=1} \langle c_\lambda\xi,\xi\rangle. \tag{2.19}$$

If $\mathcal{H}$ is finite dimensional, Lemma 2.3 implies that we can interchange the "inf" and "sup" in (2.19), and then the right side of (2.19) coincides with the right side of (2.6). Thus $\|\tau\|_h = \|E\|$ if $\mathcal{H}$ is finite dimensional.

If $\mathcal{H}$ is infinite dimensional, choose an increasing net of finite rank projections P_ν converging to the identity. We shall continue to use the notation c_λ and $w_0 = \inf_{\lambda\in\Lambda} w(c_\lambda)$ from the proof of Lemma 2.3. Since for $\nu \geq \nu_0$ we have that $\|tP_\nu bP_\nu\| \geq \|tP_{\nu_0}bP_{\nu_0}\| \to \infty$ as $|t|\to\infty$ if ν_0 is large enough, and similarly for a in place of b, we may assume (replacing the net by the subnet $\nu \geq \nu_0$) that for some positive constant κ we have

$$\inf_{t\in\mathbb{R}} \|P_\nu(a-itb)P_\nu\| > \kappa \quad\text{and}\quad \inf_{t\in\mathbb{R}} \|P_\nu(b-ita)P_\nu\| > \kappa$$

for all ν. Then by the same reasoning as that leading to (2.8), we have that $w(P_\nu c_\lambda P_\nu) \geq \max\{p,r\}\kappa^2$ for all ν and $\lambda \in \Lambda$. Hence (since $pr - q^2 = 1$) there exists a compact subset Ω of Λ such that

$$w(P_\nu c_\lambda P_\nu) > w_0$$

for all ν if $\lambda \in \Lambda\backslash\Omega$. For each ν let λ_ν be such that $w(P_\nu c_{\lambda_\nu} P_\nu) = \inf_{\lambda\in\Lambda} w(P_\nu c_\lambda P_\nu)$. Since $w(P_\nu c_{\lambda_\nu} P_\nu) \leq w_0$, all λ_ν are in Ω. Hence, by compactness and choosing a subnet, we may assume that the net (λ_ν) converges to some $\lambda_0 \in \Omega$. Then from (2.19),

$$\|\tau\|_h \leq \inf_{\lambda\in\Lambda} w(c_\lambda) \leq w(c_{\lambda_0}) = \lim \|c_{\lambda_\nu}\|. \tag{2.20}$$

But $\lim\|c_{\lambda_\nu}\| = \lim\|P_\nu c_{\lambda_\nu} P_\nu\|$ since the net (c_{λ_ν}) converges in norm and (P_ν) converges strongly to the identity. Moreover, by the already proved finite-dimensional case and the choice of λ_ν we have that $\|P_\nu c_{\lambda_\nu} P_\nu\| = \|E_\nu\|$, where E_ν is the operator on $\mathrm{B}(P_\nu\mathcal{H})$ defined by $E_\nu(x) = P_\nu a^* P_\nu x P_\nu b P_\nu + P_\nu b^* P_\nu x P_\nu a P_\nu$. Since clearly $\|E_\nu\| \leq \|E\|$ for each ν, we finally conclude from (2.20) that $\|\tau\|_h \leq \|E\|$.

It remains to consider the case when a and ib are linearly dependent over $\mathbb{R}$, say $b = tia$ for some $t\in\mathbb{R}$. Then $S_{a,b}=0$ and the theorem reduces to the identity

$$\inf\{|p+t^2r+2qt| :\ p,q,r\in\mathbb{R},\ p,r>0,\ pr-q^2=1\} = 0.$$

To verify this identity, just put $q=-rt$, $p=r^{-1}(1+r^2t^2)$ and let $r\to\infty$. □

Remark 2.4. By an application of the Kaplansky density theorem, the conclusions of Theorem 1.1 can be extended to the operator $S_{a,b}$ acting on any irreducible C*-subalgebra of $\mathrm{B}(\mathcal{H})$. In fact, the theorem can be extended to any prime C*-algebra A since each separable subalgebra of A is contained in a separable prime C*-subalgebra A_0 of A and A_0 has a faithful irreducible representation. It is not known to the author, however, to what kind of more general C*-algebras can Theorem 1.1 be extended if scalars are replaced by central elements of the corresponding multiplier algebras.

