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On real Cartan factors

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JBW*-triples can be described (modulo W*-algebras, compare [13]) by those of type I. Among these the (complex) Cartan factors are the building blocks. We determine for every complex Cartan factor U all conjugations of the underlying complex Banach space and hence all real forms (in the sense of [15]) of U, called *real Cartan factors*. We also give a concrete list of all isomorphy classes of real Cartan factors which generalizes the classification of LOOS [23] to infinite dimensions. Furthermore, we give an explicit description of the full automorphism group as well as the group of all surjective $\mathbb{R}$-linear isometries for every non-exceptional real Cartan factor and decide which of the real or complex Cartan factors are isometrically equivalent to each other as real Banach spaces.

1. Introduction

On a complex Banach space U a conjugation is a conjugate linear isometry $\tau: U \to U$ with $\tau^2 = \mathbf{1}_U$ and for every such τ the real Banach space $F := \mathsf{Fix}(\tau) \subset U$ is called a *real form* of U. Clearly τ and $F \subset U$ determine each other in a unique way. For instance, if $U = H$ is a complex Hilbert space every orthonormal basis $(e_i)_{\in I}$ of H determines a conjugation by

$$\tau\Big(\sum_{i\in I} c_i\, e_i\Big) = \sum_{i\in I} \overline{c_i}\, e_i$$

and it is easily verified that every other conjugation σ of H is equivalent to τ in the following sense: There is a surjective (complex-linear) isometry $g: H \to H$ with $\sigma = g\tau g^{-1}$, or equivalently, with $g(\mathsf{Fix}(\tau)) = \mathsf{Fix}(\sigma)$. In general, a given complex Banach space has many non-equivalent conjugations and also the case occurs that it has no conjugation at all (see [20] for an example).

In this article we study conjugations in a special class of complex Banach spaces, the so called JB*-triples. These form a fairly large class of Banach spaces. For instance every Hilbert space, every Banach space carrying the structure of a C*-algebra or more generally every closed linear subspace $A \subset \mathcal{L}(H, K)$ with $aa^*a \in A$ for every $a \in A$ is in the class, where $\mathcal{L}(H, K)$ is the space of all bounded linear operators from the Hilbert space H into the Hilbert space K. These latter operator spaces were introduced by Harris [9] under the name J*-algebras. But there are also JB*-triples which cannot be given as operator spaces, for instance the exceptional JB*-algebras.

Originally [19] the JB*-triples were introduced in connection with the study of bounded symmetric domains in infinite dimensions. These are precisely the complex Banach spaces for which the open unit ball is homogeneous with respect to the group of all biholomorphic automorphisms. A consequence of this is that JB*-triples can also be uniquely characterized by the existence of a certain ternary product $\{xyz\}$, the Jordan triple product. On a C*-algebra for instance this product just is $(xy^*z + zy^*x)/2$. The important fact is that for every JB*-triple the geometry of the Banach space and the algebraic structure given by the triple product determine each other. In particular, on every JB*-triple U the conjugations in the Banach space sense and the triple conjugations (i.e. conjugate linear endomorphisms of period 2 that respect the triple product) are precisely the same. This makes it possible to deal with isometries and conjugations entirely in the algebraic context given by the triple product.

Building blocks for JB*-triples are the Cartan factors. These come in six types, the rectangular operator spaces $\mathcal{L}(H, K)$, spaces of symmetric and of alternating operators, spin factors and two exceptional spaces of dimensions 16 and 27. On every non-exceptional Cartan factor U we determine explicitly all conjugations τ of U and also the corresponding equivalence classes of them. Calling every real form $F := \mathsf{Fix}(\tau) \subset U$ a real Cartan factor this means the classification of all real Cartan factors up to isomorphy. This result extends the classification of all real bounded symmetric domains [23] to infinite dimensions. In contrast to the complex case the group $\mathsf{Imt}(F)$ of all surjective linear isometries of F may contain the triple automorphism group $\mathsf{Aut}(F)$ as a proper subgroup. Our main result states that real or complex Cartan factors of rank > 1 (thus essentially only excluding Hilbert spaces) are equivalent with respect to a surjective $\mathbb{R}$-linear isometry if and only if they are isomorphic with respect to the triple product. Furthermore, we compute for every non-exceptional real Cartan factor F the groups $\mathsf{Imt}(F)$ and $\mathsf{Aut}(F)$ explicitly. Among these F are the spaces of all bounded linear operators between and also the spaces of all skew-hermitian operators on real or quaternionic Hilbert spaces.

Notations and Preliminaries

For (left) Banach spaces E, F over the fixed base field $\mathbb{F} = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$ (the quaternions), denote by $\mathcal{L}(E, F)$ the $\mathbb{K}$-Banach space of all bounded $\mathbb{F}$-linear operators $E \to F$ where $\mathbb{K}$ is the center of $\mathbb{F}$ (that is $\mathbb{K} = \mathbb{C}$ if $\mathbb{F} = \mathbb{C}$ and $\mathbb{K} = \mathbb{R}$ otherwise). The real Banach space underlying E will always be denoted by $E_{\mathbb{R}}$. In this sense the operator spaces $\mathcal{L}(E, F)$ and $\mathcal{L}(E_{\mathbb{R}}, F_{\mathbb{R}})$ are completely different objects – for instance in case $\mathbb{F} \neq \mathbb{R}$ the natural inclusion $\mathcal{L}(E, F) \hookrightarrow \mathcal{L}(E_{\mathbb{R}}, F_{\mathbb{R}})$ is proper in general. Denote by $\mathsf{Imt}(E, F) \subset \mathcal{L}(E, F)$ the subset of all surjective isometries (which is empty in general). In the Banach algebra $\mathcal{L}(E) := \mathcal{L}(E, E)$ the subgroup of all invertible operators is denoted by $\mathsf{GL}(E)$ and $\mathsf{Imt}(E) := \mathsf{Imt}(E, E)$ is the subgroup of all surjective isometries. All operator spaces and groups are endowed with the norm topology in the following unless otherwise stated. The standard involution of $\mathbb{F}$ is denoted by $\alpha \mapsto \overline{\alpha}$. The quaternion field will always be realized as $\mathbb{H} = \mathbb{C} \oplus \mathrm{j}\mathbb{C}$ where $\mathrm{j} \in \mathbb{H}$ satisfies $\mathrm{j}^2 = -1$ and $t\mathrm{j} = \mathrm{j}\overline{t}$ for all $t \in \mathbb{C}$. The standard involution on $\mathbb{H}$ then is $(s + \mathrm{j}t) \mapsto (\overline{s} - \mathrm{j}t)$.

