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Generalized bi-circular projections

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Abstract

Recall that a projection P on a complex Banach space X is a generalized bi-circular projection if $P+\lambda(I-P)$ is a (surjective) isometry for some λ such that $|\lambda|=1$ and $\lambda\neq 1$. It is easy to see that every hermitian projection is generalized bi-circular. A generalized bi-circular projection is said to be nontrivial if it is not hermitian. Botelho and Jamison showed that a projection P on $C([0,1])$ is a nontrivial generalized bi-circular projection if and only if $P-(I-P)$ is a surjective isometry. In this article, we prove that if P is a projection such that $P+\lambda(I-P)$ is a (surjective) isometry for some λ, then either P is hermitian or λ is an nth unit root of unity. We also show that for any nth unit root λ of unity, there are a complex Banach space X and a nontrivial generalized bi-circular projection P on X such that $P+\lambda(I-P)$ is an isometry.

Keywords: Isometry; Generalized bi-circular projection

Let X be a complex Banach space and let $\mathrm{Iso}(X)$ denote the set of all surjective isometries on X. Recall that an operator $T:X\to X$ is *hermitian* if $e^{i\alpha T}\in\mathrm{Iso}(X)$ for all $\alpha\in\mathbb{R}$ [1]. It is known that a projection $P:X\to X$ is hermitian if and only if $P+e^{i\alpha}(I-P)\in\mathrm{Iso}(X)$ for all $\alpha\in\mathbb{R}$. Note that a hermitian projection is also called *bi-circular projection* [8]. Let S^1 denote the unit circle of $\mathbb{C}$. A projection P is said to be a *generalized bi-circular projection* if $P+\lambda(I-P)\in\mathrm{Iso}(X)$ for some $\lambda\in S^1\setminus\{1\}$ [3,4]. It is easy to see that if P is a generalized bi-circular projection, then $I-P$ is also a generalized bi-circular projection. A generalized bi-circular projection is said to be nontrivial if it is not hermitian. Botelho and Jamison showed that a projection P on $C([0,1])$ is a nontrivial generalized bi-circular projection if and only if $P-(I-P)$ is a surjective isometry [2]. It is natural to ask whether for any $\lambda\in S^1\setminus\{1\}$, there is a nontrivial generalized bi-circular projection P such that $P+\lambda(I-P)$ is an isometry. In this article, we show that if λ is an nth unit root of unity, then there are a complex Banach space and a nontrivial generalized bi-circular projection P on X such that $P+\lambda(I-P)$ is an isometry.

Theorem 1. *Let X be a complex Banach space and P be a projection on X. Suppose that $P+\lambda(I-P)$ is an isometry. If λ is of infinite order in $(S^1,\cdot)$, then P is hermitian.*

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Proof. Suppose that λ is of infinite order in $(S^1, \cdot)$ and P is a projection such that $P + \lambda(I - P)$ is an isometry. Then for any $n \in \mathbb{N}$,

$$P + \lambda^n(I - P) = \big(P + \lambda(I - P)\big)^n$$

is an isometry on X. Since the set $\{\lambda^n\colon n \in \mathbb{N}\}$ is dense in S^1, $P + \alpha(I - P)$ is an isometry for all $\alpha \in S^1$. This implies that P is a hermitian projection. □

Corollary 2. *Every generalized bi-circular projection is a contraction.*

Proof. Let P be a projection such that $P + \lambda(I - P)$ is an isometry for some $\lambda \in S^1 \setminus \{1\}$. If λ is of infinite order, then P is hermitian and $\|P\| \leqslant 1$. So we may assume that $\lambda^n = 1$ for some $n \in \mathbb{N}$. Note that $\sum_{j=1}^n \lambda^j = 0$. This implies that $P = \frac{1}{n}\sum_{j=1}^n (P + \lambda^j(I - P))$ and $\|P\| \leqslant 1$. □

Theorem 3. *Let n be any integer such that $n \geqslant 2$ and let $\lambda = e^{i2\pi/n}$. Then there is a complex Banach space X and a nontrivial generalized bi-circular projection P on X such that $P + \lambda(I - P)$ is an isometry on X.*

Proof. Let C be a nonempty closed convex compact subset of a Banach space and D be a nonempty closed subset of C. It is known that if x is an extreme point of C and if x is in the closed convex hull of D, then $x \in D$. Let n be any integer such that $n \geqslant 2$ and let

$$C = \big\{(\alpha, \beta) \in \mathbb{C} \oplus \mathbb{C}\colon\ |\alpha| \leqslant 1,\ |\beta| \leqslant 1\big\},$$
$$D = \big\{(\alpha, 0)\colon\ |\alpha| = 1\big\} \cup \big\{\big(\beta, \beta e^{\frac{i2k\pi}{n}}\big)\colon\ |\beta| = 1 \text{ and } k \leqslant n\big\},$$
$$B = \overline{\mathrm{co}}(D).$$

Then $B \subset C$, and $\gamma D = D$ for all $\gamma \in S^1$. Let $\|\cdot\|$ be the norm defined on $X = \mathbb{C} \oplus \mathbb{C}$ such that

$$\big\|(\alpha, \beta)\big\| = \big(\sup\big\{\gamma > 0\colon\ (\gamma\alpha, \gamma\beta) \in B\big\}\big)^{-1},$$

and let P be the projection on X defined by

$$P(\alpha, \beta) = (\alpha, 0).$$

It is known that

(1) (α, β) is an extreme point of C if and only if $|\alpha| = 1 = |\beta|$;
(2) $(1, \beta) \in D$ with $|\beta| = 1$ if and only if $\beta = e^{\frac{i2k\pi}{n}}$ for some $k \leqslant n$.

