If $H$ is infinite-dimensional, let $\{p_n\}$ be a net of finite rank orthogonal projections increasing to the identity. Denote by $a_n$ the restriction of $p_n a$ to the range of $p_n$, and analogously for $b$. For each $n$ let $t_n$ be such that $\min_{t>0}\|ta_n^*a_n + (1/t)b_n^*b_n\| = \|t_n a_n^* a_n + (1/t_n)b_n^*b_n\|$. Then we have

$$\begin{aligned}\|S_{a,b}\| &= \sup_{\|x\|=1}\|a^*xb + b^*xa\| \geqslant \sup_{\|x\|=1}\|a^*p_nxp_nb + b^*p_nxp_na\| \\ &\geqslant \sup_{\|x\|=1}\|(p_na^*p_n)(p_nxp_n)(p_nbp_n) + (p_nb^*p_n)(p_nxp_n)(p_nap_n)\| \\ &= \Big\|t_na_n^*a_n + \frac{1}{t_n}b_n^*b_n\Big\|.\end{aligned}$$

Passing to a subnet, if necessary, assume that $t_n \to t_0$. Then

$$\lim_n\Big\|t_na_n^*a_n + \frac{1}{t_n}b_n^*b_n\Big\| = \Big\|t_0a^*a + \frac{1}{t_0}b^*b\Big\| \geqslant \min_{t>0}\Big\|ta^*a + \frac{1}{t}b^*b\Big\|.$$

Hence,

$$\|S_{a,b}\| \geqslant \min_{t>0}\Big\|ta^*a + \frac{1}{t}b^*b\Big\|.$$

Since the reverse inequality is clear, the theorem is proved. □

**PROPOSITION 4.3.** *Let $R_{a,b} : \mathcal{B}(H) \to \mathcal{B}(H)$ be (real) linear mapping defined by $R_{a,b}(x) = a^*xb + b^*x^*a$. Then*

$$\|R_{a,b}\| = \min_{t>0}\Big\|ta^*a + \frac{1}{t}b^*b\Big\|.$$

PROOF: The proof is very similar to the previous one, so we shall skip the details. Choose a unit vector $\xi$ satisfying the condition (10) in Lemma 4.1 and a unitary operator $x$ such that $xb\xi = a\xi$. Then $\|R_{a,b}\| \geqslant \langle (a^*xb + b^*x^*a)\xi, \xi\rangle = w(a^*a + b^*b) = \|a^*a + b^*b\|$. For the reverse inequality note that

$$\|R_{a,b}(x)\| = \left\|\begin{bmatrix} a^* & b^* \\ 0 & 0\end{bmatrix}\begin{bmatrix} x & 0 \\ 0 & x^*\end{bmatrix}\begin{bmatrix} b & 0 \\ a & 0\end{bmatrix}\right\| \leqslant \|a^*a + b^*b\|\,\|x\| \quad (x \in \mathcal{B}(H)).$$

□

**PROPOSITION 4.4.** *Let $a, b \in \mathcal{B}(H)$ be self-adjoint. If $H$ is real or $\dim H = 2$, then*

$$\|T_{a,b}\| = \|T_{a,b}|_{\mathcal{B}(H)_{sa}}\| = \min_{t>0}\Big\|ta^2 + \frac{1}{t}b^2\Big\|.$$

PROOF: If $H$ is real this is just Theorem 4.2 for real scalars. So let $H$ be complex with $\dim H = 2$. Choose an orthonormal basis $\{\eta_1, \eta_2\}$ of $H$ relative to which $a$ is diagonal. Since $b$ is self-adjoint, the diagonal entries of $b$ are real, and the two (in general complex conjugate) off-diagonal entries of $b$ can be made real by replacing $\eta_2$ with $\theta\eta_2$ for an appropriate scalar $\theta$ of modulus 1. Thus, we may assume that $a$ and $b$ are real matrices. As in the proof of Theorem 4.2, we may assume that $\min_{t>0}\|ta^2 + (1/t)b^2\| = \|a^2 + b^2\| = 1$. Then from Lemma 4.1 we obtain a unit vector $\xi$ such that $\|a\xi\|^2 = \|b\xi\|^2 = 1/2$. Furthermore, $\xi$ is an eigenvector of the real symmetric matrix $(a^2 + b^2)\xi = \xi$ (corresponding to the eigenvalue 1), hence $\xi$ is real. Then $\langle a\xi, b\xi\rangle \in \mathbb{R}$ and we can find a unitary self-adjoint matrix $x$ satisfying $xa\xi = b\xi$ and $xb\xi = a\xi$. The rest of the proof is the same as in Theorem 4.2 and will be omitted. □