In case E, F are Hilbert spaces over $\mathbb{F}$ we always denote the corresponding inner product by $(x \mid y)$. It satisfies in particular the identity $(\alpha x \mid \beta y) = \alpha (x \mid y) \overline{\beta}$ for all $\alpha, \beta \in \mathbb{F}$. For every $z \in \mathcal{L}(E, F)$ the adjoint $z^* \in \mathcal{L}(F, E)$ is defined as usual by $(zx \mid y) = (x \mid z^* y)$. In case $\mathbb{F} = \mathbb{R}$ we also write z' instead of z^*. In case $\mathbb{F} = \mathbb{C}$ the transposed operator z' depends on the choice of conjugations on E, F (compare section 3). With $n = \dim_{\mathbb{F}}(E)$ (the cardinality of an orthonormal basis) we denote the group $\mathsf{Imt}(E)$ of all surjective $\mathbb{F}$-linear isometries of E also by $\mathsf{O}(n)$, $\mathsf{U}(n)$ and $\mathsf{Sp}(n)$ according to the three possibilities $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$ for $\mathbb{F}$. These groups are real Banach Lie groups. $\mathsf{U}(n)$ and $\mathsf{Sp}(n)$ are always connected. $\mathsf{O}(n)$ has two connected components if n is finite and is connected otherwise.

Suppose U, V, W are complex Banach spaces. Then we call a map $\lambda: U \to V$ *semi-linear* if it is complex-linear or conjugate-linear. Denote by $\mathsf{GL}_s(U) \subset \mathsf{GL}(U_{\mathbb{R}})$ the group of all semi-linear operators on U. By a sesqui-linear mapping $U \times V \to W$ we mean always an $\mathbb{R}$-bilinear mapping that is complex-linear in the first and conjugate-linear in the second variable. The operator $\lambda \in \mathcal{L}(U)$ is called *hermitian* if $\exp(it\lambda) \in \mathsf{GL}(U)$ is isometric for every real t.

For every space S we denote by $\mathbf{1}_S$ the identity transformation on S. We also write simply $\mathbf{1}$ instead of $\mathbf{1}_S$ if the corresponding space S is obvious.

2. JB*-triples and real forms

2.1 Definition A *JB*-triple* is a complex Banach space $U \neq 0$ together with a sesqui-linear mapping $(a, b) \mapsto a \square b$ from $U \times U$ into $\mathcal{L}(U)$ such that for all $a, b, c, x, y \in U$ the following holds

(i) The triple product $\{abc\} := (a \square b)(c)$ is symmetric in a, c

(ii) $[a \square b, x \square y] = \{abx\} \square y - x \square \{yab\}$

(iii) $a \square a$ is hermitian on U and has spectrum ≥ 0

(iv) $\|a \square a\| = \|a\|^2$.

It is known that (iv) in the above definition is equivalent to $\|a^3\| = \|a\|^3$ for $a^3 := \{aaa\}$ and that always $\|a \square b\| \leq \|a\| \cdot \|b\|$ holds. For every JB*-triple the triple product can uniquely be recovered from the underlying Banach space. In this sense we may say: *A given Banach space U is a JB*-triple or not.*

The sesqui-linear mapping $(a, b) \mapsto a \square b$ may be considered as an operator-valued positive-definite hermitian form on U. Therefore we call two elements $a, b \in U$ *orthogonal* and write $a \perp b$ if $a \square b = 0$ (or equivalently $b \square a = 0$) holds. A subset $S \subset U$ is called *complete* if $S^{\perp} := \{z \in U : z \perp S\} = 0$. For every $a \in U$ denote by $Q(a)$ the conjugate-linear operator $z \mapsto \{aza\}$ on U. The Q-operator satisfies the fundamental formula: $Q(Q(a)b) = Q(a)Q(b)Q(a)$ for all $a, b \in U$.

The JB*-triple U is called *abelian* if the set of operators $U \square U \subset \mathcal{L}(U)$ is abelian. A real subtriple $F \subset U$ is called *flat* if $a \square b = b \square a$ for all $a, b \in F$. The smallest closed real (complex, respectively) subtriple of U containing a given element a is flat (abelian, respectively). A closed linear subspace $I \subset U$ is called an *ideal* in U if $\{IUU\} + \{UIU\} \subset I$ holds. Then U/I is a JB*-triple in a natural way and every ideal J in I is also an ideal in U, compare [21].

The JB*-triple U is called a JBW*-triple if U as a Banach space is the dual of another Banach space U_* called a *predual* of U – it is well known that the predual of a JBW*-triple is unique. The corresponding weak topology $\sigma(U, U_*)$ is denoted by w^*. The JBW*-triple U is called a *factor* if there does not exist a decomposition $U = V \oplus W$ into non-zero ideals V, W, or equivalently, if $\{0\}$ and U are the only w^*-closed ideals of U (for more details on JB*- and JBW*-triples compare [19], [12], [2], [7]).

The element $e \in U$ is called *unitary* if $e \square e = \mathbb{1}$ holds. Then $a \circ b := \{aeb\}$ and $a^* := \{eae\}$ define on U the structure of a JB*-algebra with unit e. The selfadjoint part $A := \{a \in U : a^* = a\}$ is a JB-algebra with $U = A \oplus iA$, compare [8]. Let us denote by $\mathcal{U}(U) \subset U$ the subset of all unitary elements. Then $\mathcal{U}(U)$ is a closed real-analytic submanifold of U (may be empty) invariant under the group $\mathsf{Aut}(U) \subset \mathsf{GL}(U)$ of all triple automorphisms of U. Every connected component of $\mathcal{U}(U)$ is a K-orbit where $K := \mathsf{Aut}(U)^0$ is the identity component of the Lie group $\mathsf{Aut}(U)$. The submanifold $\mathcal{U}(U) \subset U$ is always connected in case $\dim U < \infty$, but not in general. Example 5.7 in [4] shows that $\mathsf{Aut}(U)$ may not even be transitive on $\mathcal{U}(U)$.

Denote by $\mathsf{Aut}_s(U) \subset \mathsf{GL}_s(U)$ the group of all semi-linear triple automorphisms of U. It is known that $\mathsf{Aut}_s(U)$ consists precisely of all surjective semi-linear isometries of U. A *conjugation* on U is a conjugate-linear isometry $\tau: U \to U$ with $\tau^2 = \mathbb{1}$. Then $U^\tau := \{z \in U : \tau(z) = z\}$ is a closed real subtriple with $U = U^\tau \oplus iU^\tau$. We call U^τ a *real form* of U. Two real forms U^τ, V^σ are called *equivalent* if $\sigma = g\tau g^{-1}$ for some $g \in \mathsf{Imt}(U, V)$ (and then $g(U^\tau) = V^\sigma$ holds). A real Banach space F together with a trilinear map $\{\ \}: F^3 \to F$ is called a *real JB*-triple* if there exists an $\mathbb{R}$-linear isometry λ from F into some (complex) JB*-triple U with $\{xyz\} = \{(\lambda x)(\lambda y)(\lambda z)\}$ for all $x, y, z \in F$. It is easy to see [15] that every real JB*-triple F can be realized as real form $F = V^\sigma$ with a JB*-triple V (the *hermitification of* F) and σ uniquely determined up to isomorphy by F. In particular, every complex JB*-triple U may be considered in a natural way as a real JB*-triple $U_{\mathbb{R}}$, the *realification* of U. The real JB*-triple F is called *complexifiable* if $F \cong U_{\mathbb{R}}$ for some complex JB*-triple U (see [15] for details). $Q(e)$ is a conjugation on U for every $e \in \mathcal{U}(U)$ and the corresponding real form $U^{Q(e)}$ coincides with the tangent space to $\mathcal{U}(U)$ in the point $ie \in \mathcal{U}(U)$. We call the conjugation τ of U *unitary* if it is of the form $Q(e)$ for some unitary e. The real form F of U comes from a unitary conjugation if and only if F has the structure of a unital JB-algebra or equivalently if the closed unit ball of F has an isolated extreme point (compare [15]).

