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# Nash equilibrium and minimax theorem with $\mathcal{C}$-concavity

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**Abstract**

The purpose of this paper is to introduce a generalized $\mathcal{C}$-concave condition, and by using Himmelberg's fixed point theorem, to prove a new existence theorem of Nash equilibrium in non-compact generalized game with $\mathcal{C}$-concavity. As applications, we shall prove a minimax theorem in non-compact settings and prove a minimax inequality in compact settings.


*Keywords:* Nash equilibrium; $\mathcal{C}$-concave; Minimax theorem; Minimax inequality

## 1. Introduction

In 1951, Nash established the well-known equilibrium existence theorem for $N$-person games. Since then, the classical results of Nash [18], Debreu [2,3] and Nikaido and Isoda [19] have served as basic references for the existence of Nash equilibrium for non-cooperative games. Next, in 1977, Friedman [9] established a generalization of the Nash theorem using the quasi-concavity assumption on every payoff function. In all of them, convexity of strategy spaces, continuity and concavity/quasi-concavity of the payoff functions were assumed. Till now there have been a number of generalizations, and also many applications of those theorems have been found in several areas, e.g., see [1,9] and references therein.

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Two important concepts for removing the concavity/quasi-concavity assumptions of the payoff functions are marked by the seminal papers of Fan [5,6] for 2-person zero-sum games, and the complete abandonment of concavity in Nishimura and Friedman [20]. In fact, the concept of concavelike payoffs due to Fan [6] does not require any linear structure on the strategy space. However, Joó [13] gave a general sum 2-person game where the payoff functions are continuous and concavelike, but the game has no Nash equilibrium. Horváth and Joó [11] also show that higher smoothness of the payoff functions does not change the situation. In [8], Forgó introduced the CF-concavity by adding continuity to Fan's concavelike condition, and prove the existence of a Nash equilibrium.

In a recent paper [16], the authors introduced the $\mathcal{C}$-concavity which generalizes both concave condition and CF-concavity without assuming the linear structure, and next, they proved an existence theorem of Nash equilibrium and its applications using the $\mathcal{C}$-concavity. And, more recently, Kim and Kum [15] further generalize the $\mathcal{C}$-convexity using constraint correspondences, and they prove an equilibrium existence theorem for a compact generalized $N$-person game.

In this paper, we will introduce a $\mathcal{C}$-concave condition which generalizes both concave condition and CF-concavity without assuming the linear structure. Using this $\mathcal{C}$-concavity and the partition of unity argument, we shall prove a new existence theorem of Nash equilibrium for non-compact generalized games. And we shall give a new minimax theorem and a minimax inequality as its applications. Those results generalize the existence theorems in [4,8,15,16,18,19] to non-compact generalized games with $\mathcal{C}$-concavity. Finally we shall give an example of a game where $\mathcal{C}$-concavity can be applied; but the concavity/quasi-concavity in [9,11–14,17,20] cannot be applied.

## 2. Preliminaries

We begin with some notations and definitions. Let $A$ be a subset of a topological space $X$. We shall denote by $2^A$ the family of all subsets of $A$. Let $I$ be a countable index set. For each $i \in I$, let $X_i$ be a non-empty topological space and denote $X := \prod_{i\in I} X_i$ and $X_{\hat{i}} := \prod_{j\in I\setminus\{i\}} X_j$. If $x = (x_1, \dots, x_n, \dots) \in X$, we shall write $x_{\hat{i}} = (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n, \dots) \in X_{\hat{i}}$. If $x_i \in X_i$ and $x_{\hat{i}} \in X_{\hat{i}}$, we shall use the notation $(x_i, x_{\hat{i}}) := (x_1, \dots, x_{i-1}, x_i, x_{i+1}, \dots, x_n, \dots) = x \in X$. Denote by $[0,1]^n$ the Cartesian product of $n$ unit intervals $[0,1] \times \cdots \times [0,1]$; and denote the unit simplex in $[0,1]^n$ by $\Delta_n$, i.e.,

$$\Delta_n := \left\{ (\lambda_1, \dots, \lambda_n) \in [0,1]^n \;\middle|\; \sum_{i=1}^{n} \lambda_i = 1 \right\}.$$

Throughout this paper, all topological spaces are assumed to be Hausdorff.