**COROLLARY 4.5.** *If $a, b \in M_2$ are self-adjoint, then*

$$\|T_{a,b}\|_{cb} = \|T_{a,b}\|.$$

PROOF: By Proposition 4.4 we have

$$\min_{t>0}\Big\|ta^2 + \frac{1}{t}b^2\Big\| \geqslant \|T_{a,b}\|_{cb} \geqslant \|T_{a,b}\| = \min_{t>0}\Big\|ta^2 + \frac{1}{t}b^2\Big\|,$$

hence $\|T_{a,b}\|_{cb} = \|T_{a,b}\|$. □

The main result in [13] states that, whenever $a, b \in \mathcal{B}(H)$ are self-adjoint, $\|T_{a,b}|_{\mathcal{B}(H)_{sa}}\| \geqslant \|a\|\,\|b\|$. The following estimate is sharper.

**COROLLARY 4.6.** *Let $a, b \in \mathcal{B}(H)$ be self-adjoint. Then*

$$\|T_{a,b}|_{\mathcal{B}(H)_{sa}}\| \geqslant \sup_{\substack{p=p^*=p^2 \\ \mathrm{rank}(p)=2}} \min_{t>0}\Big\|t(pap)^2 + \frac{1}{t}(pbp)^2\Big\| \geqslant \|a\|\,\|b\|.$$

PROOF: The first inequality follows immediately from Proposition 4.4 since

$$\|T_{a,b}|\mathcal{B}(H)_{sa}\| \geqslant \|T_{pap,pbp} \mid \mathcal{B}(pH)_{sa}\|$$

for each projection $p \in \mathcal{B}(H)$. To prove the second inequality, we may assume that $\|a\| = \|b\| = 1$. Note that if $t \geqslant 1$ then $\|t(pap)^2 + (1/t)(pbp)^2\| \geqslant \|pap\|^2$ and $\|pap\|$ approximates $\|a\|$ when $p$ is the projection to the span of $\{\xi, a\xi\}$, where $\xi$ is a vector on which $a$ almost achieves its norm. A similar argument is available if $(1/t) \geqslant 1$. □

For $2 \times 2$ matrices we have a better estimate. Denote by $\|\cdot\|_2$ the Hilbert-Schmidt norm.

**COROLLARY 4.7.** *If $a, b \in M_2$ are self-adjoint, then*

$$\|T_{a,b}|_{(M_2)_{sa}}\| \geqslant \|a\|_2\,\|b\|_2.$$

PROOF: We may assume that $\min_{t>0}\|ta^2 + (1/t)b^2\| = \|a^2 + b^2\|$. Put $m = \|a^2 + b^2\|$. By Lemma 4.1 there exists a unit vector $\xi$ satisfying $(a^2 + b^2)\xi = \xi$ and $\|a\xi\|^2 = \|b\xi\|^2 = m/2$. Let $\xi^\perp$ be a unit vector orthogonal to $\xi$ and put $c = \|a\xi^\perp\|^2$. Since $a^2 + b^2 \leqslant m1$, we have $\|b\xi^\perp\|^2 \leqslant m - c$. From

$$\|a\|_2^2 = \|a\xi\|^2 + \|a\xi^\perp\|^2 = \frac{1}{2}m + c$$

and

$$\|b\|_2^2 = \|b\xi\|^2 + \|b\xi^\perp\|^2 \leqslant \frac{3}{2}m - c,$$

it follows that

$$\|a\|_2^2\|b\|_2^2 \leqslant \Big(\frac{1}{2}m + c\Big)\Big(\frac{3}{2}m - c\Big) \leqslant m^2 = \big\|T_{a,b}|_{(M_2)_s}\big\|^2. \qquad \square$$

Clearly, the inequality $\|T_{a,b}\| \geqslant \|a\|_2\|b\|_2$ can not be generalised to self-adjoint $n\times n$ matrices for $n > 2$. As an example, consider the $3\times 3$ diagonal matrices $a = \operatorname{diag}(1,1,0)$ and $b = \operatorname{diag}(0,0,1)$. (In this example we also have that $\|T_{a,b}\|_{cb} = 1 = \|a\|\,\|b\|$, hence the estimate in Theorem 2.1 can not be improved.)

EXAMPLE 4.8. If $H$ is complex and $\dim H > 2$, then $\min_{t>0} w(ta^2 + b^2/t)$ can be greater than $\big\|T_{a,b}|_{\mathcal{B}(H)_{sa}}\big\|$. To see this, first observe the following.