Thus $P + \gamma(I - P)$ is an isometry on X if and only if $\gamma = e^{\frac{i2k\pi}{n}}$ for some $k \leqslant n$. The proof is complete. □

Let (Ω, μ) be a σ-finite measure space and let X be a separable complex Banach space. For any $1 \leqslant p < \infty$, X is said to have the *trivial L^p-structure* if X is not p-direct sum of two nontrivial subspaces of X, i.e., $X = (Y \oplus Z)_p$ for some subspaces Y, Z of X implies $Y = X$ or $Z = X$. Let $L^p(\Omega, X)$ be all strongly measurable X-valued functions f such that

$$\int_\Omega \big\|f(t)\big\|^p\, dt < \infty.$$

Then $L^p(\Omega, X)$ is a Banach space with the norm

$$\|f\|_p = \bigg(\int_\Omega \big\|f(t)\big\|^p\, dt\bigg)^{1/p}.$$

Fix $1 \leqslant p < \infty$ and $p \neq 2$. Let (Ω, μ) be a σ-finite measure space, and let X be a complex Banach space with trivial L^p-structure. It is known that [9]:

(1) If H is a hermitian operator on $L^p(\Omega, X)$, then there is a hermitian valued strongly measurable mapping A of Ω to $\mathcal{B}(X)$ such that $(Hf)(\cdot) = A(\cdot)f(\cdot)$.
(2) If T is a surjective isometry on $L^p(\Omega, X)$, then

$$\begin{aligned}(Tf)(\cdot) &= S(\cdot)h(\cdot)\big(\Phi(f)\big)(\cdot)\\ &= S(\cdot)h(\cdot)f\big(\sigma(\cdot)\big) \quad \text{for all } f \in L^p(\Omega, X),\end{aligned}$$

where Φ is a set isomorphism of the measure space onto itself, $\sigma : \Omega \to \Omega$ a point isomorphism induced by Φ [6,7], $h = (d\mu \circ \Phi^{-1}/d\mu)^{1/p}$, and S is a strongly measurable mapping from Ω to $\mathrm{Iso}(X)$.

It is also known that the above results are still true for $p = 2$ if there is no subspace Y of X such that $\dim(Y) > 1$ and $Y = (Z \oplus \mathbb{C})_2$ [5].

Theorem 4. *Let (Ω, μ) be a σ-finite measure space, $1 \leqslant p < \infty$, $p \neq 2$, and X be a separable complex Banach space with trivial L^p-structure. Suppose that P is a generalized bi-circular projection on $L^p(\Omega, X)$. Then one of the following holds.*

(1) *$P - (I - P)$ is a reflection.*
(2) *There are $\lambda \in S^1 \setminus \{1\}$ and a strong measurable mapping A from Ω to $\mathcal{B}(X)$ such that for almost all $t \in \Omega$, $A(t)$ is a projection, $A(t) + \lambda(I - A(t))$ is an isometry, and $(Pf)(\cdot) = A(\cdot)f(\cdot)$.*

Proof. Let P be a generalized bi-circular projection on $L^p(\Omega, X)$. Suppose that $T = P + \lambda(I - P)$ is an isometry on $L^p(\Omega, X)$ for some $\lambda \in S^1 \setminus \{1\}$. Then for any $f \in L^p(\Omega, X)$,

$$(Tf)(\cdot) = S(\cdot)h(\cdot)\big(\Phi(f)\big)(\cdot) \quad \text{for all } f \in L^p(\Omega, X)$$

for some strongly measurable function S from Ω to $\mathcal{B}(X)$, Φ a set isomorphism of the measure space Ω to itself, and $h = (d\mu \circ \Phi^{-1}/d\mu)^{1/p}$. We claim that $\Phi^2 = I$.

Suppose that $\Phi^2 \neq I$. Then there is a measurable set A_1 such that $\mu(A_1) > 0$ and $A_1 \neq \Phi^2(A_1)$. Let $A_2 = A_1 \setminus \Phi^2(A_1)$ if $\mu(A_1 \setminus \Phi^2(A_1)) \neq 0$; otherwise let $A_2 = \Phi^{-2}(\Phi^2(A_1) \setminus A_1)$. Then $\mu(A_2 \cap \Phi^2(A_2)) = 0$ and $\mu(A_2) > 0$.