Let $I = \{1, \dots, n, \dots\}$ be a countable set of players. A *non-cooperative generalized game $\Gamma$ of normal form* is an ordered tuple $(X_1, \dots, X_n, \dots; f_1, \dots, f_n, \dots)$ where for each player $i \in I$, the non-empty set $X_i$ is the player's pure strategy space, and $f_i : X = \prod_{i\in I} X_i \to \mathbb{R}$ is the player's payoff function. The set $X$, *joint strategy space*, is the Cartesian product of the individual strategy sets, and an element of $X_i$ is called a *strategy* of the $i$th player. A strategy $\bar{x} = (\bar{x}_1, \dots, \bar{x}_n, \dots) \in X$ is called a *Nash equilibrium* for the game $\Gamma$ if the following system of inequalities holds: for each $i \in I$,

$$f_i(\bar{x}_1, \dots, \bar{x}_i, \dots, \bar{x}_n, \dots) \geqslant f_i(\bar{x}_1, \dots, \bar{x}_{i-1}, x_i, \bar{x}_{i+1}, \dots, \bar{x}_n, \dots)$$

for all $x_i \in X_i$. When $I$ is an uncountable set of players, we can similarly define the non-cooperative game $\Gamma$ of normal form, and in this case, we also call $\Gamma$ the non-cooperative generalized game. Here we remark that the model of a game in this paper is a non-cooperative game, i.e., there is no replay communicating between players, and so players act as free agents, and each player is trying to maximize his/her own payoff according to his/her strategy.

Now we recall some concepts which generalize the concavity. When $X$ and $Y$ are non-empty arbitrary sets, recall that $f: X \times Y \to \mathbb{R}$ is *concavelike on $X$ with respect to $Y$* [6] if for any $x_1, x_2 \in X$ and $\lambda \in [0, 1]$, there exists $x_0 \in X$ such that

$$f(x_0, y) \geqslant \lambda f(x_1, y) + (1 - \lambda) f(x_2, y) \quad \text{for all } y \in Y.$$

Adding the continuity to concavelike functions, Forgó [8] introduced the CF-concavity as follows: Let $X$ be a non-empty topological space, $Y$ a non-empty arbitrary set. Then $f: X \times Y \to \mathbb{R}$ is said to be *CF-concave on $X$ with respect to $Y$* if there exists a continuous function $\Psi: X \times X \times \mathbb{R} \to X$ such that for any $x_1, x_2 \in X$ and $\lambda \in [0, 1]$,

$$f\big(\Psi(x_1, x_2, \lambda), y\big) \geqslant \lambda f(x_1, y) + (1 - \lambda) f(x_2, y) \quad \text{for all } y \in Y.$$

Also note that by using the induction, we can obtain the equivalent formulations to the concavelike and CF-concave conditions in general forms, respectively, e.g., see [16, Lemma 1] and [8, Lemma 1].

Next, we will introduce a concave condition which generalizes both CF-concavity and concavity as follows:

**Definition.** Let $X$ be a topological space, $Y$ an arbitrary set and $D$ be a non-empty subset of $X$. Then $f: X \times Y \to \mathbb{R}$ is called *$\mathcal{C}$-concave on $D$* if for every $n \geqslant 2$, whenever $n$ points $x_1, \ldots, x_n \in X$ are arbitrarily given, there exists a continuous function $\phi_n: \Delta_n \to D$ such that

$$f\big(\phi_n(\lambda_1, \ldots, \lambda_n), y\big) \geqslant \lambda_1 f(x_1, y) + \cdots + \lambda_n f(x_n, y) \tag{1}$$

for all $(\lambda_1, \ldots, \lambda_n) \in \Delta_n$ and for all $y \in Y$.

**Remarks.**

(a) When $X = D$ in Definition, the $\mathcal{C}$-concavity is actually the same as the definition in [16]. In this case, the concavity clearly implies the $\mathcal{C}$-concavity by letting $\phi_n(\lambda_1, \ldots, \lambda_n) := \lambda_1 x_1 + \cdots + \lambda_n x_n$ for each $(\lambda_1, \ldots, \lambda_n) \in \Delta_n$, whenever $x_1, \ldots, x_n \in X$ are given.

(b) Note that the continuous function $\phi_n$ need not be defined globally on $\underbrace{X \times \cdots \times X}_{n \text{ times}} \times \mathbb{R}^n$ as in [8], but defined only on $\Delta_n$ in Definition. In fact, for any given $n$ points $x_1, \ldots, x_n \in X$, by defining

$$\phi_n(\lambda_1, \ldots, \lambda_n) := \Psi_n(x_1, \ldots, x_n; \lambda_1, \ldots, \lambda_n)$$

for each $(\lambda_1, \ldots, \lambda_n) \in \Delta_n$, we can see that the CF-concavity implies the $\mathcal{C}$-concavity.