2.2 Remark Let U be a JB*-triple and let $\mathcal{T} = \mathcal{T}(U)$ be the set of all conjugations of U. Then the group $\mathrm{Aut}_s(U)$ acts on $\mathcal{T}$ by $\tau \mapsto g\tau g^{-1}$ and the subgroup $\mathrm{Aut}(U)$ has the same orbits in $\mathcal{T}$ as $\mathrm{Aut}_s(U)$.

Proof For every $\tau \in \mathcal{T}$ and every $g \in \mathrm{Aut}_s(U)$ we have $g\tau g^{-1} = g\tau\tau\tau g^{-1} = (g\tau)\tau(g\tau)^{-1}$. $\square$

2.3 Conjecture *$\mathcal{T}$ is a real-analytic submanifold of the Lie group $\mathrm{Aut}_s(U)$ and the orbits of $\mathrm{Aut}(U)^0$ in $\mathcal{T}$ are just the connected components of $\mathcal{T}$.*

In case $\dim U < \infty$ the group $G = \mathrm{Aut}(U)$ is compact and $\mathcal{T}$ is a finite union of G-orbits (see section 4). From this it easily follows that conjecture 2.3 holds in this case.

For the rest of the section denote by E a real or complex JB*-triple. The element $e \in E$ is called a *tripotent* if $\{eee\} = e$ holds. Every tripotent e induces decompositions

$$E = E_1 \oplus E_{1/2} \oplus E_0 = E^1 \oplus E^{-1} \oplus E^0$$

into the corresponding Peirce spaces where $E_k = E_k(e)$ is the k-eigenspace of $e \square e$ and $E^k = E^k(e)$ is the k-eigenspace of the conjugate-linear operator $Q(e)$ in E for every $k \in \mathbb{R}$. These satisfy

$$E_1 = E^1 \oplus E^{-1}, \quad E_{1/2} \oplus E_0 = E^0$$

$$E_0 \perp E_1, \quad \{E_i E_j E_k\} \subset E_{i-j+k} \quad \text{for all } i, j, k$$

$$\{E^i E^j E^k\} \subset E^{ijk} \quad \text{if} \quad ijk \neq 0 .$$

In particular, E^1 is a JB-algebra with unit e and E^{-1} is a real subtriple of E. In case E is a complex JB*-triple obviously $E^{-1} = iE^1$ holds whereas in the real situation the dimensions of E^1 and E^{-1} are not correlated.

We call a subset $S \subset E$ *orthogonal* if $0 \notin S$ and $x \perp y$ for every $x \neq y$ in S. Denote by $\mathsf{r} = \mathsf{r}(E)$ the minimal cardinal number satisfying $\mathrm{card}(S) \leq \mathsf{r}$ for every orthogonal subset $S \subset E$ and call it the *rank* of E. The rank $\mathsf{r}(e)$ of the tripotent e then is defined as the *rank* of the Peirce space $E_1(e)$. The tripotent e is called *complete* if $E_0(e) = 0$ holds and e is called *minimal* if $E^1(e) = \mathbb{R}e \neq 0$ (or equivalently $\mathsf{r}(e) = 1$). In general, E does not contain a minimal nor a complete tripotent, for it is known that the complete tripotents are precisely the extremal points of the unit ball in E, compare [15]. In case E is a real or complex JBW*-triple therefore there always exist complete tripotents. In case E is a JBW*-triple factor we call a complete tripotent $e \in E$ *maximal* if $\dim(E_{1/2}(e)) \le \dim(E_{1/2}(\tilde{e}))$ holds for every complete tripotent $\tilde{e} \in E$. In finite dimension every complete tripotent is maximal since in this case any two complete tripotents in a factor are equivalent by an automorphism, compare [23]. Denote by $\mathcal{M} = \mathcal{M}(E) \subset E$ the subset of all maximal tripotents in the JBW*-triple factor E. This is a real analytic submanifold of E on which $\mathrm{Aut}(E)$ acts with open orbits [15]. Clearly $\mathcal{U}(E)$ is open and closed in $\mathcal{M}(E)$ and both coincide if $\mathcal{U}(E) \neq \emptyset$.

2.4 Proposition *Let H be a complex Hilbert space of dimension n and let $E \subset U := \mathcal{L}(H)$ be a real or complex subtriple. Then*

(i) *$\mathcal{U}(U)$ is precisely the group $\mathsf{U}(n)$ of all unitary operators on H,*

(ii) *$\mathcal{U}(E) = E \cap \mathcal{U}(U)$ if $\mathbf{1}_H \in E$,*

(iii) *E can be realized as subtriple $E \subset \mathcal{L}(K)$ for some complex Hilbert space K in such a way that $\mathbf{1}_K \in E$ if and only if $\mathcal{U}(E) \neq \emptyset$,*

(iv) *$\mathcal{U}(E)$ is connected if E is a complex JBW*-triple.*

Proof ad (i) Follows for instance with Lemma 3.1.

ad (ii) Suppose $e \in \mathcal{U}(E)$. Then $\mathbf{1}_H \in U_1(e) = pUq$ for the projections $p = ee^*$ and $q = e^*e$ implies $p = q = \mathbf{1}_H$, that is $e \in \mathcal{U}(U)$.

ad (iii) Fix $e \in \mathcal{U}(E)$. Then with p, q as before we have again $E \subset pUq$ and $z \mapsto e^*ze^*e|K$ defines an injective triple morphism $\varphi: E \to \mathcal{L}(K)$ with $\varphi(e) = \mathbf{1}_K$ where $K := q(H)$.

ad (iv) By (ii) and (iii) we may assume $e := \mathbf{1}_H \in E$ and hence $\mathcal{U}(E) = E \cap \mathcal{U}(U)$. Fix $c \in \mathcal{U}(E)$. Since c is a unitary and hence also a normal operator on H by (i) there is an abelian subtriple $A \subset E$ containing both e and c. We may assume that A is maximal abelian in E. This implies that A is a commutative C*-subalgebra of $\mathcal{L}(H)$ and also is w^*-closed in E, i.e. A is a commutative W*-algebra with unit e. But then $c = \exp(ih)$ for some $h = h^* \in A$ and $t \mapsto \exp(ith)$ for $0 \le t \le 1$ is a curve in $(\mathcal{U}(U) \cap A) \subset \mathcal{U}(E)$ connecting e with c. □

3. Complex Cartan factors

In the following let H be a complex Hilbert space of dimension n with fixed conjugation $x \mapsto \overline{x}$ and corresponding real form $X := \{x \in H : \overline{x} = x\}$. Then $\langle x | y \rangle := (x | \overline{y})$ defines a symmetric bilinear form $H \times H \to \mathbb{C}$ and the orthogonal group $\mathsf{O}(n) = \mathsf{Imt}(X)$ can be identified in a natural way with the subgroup of all $g \in \mathsf{U}(n) = \mathsf{Imt}(H)$ satisfying $\langle x | y \rangle = \langle gx | gy \rangle$ for all $x, y \in H$.