Since the formula in Theorem 1.1 may not be easy to apply in practical computations, we now give a simple estimate for the norm of $S_{a,b}$.

Let us denote by γ_a the minimal modulus of an operator $a \in \mathrm{B}(\mathcal{H})$, that is, the smallest point in the spectrum of $|a|$.

Corollary 2.5. *For all $a,b\in\mathrm{B}(\mathcal{H})$ the estimate*

$$\max\{\|a\|\inf_{t\in\mathbb{R}}\|b-tia\|, \|b\|\inf_{t\in\mathbb{R}}\|a-tib\|\} \leq \|S_{a,b}\| \leq 2\sqrt{\|a\|^2\|b\|^2 - \gamma^2_{\mathrm{Im}(a^*b)}}$$

holds.

Proof. The inequality $\|S_{a,b}\| \leq 2\sqrt{\|a\|^2\|b\|^2-\gamma^2_{\mathrm{Im}(a^*b)}}$ follows immediately from Theorem 1.1 and the identity (2.3). To prove the other inequality, note that from (2.7) we have for each $\xi\in\mathcal{H}$ (denoting $t=q/p$),

$$\langle c_\lambda\xi,\xi\rangle = p\|(a-tib)\xi\|^2 + \frac{1}{p}\|b\xi\|^2 \geq 2\|(a-tib)\xi\|\|b\xi\|. \tag{2.21}$$

Now observe that $\sup_{\|\xi\|=1}\|c\xi\|\|d\xi\| \geq \frac{1}{2}\|c\|\|d\|$ for all $c,d\in\mathrm{B}(\mathcal{H})$. (By the polar decomposition it suffices to prove this for positive c and d with norm 1. Then, if $\mathcal{H}$ is finite dimensional, choose a unit vector ξ such that the projections of ξ to the eigenspaces of c and d corresponding to the eigenvalue 1 both have lengths at least $1/\sqrt{2}$ and note that $\|c\xi\|\|d\xi\| \geq 1/2$. If $\mathcal{H}$ is infinite dimensional, use an approximate version of this argument.) Applying this to (2.21) we get

$$\sup_{\|\xi\|=1}\langle c_\lambda\xi,\xi\rangle \geq \|a-tib\|\|b\|;$$

hence by Theorem 1.1,

$$\|S_{a,b}\| = \inf_{\lambda\in\Lambda}\sup_{\|\xi\|=1}\langle c_\lambda\xi,\xi\rangle \geq \inf_{t\in\mathbb{R}}\|a-tib\|\|b\|.$$

Since we can interchange the roles of a and b, this concludes the proof. □

Theorem 1.1 shows that $\|S_{a,b}\|_{\mathrm{cb}} = \|S_{a,b}|\mathrm{B}(\mathcal{H})_{sa}\|$, but to compute the norm it is usually more convenient to use the formula (2.3) from Theorem 2.2.

Example 2.6. If u and v are isometries, then

$$\|S_{u,v}\| = 2\sqrt{1-\gamma^2_{\mathrm{Im}(u^*v)}};$$

hence, if in addition u or v is a unitary, $\|S_{u,v}\| = 2\|\mathrm{Re}(u^*v)\|$.

Indeed, the first equality follows immediately from (2.3) and Theorem 1.1. If, say, u is a unitary, then $w=u^*v$ is an isometry. If w contains the unilateral shift as a direct summand, then the spectra of $\mathrm{Im}(w)$ and $\mathrm{Re}(w)$ are $[-1,1]$. Hence $\gamma_{\mathrm{Im}(w)}=0$ and $\|S_{u,v}\| = 2 = 2\|\mathrm{Re}(w)\|$. On the other hand, if w does not contain the unilateral shift as a direct summand, then w is a unitary (see [3]) and the functional calculus shows that $\sqrt{1-\gamma_w^2} = \|\mathrm{Re}(w)\|$.