(c) If $f$ is $\mathcal{C}$-concave on $X$, then for any given points $x_1, x_2 \in X$ and for each $\lambda \in [0, 1]$, by defining $x_0 := \phi_2(\lambda, 1 - \lambda)$, we can see that $f$ is concavelike on $X$. Therefore, the following implication diagram holds:

$$\text{concave} \implies \text{CF-concave} \implies \mathcal{C}\text{-concave} \implies \text{concavelike}.$$

To prove the existence theorems in non-compact settings, we shall need the following special form of Himmelberg's fixed point theorem:

**Lemma 1.** [10] *Let $X$ be a convex subset of a locally convex Hausdorff topological vector space, $D$ a non-empty compact subset of $X$, and let $f : X \to D$ be a continuous mapping. Then there exists a point $\bar{x} \in D$ such that $f(\bar{x}) = \bar{x}$.*

## 3. New existence theorem of Nash equilibrium

Let $\Gamma$ be a non-cooperative generalized game where $I$ is a countable (possibly uncountable) set of players and $X_i$ is the player's pure strategy space. And let the strategy space $X := \prod_{i\in I} X_i$ be a non-empty subset of a locally convex Hausdorff topological vector space.

Now let us define the total sum of payoff functions $H : X \times X \to \mathbb{R} \cup \{\pm\infty\}$ associated with the non-cooperative generalized game $\Gamma$ as follows:

$$H(x, y) := \sum_{i\in I} f_i(y_1, \dots, y_{i-1}, x_i, y_{i+1}, \dots, y_n, \dots) \tag{2}$$

for every $x = (x_1, \dots, x_n, \dots)$, $y = (y_1, \dots, y_n, \dots) \in X = \prod_{i\in I} X_i$.

Then we shall need the following which is a general form of Lemma 3.1 in [19]:

**Lemma 2.** *Let $\Gamma$ be a non-cooperative generalized game where $I$ is a countable (possibly uncountable) set of players. If there exists a point $\bar{x} \in X$ for which*

$$H(\bar{x}, \bar{x}) \geqslant H(x, \bar{x}) \quad \textit{for any } x \in X,$$

*then $\bar{x}$ is a Nash equilibrium for $\Gamma$.*

**Proof.** For each $i \in I$, we take any $x = (\bar{x}_1, \dots, \bar{x}_{i-1}, x_i, \bar{x}_{i+1}, \dots) \in X$. Then, by substitution, we can see that

$$\begin{aligned} H(\bar{x}, \bar{x}) &= \sum_{j\in I\setminus\{i\}} f_j(\bar{x}_1, \dots, \bar{x}_i, \dots) + f_i(\bar{x}_i, \bar{x}_{\hat{i}}) \\ &\geqslant H(x, \bar{x}) = \sum_{j\in I\setminus\{i\}} f_j(\bar{x}_1, \dots, \bar{x}_i, \dots) + f_i(x_i, \bar{x}_{\hat{i}}) \end{aligned}$$

for all $x_i \in X_i$. Therefore, we have

$$f_i(\bar{x}_i, \bar{x}_{\hat{i}}) \geqslant f_i(x_i, \bar{x}_{\hat{i}}) \quad \text{for all } x_i \in X_i;$$

hence $\bar{x}$ is a Nash equilibrium for $\Gamma$. □

Using the partition of unity argument, we now prove the following existence theorem of Nash equilibrium in non-compact generalized games:

**Theorem 1.** *Let $I$ be a countable (possibly uncountable) set of index set, and let $\Gamma$ be a non-cooperative generalized game satisfying the following conditions:*

(i) *the strategy space $X := \prod_{i\in I} X_i$ is a paracompact convex subset of a locally convex Hausdorff topological vector space and $D$ be a non-empty compact subset of $X$;*
(ii) *the function $(x, y) \mapsto H(x, y)$ is continuous on $X \times X$;*
(iii) *the function $x \mapsto H(x, y)$ is $\mathcal{C}$-concave on $D$;*
(iv) *for each $x \in D$, $H(x, x) \geqslant H(y, x)$ for all $y \in X \setminus D$.*

*Then $\Gamma$ has a Nash equilibrium $\bar{x} \in D$, i.e., for each $i \in I$,*

$$f_i(\bar{x}_i, \bar{x}_{\hat{i}}) \geqslant f_i(x_i, \bar{x}_{\hat{i}}) \quad \text{for all } x_i \in X_i.$$

**Proof.** Suppose the contrary, i.e., assume that $\Gamma$ has no Nash equilibrium. Then, by Lemma 2, for all $x \in X$, there exists an $y \in X$ such that $H(x,x) < H(y,x)$.