Note that if $A_2 \subseteq \Phi(A_2)$, then $A_2 \subseteq \Phi^2(A_2)$. So $\mu(A_2 \setminus \Phi(A_2)) \neq 0$. Let $A = A_2 \setminus \Phi(A_2)$. Then

$$\mu(A) > 0 \quad \text{and} \quad \mu\big(A \cap \Phi(A)\big) = 0 = \mu\big(A \cap \Phi^2(A)\big).$$

Note that

$$P = \frac{T - \lambda I}{1 - \lambda}, \qquad \operatorname{supp}(T1_A) = \Phi(A) \quad \text{and} \quad \operatorname{supp}\big(T^2 1_A\big) = \Phi^2(A).$$

We have

$$0 = 1_A \cdot T(1_A) = 1_A \cdot T^2(1_A)$$

and

$$\begin{aligned}\frac{-\lambda 1_A}{1 - \lambda} &= \frac{1}{1 - \lambda} \cdot 1_A \cdot (T - \lambda I)(1_A)\\ &= 1_A \cdot P(1_A) = 1_A \cdot P^2(1_A)\\ &= \frac{1}{(1 - \lambda)^2} \cdot 1_A \cdot (T - \lambda I)^2(1_A)\\ &= \frac{1}{(1 - \lambda)^2} \cdot 1_A\big(T^2 1_A - 2T1_A + \lambda^2 1_A\big) = \frac{\lambda^2 1_A}{(1 - \lambda)^2}.\end{aligned}$$

This implies that $\frac{-\lambda}{1-\lambda} = 1$, a contradiction. We have proved our claim.

Suppose that $\Phi \neq I$. Then there is a measurable set A_1 such that

$$\infty > \mu(A_1) > 0 \quad \text{and} \quad A_1 \neq \Phi(A_1).$$

Let $A = A_1 \setminus \Phi(A_1)$ if $\mu(A_1 \setminus \Phi(A_1)) > 0$; otherwise let $A = \Phi^{-1}(\Phi(A_1) \setminus A_1)$. Then $\mu(A \cap \Phi(A)) = 0$. So

$$\begin{aligned} T1_A &= 1_{\Phi(A)} \cdot T1_A = 1_{\Phi(A)} \cdot (T1_A - \lambda 1_A) \\ &= 1_{\Phi(A)} \cdot (1-\lambda)P1_A = 1_{\Phi(A)}(1-\lambda)P^2 1_A \\ &= 1_{\Phi(A)} \cdot \frac{1}{1-\lambda}(T-\lambda I)^2 1_A \\ &= 1_{\Phi(A)} \cdot \frac{1}{1-\lambda}\left(T^2 1_A - 2\lambda T1_A + 1_A\right) \\ &= 1_{\Phi(A)} \cdot \frac{1}{1-\lambda}(-2\lambda T1_A) = \frac{1}{1-\lambda}(-2\lambda T1_A). \end{aligned}$$

This implies that $-2\lambda = 1 - \lambda$. We have proved that if $\Phi^2 \neq I$, then $\lambda = -1$ and $P - (I - P)$ is a reflection.

Now suppose that $\Phi = I$. Then for any $f \in L^p(\Omega, X)$,

$$(1-\lambda)Pf(\cdot) = S(\cdot)f(\cdot) - \lambda f(\cdot).$$

Thus, for almost all $t \in \Omega$, $A(t) = \frac{1}{1-\lambda}(S(t) - \lambda I)$ is a projection and $A(t) - \lambda(I - A(t))$ is an isometry on X. The proof is complete. □

Corollary 5. *Let (Ω, μ) be a σ-finite measure space, $1 \leqslant p < \infty$, $p \neq 2$, and X be a separable complex Banach space with trivial L^p-structure. Suppose that there is a nontrivial generalized bi-circular projection P on $L^p(X)$ that is not reflection. Then there is a nontrivial generalized bi-circular projection on X.*

Remark 6. Let H be a separable Hilbert space and E be a rearrangement invariant space over [0, 1] that is not equal to L_p for any $1 \leqslant p \leqslant \infty$. The Köthe–Bochner function space $E(H)$ is the set of all strongly measurable functions $F : [0,1] \to H$ such that $\|f\|_{E(H)} = \| \|f(\cdot)\|_H \|_E < \infty$. B. Randrianantoanina [7, Theorem 11] proved that for any surjective isometry T from the Köthe function space $E(H)$ onto itself, there are an invertible Borel mapping $\sigma : [0,1] \to [0,1]$ and a strong measurable mapping $S : [0,1] \to \mathrm{Iso}(H)$ such that

$$Tf(t) = S(t)f\big(\sigma(t)\big).$$

Suppose that H is a separable complex space and E is a complex rearrangement invariant space over [0, 1]. If P is a generalized bi-circular projection on $E(H)$ and if $P - (I - P)$ is not a reflection, then there are $\lambda \in S^1 \setminus \{1\}$ and a strong measurable mapping A from Ω to $\mathcal{B}(H)$ such that for almost all $t \in \Omega$, $A(t)$ is a projection, $A(t) + \lambda(I - A(t))$ is an isometry, and $(Pf)(\cdot) = A(\cdot)f(\cdot)$.

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