Now let K be a further complex Hilbert space with fixed conjugation also written as $y \mapsto \overline{y}$. Denote by m the dimension of K and assume for easier notation that $H = K$ in case $n = m$. For every $z \in \mathcal{L}(H,K)$ we get operators $\overline{z} \in \mathcal{L}(H,K)$ and $z' \in \mathcal{L}(K,H)$ uniquely determined by $\overline{z}(\overline{x}) = \overline{zx}$ and $\langle zx \mid y\rangle = \langle x \mid z'y\rangle$ for all $x \in H$ and $y \in K$. Clearly $z^* = \overline{z}'$ holds and $z \mapsto \overline{z}$ defines a conjugation on the Banach space $\mathcal{L}(H,K)$.

We will frequently use the following elementary result

3.1 Lemma *Let $H, K \neq 0$ be arbitrary complex Banach spaces. Then*

(i) *$a \in \mathcal{L}(K)$, $b \in \mathcal{L}(H)$ with $az = zb$ for every rank-1-operator $z \in \mathcal{L}(H,K)$ implies that a, b are multiples of the respective identity operators.*

In case H, K are complex Hilbert spaces with distinguished conjugation and $\varepsilon \in \{1, -1\}$ is fixed also the following holds

(ii) *$a, b \in \mathcal{L}(H,K)$ with $a'z = z'b$ for all $z \in \mathcal{L}(H,K)$ implies $a = b = 0$, provided* $\dim H > 1$.

(iii) *$a, b \in \mathcal{L}(H)$ with $az = zb$ for all $z = \varepsilon z' \in \mathcal{L}(H)$ implies that a, b are multiples of the identity, provided* $\dim H > (1 - \varepsilon)$.

Proof ad(i) Easy consequence of the Hahn-Banach Theorem.

ad (ii) By the assumption $\dim H > 1$ there is an orthogonal decomposition $H = H_1 \oplus H_2$ into closed linear subspaces H_i of positive dimension that are invariant under the conjugation of H. For every $z \in \mathcal{L}(H,K)$ and $i = 1,2$ put $z_i := z \mid H_i \in \mathcal{L}(H_i, K)$. Then $a_i'z_j = z_i'b_j$ for all $z \in \mathcal{L}(H,K)$ implies the claim.

ad(iii) Suppose $\varepsilon = -1$ (the case $\varepsilon = 1$ is similar). By assumption there is an orthogonal decomposition $H = H_1 \oplus H_2$ with H_i invariant under the conjugation of H and $\dim H_1 = 2$. Fix $j \in \mathrm{GL}(H_1)$ with $j' = -j$ and write every $z \in \mathcal{L}(H)$ as operator matrix $z = (z_{ij})$ with $z_{ij} \in \mathcal{L}(H_j, H_i)$. Applying the assumption of (iii) for

$$z = \begin{pmatrix} j & 0 \\ 0 & 0 \end{pmatrix} \qquad \text{we derive} \qquad a_{21} = 0 \quad \text{and} \quad b_{12} = 0\,.$$

Applying it again for all

$$z = \begin{pmatrix} 0 & -x' \\ x & 0 \end{pmatrix} \qquad \text{we derive} \qquad \begin{pmatrix} a_{12}x & -a_{11}x' \\ a_{22}x & 0 \end{pmatrix} = \begin{pmatrix} -x'b_{21} & -x'b_{22} \\ xb_{11} & 0 \end{pmatrix}$$

for all $x \in \mathcal{L}(H_1, H_2)$. From (ii) we get $a_{12} = 0$, $b_{21} = 0$ and the claim follows with (i). □

An important class of JBW*-triples is given by the *Cartan factors*. These come in six types as follows (always denoted by U) †

† One has to distinguish the notions 'Cartan factor of type I, II,... or VI' and 'JBW*-triple of type I, II or III'. Every Cartan factor as a JBW*-triple is of type I.

$\mathbf{I}_{n,m} := \mathcal{L}(H,K)$.
The transposition $z \mapsto z'$ defines a triple isomorphism $\mathcal{L}(H,K) \to \mathcal{L}(K,H)$, i.e. we may assume $n \le m$. The group $\mathsf{Aut}(U)^0$ is given by all transformations $z \mapsto uzv$ with $u \in \mathsf{U}(m)$, $v \in \mathsf{U}(n)$ (compare [17] p. 91). In case $n = m > 1$ there is precisely a second connected component of $\mathsf{Aut}(U)$ given by all $z \mapsto uz'v$ with $u, v \in \mathsf{U}(n)$ – in all other cases $\mathsf{Aut}(U)$ is connected. Abstractly, $\mathsf{Aut}(U)^0$ is just the Lie group $\big(U(n) \times U(m)\big)/U(1)$ where the embedding of $U(1)$ is given by $\lambda \mapsto (\lambda, \overline{\lambda})$. This follows for instance from 3.1.i. Clearly $\mathcal{M}(U) = \mathcal{U}(U) = \mathsf{Imt}(H,K)$ is connected and nonempty in case $n = m$. In all other cases $\mathcal{U}(U)$ is empty and $\mathcal{M}(U)$ is the space of all (necessarily not surjective) complex-linear isometries $H \to K$.

$\mathbf{II}_n := \{z \in \mathcal{L}(H) : z' = -z\}$ for $n \ge 2$ as subtriple of $\mathcal{L}(H)$. All transformations $z \mapsto uzu'$, $u \in \mathsf{U}(n)$, form a connected subgroup G of $\mathsf{Aut}(U)$. Actually, $G = \mathsf{Aut}(U)$ holds for all $n \ne 4$, compare f.i. [11] – the proof there works for all even cardinal numbers $n \ge 6$ and for all finite odd n an induction argument can be given ($\mathbf{II}_3$ is a complex Hilbert space). $\mathbf{II}_4$ is isomorphic to the spin factor $\mathbf{IV}_6$ defined below and its automorphism group has two connected components (for an explicit isomorphism to $\mathbf{IV}_6$ compare [22] p. 200). In this case $\mathsf{Aut}(U) = G \cup G\theta$ with

$$\theta\begin{pmatrix} a & b \\ c & d \end{pmatrix} := \begin{pmatrix} a & c \\ b & d \end{pmatrix} \quad \text{for all} \quad a,b,c,d \in \mathbb{C}^{2\times 2} \tag{3.2}$$

satisfying $a + a' = b + c' = d + d' = 0$. Abstractly, for all n we have $\mathsf{Aut}(U)^0 \cong \mathsf{U}(n)/\mathsf{O}(1)$ where $\mathsf{O}(1)$ is identified with $\{\pm\mathbf{1}\} \subset \mathsf{U}(n)$. Finally, $\mathcal{U}(U) = \mathsf{Imt}(H) \cap U$ is empty if and only if n is finite and odd.

$\mathbf{III}_n := \{z \in \mathcal{L}(H) : z' = z\}$ as subtriple of $\mathcal{L}(H)$. Always $\mathsf{Aut}(U) = \{z \mapsto uzu' : u \in \mathsf{U}(n)\} \cong \mathsf{U}(n)/\mathsf{O}(1)$ is connected (the proof in [11] works also in the finite dimensional case). Moreover $\mathcal{U}(U) = \mathsf{Imt}(H) \cap U$ is nonempty.