For any $z \in X$, we let

$$U(z) := \bigl\{x \in X \mid H(x,x) < H(z,x)\bigr\}.$$

Then, since $H$ is continuous, each $U(z)$ is open (possibly empty) in $X$; and also we have $\bigcup_{z\in X} U(z) = X$. Here, without loss of generality, we may assume that $X \setminus D$ is non-empty. By the assumption (iv), for each $z \in X \setminus D$, we have that $U(z) \subset X \setminus D$. Since

$$X = \bigcup_{z\in X} U(z) = \Bigl(\bigcup_{z\in D} U(z)\Bigr) \cup \Bigl(\bigcup_{z\in X\setminus D} U(z)\Bigr),$$

we obtain that $D \subset \bigcup_{z\in D} U(z)$. Since $D$ is compact and each $U(z)$ is open, there exists a finite number of non-empty open sets $U(z_1), \ldots, U(z_n)$ such that $D \subset \bigcup_{i=1}^n U(z_i)$, where $\{z_1, \ldots, z_n\} \subset D$. Since $X \setminus D$ is non-empty, if possible, let $z_{n+1} \in X \setminus D$ should be chosen satisfying that $z_{n+1} \notin U(z_i)$ for each $i \in \{1, \ldots, n\}$. And denote an open set $U(z_{n+1}) := X \setminus D$. Then $\{U(z_1), \ldots, U(z_{n+1})\}$ is a finite open covering of $X$. Since $X$ is paracompact, there exists a partition of unity $\{\alpha_1, \ldots, \alpha_{n+1}\}$ subordinate to the open covering $\{U(z_1), \ldots, U(z_{n+1})\}$, i.e.,

$$0 \leqslant \alpha_i(x) \leqslant 1, \quad \sum_{i=1}^{n+1} \alpha_i(x) = 1 \quad \text{for all } x \in X,\ i = 1, \ldots, n+1;$$
$$\text{and if } x \notin U(z_j) \text{ for some } j, \text{ then} \quad \alpha_j(x) = 0.$$

For such $\{z_1, \ldots, z_{n+1}\} \subset X$, since $H$ is $\mathcal{C}$-concave on $D$, there exists a continuous mapping $\phi_{n+1} : \Delta_{n+1} \to D$ satisfying the condition

$$H\bigl(\phi_{n+1}(\lambda_1, \ldots, \lambda_{n+1}), x\bigr) \geqslant \lambda_1 H(z_1, x) + \cdots + \lambda_{n+1} H(z_{n+1}, x)$$
$$\text{for all } (\lambda_1, \ldots, \lambda_{n+1}) \in \Delta_{n+1} \text{ and for all } x \in X.$$

Next we consider a continuous mapping $\Psi : X \to D$, defined by

$$\Psi(z) := \phi_{n+1}\bigl(\alpha_1(z), \ldots, \alpha_{n+1}(z)\bigr) \quad \text{for all } z \in X.$$

Since $\phi_{n+1}$ and each $\alpha_i$ are continuous, $\Psi$ is continuous on $X$. Moreover, $\Psi$ maps a non-empty convex set $X$ into a compact subset $D$ in a locally convex Hausdorff topological vector space. Therefore, by Lemma 1, there exists a fixed point $\bar{x} \in D$ such that $\Psi(\bar{x}) = \bar{x}$. Since $H$ is $\mathcal{C}$-concave on $D$, we have

$$H\bigl(\Psi(\bar{x}), x\bigr) \geqslant \alpha_1(\bar{x}) H(z_1, x) + \cdots + \alpha_n(\bar{x}) H(z_n, x) + \alpha_{n+1}(\bar{x}) H(z_{n+1}, x)$$

for all $x \in X$; and so by putting $x := \bar{x}$, we have

$$H(\bar{x}, \bar{x}) \geqslant \alpha_1(\bar{x}) H(z_1, \bar{x}) + \cdots + \alpha_n(\bar{x}) H(z_n, \bar{x}) + \alpha_{n+1}(\bar{x}) H(z_{n+1}, \bar{x}). \tag{3}$$