If $H$ is finite dimensional and $a, b \in \mathcal{B}(H)$ are such that

$$\big\|T_{a,b}|_{\mathcal{B}(H)_{sa}}\big\| = w(a^2 + b^2) = 1,$$

then there exists a unit vector $\xi \in H$ such that $(a^2+b^2)\xi = \xi$ and $\langle a\xi, b\xi\rangle \in \mathbb{R}$.

Indeed, choose $x = x^* \in \mathcal{B}(H)$ with $\|x\| = 1$ and a unit vector $\xi \in H$ such that $\big|\langle (axb + bxa)\xi, \xi\rangle\big| = 1$. Using the fact that equality holds in the Schwarz inequality only if the two vectors are linearly dependent, we deduce from

$$\begin{aligned} 1 = \big|\langle (axb + bxa)\xi, \xi\rangle\big| &= |\langle xb\xi, a\xi\rangle + \langle xa\xi, b\xi\rangle| \leqslant 2\,\|a\xi\|\,\|b\xi\| \\ &\leqslant \|a\xi\|^2 + \|b\xi\|^2 = \langle (a^2+b^2)\xi, \xi\rangle \leqslant 1 \end{aligned}$$

that $(a^2+b^2)\xi = \xi$ and then $\|a\xi\|^2 = \|b\xi\|^2 = 1/2$ and $xb\xi = \beta a\xi$, $xa\xi = \alpha b\xi$ for some complex numbers $\alpha$ and $\beta$ of modulus 1. Then from $|\langle xb\xi, a\xi\rangle + \langle xa\xi, b\xi\rangle| = |(1/2)\alpha + (1/2)\beta| = 1$ it follows that $\alpha = \beta$. Since $x$ is a contraction, we have $\|x(a\xi + \lambda b\xi)\| \leqslant \|a\xi + \lambda b\xi\|$ for each complex number $\lambda$. But this is equivalent to the condition $\operatorname{Re}\big(\lambda\langle a\xi, b\xi\rangle\big) \leqslant \operatorname{Re}\big(\lambda\langle b\xi, a\xi\rangle\big)$, which implies $\langle a\xi, b\xi\rangle \in \mathbb{R}$.

Now let

$$a = \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 0 & -\frac{i}{2} & \frac{1}{2} \\ \frac{i}{2} & 0 & 0 \\ \frac{1}{2} & 0 & 0 \end{bmatrix}.$$

Then

$$a^2 + b^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{3}{4} & \frac{i}{4} \\ 0 & -\frac{i}{4} & \frac{1}{4} \end{bmatrix}$$



and since the norm of $2\times 2$ matrix in the lower right corner of $a^2+b^2$ is $(2+\sqrt{2})/4 < 1$, we have $\|a^2+b^2\| = 1$. If $\big\|T_{a,b}|_{\mathcal{B}(H)_{sa}}\big\| = 1$, then by the above observation we would have $\langle a\xi, b\xi\rangle \in \mathbb{R}$, where $\xi = [1\ 0\ 0]^t$ is the only eigenvector of $a^2+b^2$ corresponding to the eigenvalue $1 = w(a^2+b^2)$. However, in our case $\langle a\xi, b\xi\rangle = -(i/2\sqrt{2}) \notin \mathbb{R}$.

In view of Corollary 4.5 one may ask if $\|T_{a,b}\|_{cb} = \|T_{a,b}\|$ for all $2\times 2$ matrices $a$ and $b$. The following example shows that this is not the case.

EXAMPLE 4.9. Put $a = e - iu$ and $b = (e+iu)/2$, where

$$e = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad u = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Let $x = [x_{ij}]$. Then

$$T_{a,b}(x) = \begin{bmatrix} x_{11} & x_{21} \\ 0 & 0 \end{bmatrix},$$

so $\|T_{a,b}\| = 1$. We shall show that $\|T_{a,b}\|_{cb} = \sqrt{2}$. First we note that $T_{a,b}(x) = axb + bxa = exe + uxu$. Furthermore, as in the proof of Theorem 2.1, denote by $\Lambda = \begin{bmatrix} \alpha & \beta \\ \overline{\beta} & \gamma \end{bmatrix}$ a positive matrix with $\det\Lambda = 1$. Let $A = |\beta|^2 + \gamma^2$, $B = \beta(\alpha+\gamma)$, $C = \alpha^2 + |\beta|^2$ and note that $\det\Lambda = 1$ is equivalent to the condition $AC - |B|^2 = 1$. Then, as in the proof of Theorem 2.1, to compute $\|T_{a,b}\|_{cb}$, that is, the Haagerup norm of $w = e\otimes e + u\otimes u$, it suffices to consider the representations of $w$ of the form

$$w = (\gamma e - \overline{\beta}u)\otimes(\alpha e + \beta u) + (-\beta e + \alpha u)\otimes(\overline{\beta}e + \gamma u).$$