$\mathbf{IV}_n$ Denote by $\|\cdot\|_2$ the Hilbert norm of H and define a triple product on H by

$$\{xyz\} := (x|y)z + (z|y)x - \langle x|z\rangle\overline{y}$$

for all $x, y, z \in H$. Let U be the Banach space obtained from H with the equivalent norm defined by

$$\|z\|^2 := (z|z) + \sqrt{(z|z)^2 - |\langle z|z\rangle|^2}\ .$$

Then U together with the above triple product is a JBW*-triple and is called the (complex) *spin factor* of dimension n if $n > 2$. On the real form X of H the norms $\|\cdot\|_2$ and $\|\cdot\|$ coincide, in particular $\mathcal{U}(U)\cap X = S := \{x \in X : \|x\| = 1\}$ is the unit sphere in X. Clearly,

$$\mathcal{U}(U) = \{\lambda s : \lambda \in \mathsf{U}(1), s \in S\}$$

is nonempty. $\mathsf{Aut}(U) \cong (\mathsf{U}(1) \times \mathsf{O}(n))/\mathsf{O}(1)$ has 2 connected components for $n \in 2\mathbb{N}$ and is connected in all other cases.

V$:= M_{12}(\mathbb{O}^{\mathbb{C}})$ the 1×2-matrices over the complex Cayley numbers $\mathbb{O}^{\mathbb{C}}$. Then $\mathcal{U}(U)$ is empty.

VI$:= \mathcal{H}_3(\mathbb{O}^{\mathbb{C}})$ the hermitian 3×3-matrices over $\mathbb{O}^{\mathbb{C}}$. Here $\mathcal{U}(U)$ is nonempty.

The types **V** and **VI** are the *exceptional Cartan factors*. They have dimensions 16 and 27 – for details in the finite dimensional case compare [23], [10]. The Cartan factors of types **I** – **IV** are called the *classical Cartan factors*. It is known (compare f.i. [18] p. 475) that every Cartan factor occurs (up to isomorphy) in the following list precisely once:

$\mathbf{I}_{n,m}$ for $m \geq n \geq 1$, $\mathbf{II}_n$ for $n \geq 5$, $\mathbf{III}_n$ for $n \geq 2$, $\mathbf{IV}_n$ for $n \geq 5$, **V**, **VI**.

As a consequence we have in particular

3.3 Lemma *Let U be a Cartan factor. Then the set $\mathcal{M}(U)$ of all maximal tripotents in U is a non-void connected real-analytic submanifold of U on which the group $\mathsf{Aut}^0(U)$ acts transitively.*

Proof In finite dimensions this follows from [23]. The remaining cases follow from the above considerations. □

The Cartan factors are precisely the JBW*-triple factors U containing a minimal tripotent. Every tripotent $e \in U$ has a representation

$$e = \sum_{i \in I} e_i$$

where $(e_i)_{i\in I}$ is an orthogonal family of minimal tripotents in U with $\mathsf{card}(I) = \mathsf{r}(e)$ and the sum converges with respect to the w^*-topology. Define the *rank* $\mathsf{r}(x)$ of an arbitrary element $x \in U$ as the minimum of all ranks $\mathsf{r}(e)$ where e runs through all tripotents e with $x \in U^1(e)$. In case U is a Cartan factor of type **I** – **III** as defined above, every $x \in U$ may be considered as an operator between complex Hilbert spaces and $\mathsf{r}(x)$ is essentially the dimension of the range of this operator (more precisely, the dimension of the closure of its range in the case of types **I** and **III** and half of this dimension in the case of type **II**). Cartan factors of infinite rank only occur in the types **I** – **III**.

It is well known that every Cartan factor U of finite rank is simple, i.e. $\{0\}$ and U are the only ideals in U. On the other hand, in every Cartan factor U of infinite rank the elements of finite rank form an ideal $\mathcal{F}$ that is not closed but is contained in every non-zero ideal of U. Furthermore, the closure $\mathcal{K}$ of $\mathcal{F}$ (the space of all 'compact operators') is a closed ideal with $\{0\} \neq \mathcal{K} \neq U$. For every infinite cardinal number c denote by $\mathcal{F}_\mathsf{c}$ the set of all $x \in U$ with $\mathsf{r}(x) < \mathsf{c}$. It is easily verified that $\mathcal{F}_\mathsf{c}$ is an ideal in U. The closure $\mathcal{K}_\mathsf{c}$ of $\mathcal{F}_\mathsf{c}$ is the smallest closed ideal in U containing all tripotents of rank $< \mathsf{c}$.

3.4 Proposition *For every Cartan factor U of type* **I** *the assignment $\mathsf{c} \mapsto \mathcal{K}_\mathsf{c}$ defines a 1-1-correspondence between all cardinal numbers c with $\aleph_0 \leq \mathsf{c} \leq \mathsf{r}(U)$ and the set of all closed ideals I of U with $\{0\} \neq I \neq U$. In particular, the set of all closed ideals of U is well ordered by inclusion.*

Proof By assumption we may assume that $U = \mathcal{L}(H, K)$ for complex Hilbert spaces H, K of infinite dimensions. Let I be a closed ideal with $\{0\} \neq I \neq U$. Denote by c the smallest cardinal number such that $\mathsf{r}(e) < \mathsf{c}$ for every tripotent $e \in I$. Then $\mathcal{F} \subset I$ implies $\mathsf{c} \geq \aleph_0$. For every $a \in \mathcal{L}(K)$ and $b \in \mathcal{L}(H)$ the operator $\lambda \in \mathcal{L}(U)$ defined by $\lambda(x) = axb$ is a complex-linear combination of derivations of U and hence leaves the ideal $I \subset U$ invariant by [14] Proposition 1.8. From this the following property can be derived easily: *For every tripotent $e \in I$ and every tripotent $f \in U$ with $\mathsf{r}(f) \leq \mathsf{r}(e)$ also $f \in I$ holds.* But then $\mathsf{c} \leq \mathsf{r}(U)$ must hold – otherwise U would contain a tripotent e with $\mathsf{r}(e) = \mathsf{r}(U)$ and hence also would contain a complete tripotent of U, i.e. $I = U$ contrary to our assumption. Also $\mathcal{K}_\mathsf{c} \subset I$ follows from the above property. For the proof of the opposite inclusion $I \subset \mathcal{K}_\mathsf{c}$ fix an arbitrary element $x \in I$ and denote by V the smallest w^*-closed complex subtriple of U containing x. It is known that V has the structure of a commutative W*-algebra in such a way that the triple product is given by $\{abc\} = a\bar{b}c$ on V. Representing V as $L^\infty(\Sigma, \mu)$ for some finite measure space (Σ, μ) shows that to every $\varepsilon > 0$ there is a tripotent $e \in V$ together with elements $y, z \in V$ such that $y = \{eex\}$, $e = \{yez\}$ and $\|x - y\| < \varepsilon$. But this implies $y, e \in I$ and hence $e, y \in \mathcal{K}_\mathsf{c}$, i.e. $x \in \mathcal{K}_\mathsf{c}$ and consequently $I = \mathcal{K}_\mathsf{c}$. Now suppose that also $I = \mathcal{K}_\mathsf{d}$ holds for some infinite cardinal number $\mathsf{d} < \mathsf{c}$. Fix a tripotent e of rank d in I. Then there is a sequence (x_i) in $\mathcal{F}_\mathsf{d}$ with $e = \lim x_i$, a contradiction since $\aleph_0 \mathsf{d} = \mathsf{d} < \mathsf{c}$. □