However, if $\bar{x} \in U(z_j)$ for some $1 \leqslant j \leqslant n$, then $H(\bar{x}, \bar{x}) < H(z_j, \bar{x})$ and $\alpha_j(\bar{x}) > 0$; and if $\bar{x} \notin U(z_k)$ for some $1 \leqslant k \leqslant n$, $\alpha_k(\bar{x}) = 0$. Also note that since $\bar{x} \in D$, $\bar{x} \notin X \setminus D = U(z_{n+1})$; and so $\alpha_{n+1}(\bar{x}) = 0$. Therefore, we have

$$\sum_{i=1}^{n+1}\alpha_i(\bar{x})H(z_i,\bar{x}) > \sum_{i=1}^{n+1}\alpha_i(\bar{x})H(\bar{x},\bar{x}) = H(\bar{x},\bar{x}),$$

which contradicts (3). This completes the proof. □

**Remarks.**

(1) Theorem 1 generalizes the equilibrium existence theorems due to Nash [18] and Forgó [8] in the following aspects:
   (i) for each $i \in I$, the strategy set $X_i$ need not be compact; but the product space $X = \prod_{i\in I} X_i$ must be a paracompact convex subset of a locally convex Hausdorff topological vector space;
   (ii) for each $i \in I$, every payoff function $f_i$ need not be concave nor continuous, and $H$ need not be CF-concave;
   (iii) the set $I$ of players need not be finite.

(2) Theorem 1 can be further generalized by using the constraint correspondences $T_i$ as in Definition 1 in [15]. Also it should be noted that in our Theorem 1, the set of players $I$ is a countable (possibly uncountable) set; however, in Theorem 1 in [15], the set of players $I$ is a finite set.

When the strategy space $X = D$ is compact in Theorem 1, the total sum of payoff functions $H(x, y)$ must be bounded on $X \times X$. In this case, the coercive condition (iv) is automatically satisfied, and so we have the following:

**Theorem 2.** *Let $I$ be a countable (possibly uncountable) set of players, and let $\Gamma$ be a non-cooperative generalized game satisfying the following*:

(i) *the strategy space $X := \prod_{i\in I} X_i$ is non-empty compact convex subset of locally convex Hausdorff topological vector space*;
(ii) *the function $(x, y) \mapsto H(x, y)$ is continuous on $X \times X$*;
(iii) *the function $x \mapsto H(x, y)$ is $\mathcal{C}$-concave on $X$.*

*Then $\Gamma$ has at least one Nash equilibrium.*

## 4. Some applications

As an application of Theorem 1, we shall prove the following minimax theorem in non-compact settings:

**Theorem 3.** *Let $X$ and $Y$ be non-empty sets such that $X \times Y$ is a paracompact convex in a locally convex Hausdorff topological vector space, $D$ a non-empty compact subset of $X$, and $E$ a non-empty compact subset of $Y$. Assume that*

(a) *the function $f : X \times Y \to \mathbb{R}$ is continuous on $X \times Y$*;
(b) *for each $y \in Y$, the function $x \mapsto -f(x, y)$ is $\mathcal{C}$-concave on $D$*;
(c) *for each $x \in X$, the function $y \mapsto f(x, y)$ is $\mathcal{C}$-concave on $E$*;
(d) *for each $(x, y) \in D \times E$, $f(x, v) - f(u, y) \leqslant 0$ for all $(u, v) \in X \times Y \setminus D \times E$.*

*Then we have*

$$\sup_{y\in Y}\inf_{x\in X} f(x,y)=\inf_{x\in X}\sup_{y\in Y} f(x,y).$$

**Proof.** Let $f_1(x,y):=-f(x,y)$ and $f_2(x,y):=f(x,y)$. In order to apply Theorem 1, we first note that the mapping $H:(X\times Y)\times(X\times Y)\to\mathbb{R}$ is given by

$$H\big((x_1,y_1),(x_2,y_2)\big):=f_1(x_1,y_2)+f_2(x_2,y_1)\quad\text{for each } (x_1,y_1),(x_2,y_2)\in X\times Y.$$

Then $H$ is clearly continuous, so it suffices to show that the assumptions (iii) and (iv) of Theorem 1 are satisfied. Let two points $(x_1,y_1),(x_2,y_2)\in X\times Y$ be given arbitrarily. Then for $\{x_1,x_2\}$, by the assumption (b), there exists a continuous function $\phi_1:\Delta_2\to D$ such that

$$f_1\big(\phi_1(\lambda,1-\lambda),v\big)\geqslant\lambda f_1(x_1,v)+(1-\lambda)f_1(x_2,v)$$

for every $\lambda\in[0,1]$ and every $v\in Y$. Also, for $\{y_1,y_2\}$, by the assumption (c), there exists a continuous function $\phi_2:\Delta_2\to E$ such that

$$f_2\big(u,\phi_2(\lambda,1-\lambda)\big)\geqslant\lambda f_2(u,y_1)+(1-\lambda)f_2(u,y_2)$$

for every $\lambda\in[0,1]$ and every $u\in X$.