Note that $S_{u,v}(1) = 2\mathrm{Re}(u^*v)$; hence $S_{u,v}$ attains its norm at the identity operator if u and v are unitaries.

It may also be interesting to observe that in the case of isometries the upper bound in Corollary 2.5 agrees with the norm of $S_{u,v}$, while the lower bound is $\frac{1}{2}\|S_{u,v}\|$ (after a short calculation). There are, however, examples showing that the lower bound cannot be improved in general. Take, for instance, two non-zero orthogonal projections e, f with $ef = 0$; using (2.3) one can compute that $\|S_{e,f}\| = 1$, which in this case agrees with the lower bound in Corollary 2.5.

Note. After this paper was submitted for publication, we received two preprints from Richard M. Timoney, Trinity College Dublin, in which the relation between the completely bounded norm and the k-norms of elementary operators is investigated; his results also show that $\|S_{a,b}\|_{\text{cb}} = \|S_{a,b}\|$.

Note added in proof. Mathieu's conjecture mentioned in the introduction has been recently confirmed by Blanco, Boumazgour and Ransford and independently by Timoney.

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BACKWARD UNIQUENESS FOR SOLUTIONS OF LINEAR PARABOLIC EQUATIONS

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(Communicated by David S. Tartakoff)

Abstract. We address the backward uniqueness property for the equation $u_t - \Delta u = w_j \partial_j u + vu$ in $\mathbb{R}^n \times (T_0, 0]$. We show that under rather general conditions on v and w, $u|_{t=0} = 0$ implies that u vanishes to infinite order for all points $(x, 0)$. It follows that the backward uniqueness holds if $w = 0$ and $v \in L^\infty([0, T_0], L^p(\mathbb{R}^n))$ when $p > n/2$. The borderline case $p = n/2$ is also covered with an additional continuity and smallness assumption.

1. Introduction

A question of a solution semigroup of an equation on $\mathbb{R}^n$,

$$u_t - \Delta u + f(x, u, \nabla u) = 0,$$

being one-to-one is clearly reduced to the following problem: If a solution u of

$$u_t - \Delta u + w_j \partial_j u + vu = 0 \tag{1.1}$$

on $\mathbb{R}^n \times (-T_0, 0]$ vanishes for $t = 0$, does it necessarily vanish for $t < 0$? This question of backward uniqueness was raised by Lax [L] and addressed by various authors ([L], [P], [LP], [G], [AN1], [AN2]). The approach by weighted inequalities, resembling the Carleman inequality method, has been employed by Lees and Protter [P], [LP]. In particular, the backward uniqueness property was proved for the differential inequality

$$(u_t - \Delta u)^2 \leq \frac{C}{t^2} u^2 + C|\nabla u|^2.$$

When comparing with the equation (1.1), this covers the case $v \in L^\infty((T_0, 0), L^{p_1}(\mathbb{R}^n))$ and $w \in L^\infty((T_0, 0), L^{p_2}(\mathbb{R}^n))$ when $p_1 \geq n$ and $p_2 = \infty$. The logarithmic convexity approach, introduced by Agmon and Nirenberg [AN1], [AN2] and used in [A], [BT], [G], is a lot simpler. However, it leads to the same range of exponents p_1, p_2 (cf. [G]). Based on forward uniqueness results, it seems that backward uniqueness should hold when $p_1 \geq n/2$ and $p_2 \geq n$ with sufficiently small norms when $p_1 = n/2$ and $p_2 = n$.

Recently, Escauriaza and Vega (cf. [E], [EV]) showed that if u, a solution of (1.1), vanishes of infinite order in space-time at $(x, t) = (0, 0)$, and if $w = 0$ and $v \in L^\infty((T_0, 0), L^{p_1}(\mathbb{R}^n))$ where $p_1 \geq n/2$ (with sufficiently small norm when $p_1 =$

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