Every spin factor can be realized as a closed $*$-invariant subtriple of some $\mathcal{L}(H)$ containing the identity $e := \mathbf{1}_H$ of $\mathcal{L}(H)$, compare [9]. Clearly, e has rank 2 when considered as an element of U, but it has rank $\dim H$ when considered as an element of $\mathcal{L}(H)$. The Cartan factor $\mathbf{I}_{n,m}$ may be identified in a natural way with the Banach space $\mathcal{L}^2(H \times K, \mathbb{C})$ of all continuous bilinear forms $\lambda: H \times K \to \mathbb{C}$ where $\|\lambda\|:= \sup|\lambda(h,k)|$ with $h \in H$ and $k \in K$ running over all unit vectors. The Cartan factor $\mathbf{III}_n$ ($\mathbf{II}_n$ resp.) is isomorphic to the subspace of all symmetric (alternating, resp.) forms in $\mathcal{L}^2(H \times H, \mathbb{C})$.

4. Real Cartan factors

As in section 3 let H, K be complex Hilbert spaces of dimensions n, m with fixed conjugations. In case of $n = m$ we assume $H = K$ for easier notation. Denote by $X := \{x \in H : \overline{x} = x\}$ and $Y := \{y \in K : \overline{y} = y\}$ the corresponding real forms.

A real JB*-triple E is called a *real Cartan factor* if it is a real form of a (complex) Cartan factor. Also, the real forms of (complex) spin factors (JBW*-triples respectively) are called *real spin factors* (*real JBW*-triples* respectively). The real Cartan factors of finite dimension have been completely classified by Loos (compare 11.4 in [23]) in terms of 12 different types (8 classical series and 4 exceptional types)

$$\mathbf{I}^{\mathbb{R}}_{n,m}, \mathbf{I}^{\mathbb{H}}_{2p,2q}, \mathbf{I}^{\mathbb{C}}_{n,n}, \dots, \mathbf{VI}^{\mathbb{O}}.$$

The notation has the nice property that erasing the superscripts gives the corresponding hermitification – for instance $\mathbf{I}_{n,m}$ in case of the real Cartan factor $\mathbf{I}^{\mathbb{R}}_{n,m}$. In the following we extend Loos' classification to infinite dimensions by determining all conjugations on classical Cartan factors and then finding the associated equivalence classes. This is done in the following way: Suppose for $n \neq m$ we have $U = \mathcal{L}(H, K) = \mathbf{I}_{n,m}$ as an example. Then every conjugation τ of U must be of the form $\tau(z) = u\overline{z}v$ for suitable $v \in \mathsf{U}(n)$ and $u \in \mathsf{U}(m)$ since $z \mapsto \overline{z}$ is already known to be a conjugation of U. Now $\tau^2(z) = u\overline{u}z\overline{v}v = z$ for all $z \in U$ implies $az = zb$ for $a := u\overline{u}$, $b := v^*v'$ and all $z \in U$. By 3.1.i the unitary operators a, b are real multiples of the identity, i.e. there are only two cases possible: (i) Either u, v are both symmetric or (ii) they are both alternating. Case (ii) can only happen if n, m are even (i.e. not contained in $2\mathbb{N} + 1$) and then H, K both can be made into quaternionic Hilbert spaces. Consider a second conjugation $\tilde{\tau}(z) = \tilde{u}\overline{z}\tilde{v}$ on U. Then $\tau, \tilde{\tau}$ are equivalent if and only if for some $c \in \mathsf{U}(m), d \in \mathsf{U}(n)$ the relations $\tilde{u} = cuc'$ and $\tilde{v} = dvd'$ hold. The set $\{v \in \mathsf{U}(n) : v = v'\} = \mathcal{U}(\mathbf{III}_n)$ is connected and a $\mathsf{U}(n)$-orbit under the action $v \mapsto dvd'$ (compare section 3), i.e. all conjugations of case (i) form a complete equivalence class of conjugations of U and this class is connected. By similar reasoning in all other cases we get (compare also the proof of Proposition 5.1)

4.1 Theorem *Let U be a classical Cartan factor and let $\mathcal{T}$ be the set of all conjugations of U. Then the equivalence classes of $\mathcal{T}$ consist precisely of all conjugations τ of the following form:*

$\mathbf{I}_{n,m}$ for $m \geq n \geq 1$

(i) $\tau(z) = u\bar{z}v$ with $v = v' \in \mathsf{U}(n)$, $u = u' \in \mathsf{U}(m)$

(ii) $\tau(z) = u\bar{z}v$ with $v = -v' \in \mathsf{U}(n)$, $u = -u' \in \mathsf{U}(m)$ if n, m are even

(iii) $\tau(z) = uz^*u$ with $u \in \mathsf{U}(n)$ if $n = m > 1$

$\mathbf{II}_n$ for $n \geq 2$, for $n = 4$ see Remark 4.2

(iv) $\tau(z) = u\bar{z}u$ with $u = u' \in \mathsf{U}(n)$

(v) $\tau(z) = u\bar{z}u$ with $u = -u' \in \mathsf{U}(n)$ if $n > 2$ is even

$\mathbf{III}_n$ for $n \geq 1$

(vi) $\tau(z) = u\bar{z}u$ with $u = u' \in \mathsf{U}(n)$

(vii) $\tau(z) = u\bar{z}u$ with $u = -u' \in \mathsf{U}(n)$ if n is even

$\mathbf{IV}_n$ for $n \geq 3$ and cardinal numbers $r \geq s$ with $r + s = n$, for $n = 2$ see Remark 4.3,

(viii) $\tau(z) = \lambda g(\bar{z})$ *with* $\lambda \in U(1)$, $g \in \mathsf{O}(n)$ *such that* $g^2 = \mathbf{1}$ *and such that the* $+1$*-eigenspace of* g *has dimension* r *and the* -1*-eigenspace has dimension* s.