Now we define a continuous function $\Phi_2:\Delta_2\to D\times E$ by

$$\Phi_2(\lambda,1-\lambda):=\big(\phi_1(\lambda,1-\lambda),\phi_2(\lambda,1-\lambda)\big)\quad\text{for every } \lambda\in[0,1].$$

Then it is easy to see that $\Phi_2$ is a continuous function on $\Delta_2$. Also, for every $\lambda\in[0,1]$, we have

$$\begin{aligned}
&\lambda H\big((x_1,y_1),(u,v)\big)+(1-\lambda)H\big((x_2,y_2),(u,v)\big)\\
&\quad=\lambda\big(f_1(x_1,v)+f_2(u,y_1)\big)+(1-\lambda)\big(f_1(x_2,v)+f_2(u,y_2)\big)\\
&\quad=\big[\lambda f_1(x_1,v)+(1-\lambda)f_1(x_2,v)\big]+\big[\lambda f_2(u,y_1)+(1-\lambda)f_2(u,y_2)\big]\\
&\quad\leqslant f_1\big(\phi_1(\lambda,1-\lambda),v\big)+f_2\big(u,\phi_2(\lambda,1-\lambda)\big)\\
&\quad=H\big(\Phi_2(\lambda,1-\lambda),(u,v)\big)\quad\text{for all } (u,v)\in X\times Y.
\end{aligned}$$

For arbitrarily given $n$ points $(x_1,y_1),\dots,(x_n,y_n)\in X\times Y$, we can similarly define a continuous function $\Phi_n:\Delta_n\to D\times E$ by

$$\Phi_n(\lambda_1,\dots,\lambda_n):=\big(\psi_1(\lambda_1,\dots,\lambda_n),\psi_2(\lambda_1,\dots,\lambda_n)\big)$$

for every $(\lambda_1,\dots,\lambda_n)\in\Delta_n$, where $\psi_1:\Delta_2\to D$ is a continuous function suitable for $f_1$ with respect to $\{x_1,\dots,x_n\}$, and $\psi_2:\Delta_2\to E$ is a continuous function suitable for $f_2$ with respect to $\{y_1,\dots,y_n\}$ in the $\mathcal{C}$-concavity condition. Thus we can also show the condition (1) for the $\mathcal{C}$-concavity of $H$; and hence $H$ is $\mathcal{C}$-concave on $D\times E$. It remains to show that $H$ satisfies the coercive condition (iv) in Theorem 1. For each $(x,y)\in D\times E$, $H((x,y),(x,y))=f_1(x,y)+f_2(x,y)=-f(x,y)+f(x,y)=0$. And for each $(x,y)\in D\times E$, $H((u,v),(x,y))=f_1(u,y)+f_2(x,v)=f(x,v)-f(u,y)$. Therefore, by assumption (d), we have that for each $(x,y)\in D\times E$, $H((x,y),(x,y))\geqslant H((u,v),(x,y))$ for all $(u,v)\in X\times Y\setminus D\times E$, which implies the assumption (iv) of Theorem 1.

Therefore, by Theorem 1, there exists a Nash equilibrium $(x_0,y_0)\in D\times E$ such that

$$f_1(x_0,y_0)=\sup_{x\in X} f_1(x,y_0)\quad\text{and}\quad f_2(x_0,y_0)=\sup_{y\in Y} f_2(x_0,y).$$

Therefore, we have

$$-f(x_0, y_0) = f_1(x_0, y_0) \geqslant f_1(x, y_0) = -f(x, y_0) \quad \text{for all } x \in X,$$

and

$$f(x_0, y_0) = f_2(x_0, y_0) \geqslant f_2(x_0, y) = f(x_0, y) \quad \text{for all } y \in Y.$$

Hence

$$\sup_{y\in Y} f(x_0, y) \leqslant f(x_0, y_0) \leqslant \inf_{x\in X} f(x, y_0),$$

which implies

$$\inf_{x\in X} \sup_{y\in Y} f(x, y) \leqslant f(x_0, y_0) \leqslant \sup_{y\in Y} \inf_{x\in X} f(x, y).$$

And the reverse inequality

$$\sup_{y\in Y} f(x, y) \geqslant \sup_{y\in Y} \inf_{x\in X} f(x, y)$$

is trivial, and so we obtain the conclusion. □

As another application of Theorem 2, we shall prove the following which is comparable to the well-known minimax inequality due to Fan [7]:

**Theorem 4.** *Let $X$ be a non-empty compact convex set in a locally convex Hausdorff topological vector space $E$ and let $f : X \times X \to \mathbb{R}$ be a real-valued function on $X \times X$ such that*