Then by a short computation,

$$\|w\|_h = \inf\Big\{\big\|(A+C)e\big\|^{1/2}\big\|Ae^\perp + Ce + 2\operatorname{Re}(Bu^*)\big\|^{1/2} : AC - |B|^2 = 1\Big\},$$

where $e^\perp = 1 - e$. Furthermore,

$$Ae^\perp + Ce + 2\operatorname{Re}(Bu^*) = \begin{bmatrix} C & \overline{B} \\ B & A \end{bmatrix}$$

and the norm of the last matrix is equal to $\Big(A + C + \sqrt{(A-C)^2 + 4|B|^2}\Big)/2$. By symmetry we may assume that $A \geqslant C$, hence

$$\big\|Ae^\perp + Ce + 2\operatorname{Re}(Bu^*)\big\| \geqslant \frac{1}{2}\big(A + C + |A - C|\big) = A.$$

Therefore

$$\|T_{a,b}\|_{cb} = \|w\|_h \geqslant (A+C)^{1/2}A^{1/2} \geqslant (2AC)^{1/2} = \Big(2\big(1+|B|^2\big)\Big)^{1/2} \geqslant \sqrt{2}.$$

In fact $\|T_{a,b}\|_{cb} = \sqrt{2}$, since $\|w\|_h \leqslant \|e^2 + uu^*\|^{1/2}\,\|e^2 + u^*u\|^{1/2} = \sqrt{2}$.

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# COBOUNDARY EQUATIONS OF EVENTUALLY EXPANDING TRANSFORMATIONS

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Let $T$ be an eventually expansive transformation on the unit interval satisfying the Markov condition. Then $T$ is an ergodic transformation on $(X, \mathcal{B}, \mu)$ where $X = [0,1)$, $\mathcal{B}$ is the Borel $\sigma$-algebra on the unit interval and $\mu$ is the $T$ invariant absolutely continuous measure. Let $G$ be a finite subgroup of the circle group or the whole circle group and $\phi : X \to G$ be a measurable function with finite discontinuity points. We investigate ergodicity of skew product transformations $T_\phi$ on $X \times G$ by showing the solvability of the coboundary equation $\phi(x)g(Tx) = \lambda g(x)$, $|\lambda| = 1$. Its relation with the uniform distribution mod $M$ is also shown.

## 1. Introduction

Let $(X, \mathcal{B}, \mu)$ be a probability space and $T$ be a measure preserving transformation on $X$. A transformation $T$ on $X$ is called ergodic if the constant function is the only $T$-invariant function and it is called weakly mixing if the constant function is the only eigenfunction with respect to $T$. A measure preserving transformation $T$ is called exact if $\bigcap_{n=0}^{\infty} T^{-n}\mathcal{B}$ is the trivial $\sigma$-algebra consisting of empty set and whole set modulo measure zero sets. So exact transformation are as far from being invertible as possible. Recall that if a transformation is exact then that transformation is weakly mixing ([**11**]).

A piecewise differentiable transformation $T : [0,1) \to [0,1)$ is said to be *eventually expansive* if some iterate of $T$ has its derivative bounded away from 1 in modulus, that is, $|(T^n)'| > 1$ everywhere for some $n$. Let $\{\Delta_i\}$ be a countable (or finite) partition of the unit interval $[0,1)$ by subintervals. Suppose that an eventually expansive map $T$ on the interval $[0,1)$ satisfies

(i) $T|_{\operatorname{Int}\Delta_i}$ has a $C^2$-extension to the closure of $\Delta_i$,
(ii) $T|_{\operatorname{Int}\Delta_i}$ is strictly monotone,
(iii) $\overline{T(\Delta_i)} = [0,1]$, and in the case that the number of subintervals in the partition is infinite
(iv) $\sup_i \left\{ \sup_{x_1 \in \operatorname{Int}\Delta_i} |T''(x_1)| \Big/ \inf_{x_2 \in \operatorname{Int}\Delta_i} |T'(x_2)|^2 \right\} < \infty.$

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