The unitary conjugations are precisely those in (iii), (v), (vi) and (viii) *with* $s = 1$. *Each equivalence class* (i) - (viii) *can be represented by the following real Cartan factors* E, *where* H *is a complex Hilbert space of dimension* n, X, Y *are real Hilbert spaces of dimensions* n, m *and* P, Q *are Hilbert spaces of dimensions* p, q *over the quaternion field* $\mathbb{H}$. *The triple product in cases* (i) – (vii) *is given by* $\{xyx\} = xy^*x$

(i) $\mathbf{I}^{\mathbb{R}}_{n,m} = \mathcal{L}(X,Y)$ (ii) $\mathbf{I}^{\mathbb{H}}_{2p,2q} = \mathcal{L}(P,Q)$

(iii) $\mathbf{I}^{\mathbb{C}}_{n,n} = \{z \in \mathcal{L}(H) : z^* = z\}$ (iv) $\mathbf{II}^{\mathbb{R}}_n = \{x \in \mathcal{L}(X) : x' = -x\}$

(v) $\mathbf{II}^{\mathbb{H}}_{2p} = \{w \in \mathcal{L}(P) : w^* = w\}$ (vi) $\mathbf{III}^{\mathbb{R}}_n = \{x \in \mathcal{L}(X) : x' = x\}$

(vii) $\mathbf{III}^{\mathbb{H}}_{2p} = \{w \in \mathcal{L}(P) : w^* = -w\}$

(viii) $\mathbf{IV}^{r,s}_n = E$ *where the Banach space* E *is the* ℓ^1*-sum* $E = X_1 \oplus^1 X_2$ *of closed linear subspaces* $X_1, X_2 \subset X$ *of dimensions* r, s *with* $X_2 = X_1^{\perp}$. *Then with the inner product* $(x|y)$ *from* X *and the involution* $x \to \overline{x}$ *on* E *defined by* $\overline{x} = (x_1, -x_2)$ *for every* $x = (x_1, x_2)$ *the Jordan triple* *product on E is given by*

$$\{xyz\} = (x|y)z + (z|y)x - (x|\overline{z})\overline{y}\,. \qquad \square$$

4.2 Remark Besides (iv) and (v) the factor $U = \mathbf{II}_4 \approx \mathbf{IV}_6$ has two additional equivalence classes represented by the two conjugations $\tau(z) = \theta(\overline{z})$ and $\sigma(z) = \theta(j\overline{z}j)$ where $j = \left(\begin{smallmatrix} 0 & e \\ -e & 0 \end{smallmatrix}\right)$, e is the unit 2×2-matrix and θ is defined as in (3.2). The corresponding real forms satisfy $U^\tau \approx \mathbf{IV}_6^{4,2}$ and $U^\sigma \approx \mathbf{IV}_6^{6,0}$.

4.3 Remark The spin spaces $\mathbf{IV}_2 \approx \mathbb{C} \oplus \mathbb{C}$ and $\mathbf{IV}_2^{1,1} \approx \mathbb{R} \oplus \mathbb{R}$ are not factors and $\mathbf{IV}_2^{2,0} \approx \mathbb{C}_\mathbb{R}$ is a complexifiable factor in contrast to all real Cartan factors.

For p, q finite in Theorem 4.1 the real Cartan factor $\mathcal{L}(P, Q)$ can be identified with the space $\mathbb{H}^{p \times q}$ of all quaternionic $p \times q$-matrices $a = (a_{ij})$ where the triple product is given by $\{zaz\} = za^*z$ with $a^* = (\overline{a_{ji}})$, more precisely: Consider $P \approx \mathbb{H}^{1 \times p}$ as space of row vectors $z = (z_1, \ldots, z_p)$ with inner product given by $(z|w) := zw^*$ and scalar multiplication given by $\lambda z = (\lambda z_1, \ldots, \lambda z_p)$ for every $\lambda \in \mathbb{H}$. Then every matrix $a \in \mathbb{H}^{p \times q}$ induces an $\mathbb{H}$-linear map $P \to Q \approx \mathbb{H}^{1 \times q}$ by matrix multiplication $z \mapsto za$ from the right.

Together with the classification of the exceptional types in [23] we get

4.4 Corollary *Every real Cartan factor occurs up to isomorphy precisely once in the list:*

$\mathbf{I}_{n,m}^{\mathbb{R}}, \mathbf{I}_{2n,2m}^{\mathbb{H}}, \mathbf{III}_{2n}^{\mathbb{H}}$ for $m \geq n \geq 1$, $\qquad \mathbf{I}_{n,n}^{\mathbb{C}}, \mathbf{III}_n^{\mathbb{R}}$ for $n \geq 2$,

$\mathbf{II}_{2n}^{\mathbb{H}}$ for $n \geq 3$, $\qquad \mathbf{II}_n^{\mathbb{R}}$ for $n \geq 5$,

$\mathbf{IV}_n^{r,s}$ for $r \geq s$ and $r + s = n \geq 5$, $\qquad \mathbf{V}^{\mathbb{O}}, \mathbf{V}^{\mathbb{O}_0}, \mathbf{VI}^{\mathbb{O}}, \mathbf{VI}^{\mathbb{O}_0}$.

4.5 Lemma *For every real JBW*-triple factor E the following two conditions ar equivalent*

(i) *E is a real Cartan factor or the realification of a complex Cartan factor,*

(ii) *E contains a minimal tripotent e.*

Proof Suppose, $E = U^\tau$ for some complex JBW*-triple U with conjugation τ.
Case 1: U is a factor. Then the tripotent e has rank ≤ 2 with respect to U. In particular, U has a minimal tripotent. Therefore U is a complex Cartan factor and E is a real Cartan factor.
Case 2: U is not a factor. Then $U = U_1 \oplus U_2$ is the direct sum of two w^*-closed ideals U_i with $U_2 = \tau(U_1)$. Then E is isomorphic to the realification of U_1. Also, U_1 must be a factor and writing $e = e_1 + e_2$ gives a minimal tripotent $e_1 \in U_1$, that is, U_1 is a complex Cartan factor. $\square$

With Theorem 4.1 also all real forms of an arbitrary ℓ^∞-sum of Cartan factors could easily be determined. Instead of doing this let us discuss the case of certain continuous products of Cartan factors. Let S be a locally compact topological space and let V be a Cartan factor. Then the Banach space $U := C_0(S, V)$ of all continuous V-valued functions f on S vanishing at infinity is a JB*-triple itself – put $\|f\| := \sup\{\|f(s)\| : s \in S\}$ and define the triple product pointwise. By [20] the conjugations of $U = C_0(S, V)$ can be described in terms of the conjugations of V and the topology of S in the following way: Denote by G the group $\mathsf{Aut}(V)$ endowed with the topology of pointwise convergence on V. Fix a conjugation τ on V. Then $g \mapsto g^\tau := \tau g \tau$ defines a group automorphism of G and $\{\tau c : c \in C\}$ with $C := \{c \in G : c^\tau c = \mathbf{1}_V \in G\}$ is the set of all conjugations of V. Now suppose that σ is a conjugation of $U = C_0(S, V)$. Then, by [20] proposition 5.1 there is a homeomorphism $\psi: S \to S$ and a continuous map $c: S \to G$ such that for every $s \in S$ and every $f \in U$

$$(4.6) \qquad (\sigma f)(s) \;=\; \tau\big(c(s)(f \circ \psi(s))\big)\,.$$

Clearly, ψ and c satisfy

$$(4.7) \qquad \psi^2 = \mathbf{1}_S \quad \text{and} \quad c(s)^\tau c(\psi(s)) \;\equiv\; \mathbf{1}_V \in G\,,$$

and every pair of continuous maps $\psi: S \to S$, $c: S \to G$ satisfying (4.7) defines a conjugation σ of U via (4.6). Denote by $E := U^\sigma$ the real form of U with respect to the conjugation σ. For every $s \in S$ define the triple homomorphism $\varepsilon_s: E \to V_{\mathbb{R}}$ by $\varepsilon_s(f) = f(s)$. Then

$$\varepsilon_s(E) = \begin{cases} V^{\tau c(s)} & \text{if } \psi(s) = s \\ V_{\mathbb{R}} & \text{otherwise}\,. \end{cases}$$