(a) *for each $y \in X$, the function $x \mapsto f(x, y)$ is lower semicontinuous on $X$;*
(b) *for each $x \in X$, the function $y \mapsto f(x, y)$ is $\mathcal{C}$-concave on $X$.*

*Then the minimax inequality*

$$\min_{x\in X} \sup_{y\in X} f(x, y) \leqslant \sup_{x\in X} f(x, x)$$

*holds.*

**Proof.** Let $\mu := \sup_{x\in X} f(x, x)$. Clearly we may assume that $\mu < \infty$. Suppose the contrary, i.e.,

$$\min_{x\in X} \sup_{y\in X} f(x, y) > \sup_{x\in X} f(x, x) = \mu.$$

Then, for each $x \in X$, there exists $y \in X$ such that $f(x, y) > \mu$. For any $y \in X$, we let

$$U(y) := \{x \in X \mid f(x, y) > \mu\}.$$

Then, by the assumption (a), each $U(y)$ is (possibly empty) open in $X$ and also we have $\bigcup_{y\in X} U(y) = X$. Since $X$ is compact, there exists a finite number of non-empty open sets $U(y_1), \ldots, U(y_n)$ such that $\bigcup_{i=1}^{n} U(y_i) = X$. Let $\{\alpha_i \mid i = 1, \ldots, n\}$ be the partition of unity subordinate to the open covering $\{U(y_i) \mid i = 1, \ldots, n\}$ of $X$, i.e.,

$$0 \leqslant \alpha_i(x) \leqslant 1, \quad \sum_{i=1}^{n} \alpha_i(x) = 1 \quad \text{for all } x \in X,\ i = 1, \ldots, n;$$

$$\text{and if } x \notin U(y_j) \text{ for some } j, \text{ then } \quad \alpha_j(x) = 0.$$

For such $\{y_1, \dots, y_n\} \subset X$, since $y \mapsto f(x, y)$ is $\mathcal{C}$-concave, there exists a continuous mapping $\phi_n : \Delta_n \to X$ satisfying the condition

$$f\big(x, \phi_n(\lambda_1, \dots, \lambda_n)\big) \geqslant \lambda_1 f(x, y_1) + \cdots + \lambda_n f(x, y_n)$$

for all $(\lambda_1, \dots, \lambda_n) \in \Delta_n$ and for all $x \in X$.

Now consider a continuous mapping $\Psi : X \to X$, defined by

$$\Psi(x) := \phi_n\big(\alpha_1(x), \dots, \alpha_n(x)\big) \quad \text{for all } x \in X.$$

Since $\phi_n$ and each $\alpha_i$ are continuous, $\Psi$ is continuous on $X$. Moreover, $\Psi$ maps $X$, which is a compact convex subset of a locally convex Hausdorff topological vector space, into itself. Therefore, by Lemma 1, there exists a fixed point $\bar{x} \in X$ such that $\Psi(\bar{x}) = \bar{x}$.

On the while, by the $\mathcal{C}$-concavity of $f$, we have

$$f\big(x, \Psi(\bar{x})\big) \geqslant \alpha_1(\bar{x}) f(x, y_1) + \cdots + \alpha_n(\bar{x}) f(x, y_n) \quad \text{for all } x \in X;$$

and so we have

$$f(\bar{x}, \bar{x}) \geqslant \sum_{i=1}^{n} \alpha_i(\bar{x}) f(\bar{x}, y_i). \tag{4}$$

However, if $\bar{x} \in U(y_j)$ for some $1 \leqslant j \leqslant n$, then we have $f(\bar{x}, y_j) > \mu$ and $\alpha_j(\bar{x}) > 0$; and if $\bar{x} \notin U(y_k)$ for some $1 \leqslant k \leqslant n$, then $\alpha_k(\bar{x}) = 0$. Thus we have

$$\mu = \sup_{x \in X} f(x, x) \geqslant f(\bar{x}, \bar{x}) \geqslant \sum_{i=1}^{n} \alpha_i(\bar{x}) f(\bar{x}, y_i) > \mu,$$

which is a contradiction. This completes the proof. □

As we mentioned before, the generalized game described in [8,19] has an equilibrium if the payoff function $f_i$ satisfies either CF-concavity or concavity. Indeed, many of the assumptions made in the preceding theorems in [8,19] have been weakened and the existence of equilibrium has been proved; however, it is hard to improve the equilibrium theorem by relaxing quasi-concavity assumption of the payoff functions and the convexity assumption on the strategy space. On the other hand, in this paper, we introduce a meaningful $\mathcal{C}$-concavity, and prove a new Nash equilibrium existence theorem. Since the Nash equilibrium is an useful tool in many areas of mathematical economics including oligopoly theory, general equilibrium and social choice theory, the $\mathcal{C}$-concavity should be helpful in developing the theory of Nash equilibrium. Also note that Theorem 1 can be improved to more general spaces by using Eilenberg–Montgomery's fixed point theorem without assuming the linear structure on $X$.

Finally, we shall give an example where Theorem 1 can be applied but previous results due to Nash [18], Nikaido and Isoda [19], and Friedman [9] can not be applied.

**Example.** Let $\Gamma = \{X_1, X_2; f_1, f_2\}$ be a 2-person game where $X_1 = (-1, 1]$, $X_2 = [0, 1]$, $D = [0, 1] \subset X_1$, $E = [0, 1] = X_2$, and payoff functions be given as follows:

$$f_1(x_1, x_2) := x_1^2 x_2 \quad \text{for every } (x_1, x_2) \in X = X_1 \times X_2,$$
$$f_2(y_1, y_2) := y_1 \sqrt{y_2} \quad \text{for every } (y_1, y_2) \in X = X_1 \times X_2.$$

Clearly, $f_1(\cdot, x_2)$ is not quasi-concave for any $x_2 \in [0,1]$, and thus theorems of Nash [18], Nikaido and Isoda [19], and Friedman [9] cannot be applied. For this game, the related total sum of payoff functions $H : X \times X \to \mathbb{R}$ is given by

$$H\big((x_1,x_2),(y_1,y_2)\big) = f_1(x_1,y_2) + f_2(y_1,x_2) = x_1^2 y_2 + y_1\sqrt{x_2},$$

for every $((x_1,x_2),(y_1,y_2)) \in X \times X$. Then $H(x,y)$ is continuous on $X \times X$. For arbitrarily given two points $(x_1,x_2),(x_3,x_4) \in X$, we now define a continuous function $\phi_2 : \Delta_2 \to D \times E$ by

$$\phi_2(\lambda, 1-\lambda) := \Big(\sqrt{\lambda x_1^2 + (1-\lambda)x_3^2}, \big[\lambda\sqrt{x_2} + (1-\lambda)\sqrt{x_4}\,\big]^2\Big) \quad \text{for all } \lambda \in [0,1].$$

Then it is easy to see that $\phi_2$ is a continuous function on $\Delta_2$. Also, for every $\lambda \in [0,1]$ and $(y_1,y_2) \in X$, we have

$$\begin{aligned}
&H\big(\phi_2(\lambda,1-\lambda),(y_1,y_2)\big)\\
&\quad = H\Big(\Big(\sqrt{\lambda x_1^2 + (1-\lambda)x_3^2}, \big[\lambda\sqrt{x_2} + (1-\lambda)\sqrt{x_4}\,\big]^2\Big),(y_1,y_2)\Big)\\
&\quad = \big(\lambda x_1^2 + (1-\lambda)x_3^2\big)y_2 + \big(\lambda\sqrt{x_2} + (1-\lambda)\sqrt{x_4}\big)y_1\\
&\quad \geqslant \lambda\big(x_1^2 y_2 + y_1\sqrt{x_2}\big) + (1-\lambda)\big(x_3^2 y_2 + y_1\sqrt{x_4}\big)\\
&\quad = \lambda H\big((x_1,x_2),(y_1,y_2)\big) + (1-\lambda)H\big((x_3,x_4),(y_1,y_2)\big).
\end{aligned}$$

For arbitrarily given $n$ points $(x_1,x_2),\dots,(z_1,z_2) \in X$, we can similarly define a continuous function $\phi_n : \Delta_n \to D \times E$ by

$$\phi_n(\lambda_1,\dots,\lambda_n) := \Big(\sqrt{\lambda_1 {x_1}^2 + \dots + \lambda_n {z_1}^2}, \big[\lambda_1\sqrt{x_2} + \dots + \lambda_n\sqrt{z_2}\,\big]^2\Big)$$

for all $(\lambda_1,\dots,\lambda_n) \in \Delta_n$; then we can show the $\mathcal{C}$-concave condition (1); and hence $H$ is $\mathcal{C}$-concave on $D \times E$. Therefore, we can apply the Theorem 1 to the game $\Gamma$; and clearly, $(1,1)$ is a Nash equilibrium for $\Gamma$. In fact,

$$\begin{aligned}
&1 = f_1(1,1) \geqslant f_1(x_1,1) = x_1^2 \quad \text{for every } x_1 \in X_1,\\
&1 = f_2(1,1) \geqslant f_2(1,y_2) = \sqrt{y_2} \quad \text{for every } y_2 \in X_2.
\end{aligned}$$

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