In general, the fixed point set $\mathsf{Fix}(\psi)$ of ψ is not open in S and even if $\mathsf{Fix}(\psi) = S$ and all real forms $V^{\tau c(s)}$, $s \in S$, are equivalent to the real form F of V the real form $E = U^\sigma$ may not be equivalent to $C_0(S, F)$ (see [20] for examples). On the other hand, as an example the following is clear. *For $I := \{s \in \mathbb{R} : 0 < s \leq 1\}$ every real form of $C_0(I, V)$ (the space of all continuous curves in V starting at the origin) is equivalent to $C_0(I, F)$ for some real form F of V.*

For the rest of the section assume that the space S is compact. Then $U = C_0(S, V)$ is the same as the space $C(S, V)$ of all continuous V-valued functions on S. For convenience we put $C(\emptyset, V) := \{0\}$ for the empty space. By definition, S is called *stonean* if the closure of every open subset again is open in S. In particular, every hyperstonean space Ω (compare [24] for details) is stonean, that is precisely the case when $C(\Omega)$ has a predual (i.e. is a W*-algebra or in our setting is a JBW*-triple) and hence is of the form $L^\infty(\Sigma,\mu)$. For every JB*-triple V of finite dimension then also $C(\Omega, V) \simeq L^\infty(\Sigma,\mu,V)$ is a JBW*-triple.

4.8 Proposition *Let V be a Cartan factor of finite dimension and let S be a stonean space. Then to every real form E of $U = C(S,V)$ there is a finite system $S_0, S_1, \dots, S_n$ of open closed subsets of S such that*

(i) *S is the disjoint union of $S_0, S_1, \dots, S_n$,*

(ii) *S_0 and S_1 are homeomorphic,*

(iii) *E is isomorphic to the ℓ^∞-sum*

$$\bigoplus_{1\le k\le n}^{\infty} C(S_k, F_k)$$

where $F_1 = V_{\mathbb{R}}$ and $F_2, \dots, F_n$ are mutually inequivalent real forms of V.

Proof Fix a conjugation σ of U and choose τ, ψ, c as above. Let $S_1 \subset S$ be a subset which is maximal with respect to the property: *S_1 is open in S and disjoint from $S_0 := \psi(S_1)$*. Then S is the disjoint union $S = P \cup Q$ of open ψ-invariant subsets, where P is the closure of $S_0 \cup S_1$ in S. From the maximality of S_1 the inclusion $Q \subset \mathsf{Fix}(\psi)$ easily follows. For the proof of the opposite inclusion define $h: S_0 \cup S_1 \to \mathbb{R}$ by $h(s) = -h(\psi s) = 1$ for all $s \in S_0$. Since the domain of h is open and dense in the stonean space P and h is bounded and continuous this function has a continuous extension $\overline{h}$ to all of P, compare [24] p. 105. But $\overline{h}$ takes only the values $\{\pm 1\}$ and satisfies $\overline{h} = -\overline{h}\circ\psi$, i.e. ψ cannot have a fixed point in P. Therefore $Q = \mathsf{Fix}(\psi)$ is open in S and via $f \mapsto (f|S_1, f|Q)$ the real form E is the ℓ^∞-sum of $C(S_1, V_{\mathbb{R}})$ and the real form of $C(Q,V)$ induced by σ, i.e. for the rest of the proof we may assume without loss of generality that $S = Q = \mathsf{Fix}(\psi)$. Denote again by $\mathcal{T} \subset \mathsf{Aut}_s(V)$ the space of all conjugations of V. Then $\mathcal{T} = \mathcal{T}_2 \cup \dots \cup \mathcal{T}_n$ is a finite disjoint union of G-orbits for the compact group $G = \mathsf{Aut}(V)$. Every $\mathcal{T}_k$ is open and closed in $\mathcal{T}$, i.e. $S_k := \{s \in S : \tau c(s) \in \mathcal{T}_k\}$ is also open and closed in S. Without loss of generality we may therefore assume that $n = 2$ – or equivalently – that all conjugations $\tau c(s)$ are equivalent to a single conjugation τ_0. Consider the set $\mathcal{R}$ of all pairs (R, α) where R is an open subset of S and $\alpha: R \to G$ is a continuous mapping with

$$\tau c(s) = \alpha(s)\tau_0\alpha(s)^{-1} \quad \text{for all} \quad s \in R\,.$$

$\mathcal{R}$ is partially ordered with respect to $(R,\alpha) \le (\tilde{R},\tilde{\alpha})$: $\iff$ ($R \subset \tilde{R}$ and $\alpha = \tilde{\alpha}|R$). Fix a maximal element (R,α) of $\mathcal{R}$. Since the action of G on $\mathcal{T}$ admits local continuous cross sections the open set R is dense in S. Again, α extends to a continuous map $\alpha: S \to \mathcal{L}(V)$ taking values in G, i.e. $R = S$ by the maximality of R. Consider the conjugation σ_0 of U defined by $(\sigma_0 f)(s) = \tau_0(f(s))$. The corresponding real form of U then is just $\mathcal{C}(S, V^{\tau_0})$. But then $\sigma = g\sigma_0 g^{-1}$ where $g \in \mathsf{Aut}(U)$ is defined by $(gf)(s) = \alpha(s)f(s)\alpha(s)^{-1}$ for all $s \in S$. □

5. Automorphism groups and isometric equivalence

5.1 Proposition *With the notations of Theorem 4.1 let E be one of the real Cartan factors* (i) - (viii). *Then the following transformations form an open subgroup $G \subset \mathsf{Aut}(E)$ of index ≤ 2.*

(i) $x \mapsto uxv$ with $u \in \mathsf{O}(m)$, $v \in \mathsf{O}(n)$

(ii) $w \mapsto uwv$ with $u \in \mathsf{Sp}(q)$, $v \in \mathsf{Sp}(p)$

(iii) $z \mapsto \pm uzu^*$ with $u \in \mathsf{U}(n)$

(iv),(vi) $x \mapsto \pm uxu'$ with $u \in \mathsf{O}(n)$

(v),(vii) $w \mapsto \pm uwu^*$ with $u \in \mathsf{Sp}(p)$

(viii) $(x_1, x_2) \mapsto (ux_1, vx_2)$ with $u \in \mathsf{O}(r)$, $v \in \mathsf{O}(s)$

G coincides with $\mathsf{Aut}(E)$ *except in the following cases where* $\mathsf{Aut}(E) = G \cup Gg$

(i) for $n = m > 1$ and $g(x) = x'$

(ii) for $p = q$ and $g(w) = w^*$

(iii) for $n > 1$ and $g(z) = z'$

(iv) for $n = 4$ and $g = \theta$ on $\{x \in \mathbb{R}^{4\times 4} : x' = -x\}$, compare (3.2)

(v) for $p = 2$ and $g(z) = z'$ on $\{z \in \mathbb{H}^{2\times 2} : z^* = z\}$

(viii) for $r = s$ and $g(x_1, x_2) = (x_2, x_1)$ if X_1 and X_2 are identified.

Proof We choose the case (ii) for $n = m = 2p$ as an example. Then we may take $E = U^\tau$ for $U = \mathbf{I}_{n,n}$ with conjugation $\tau(z) = -j\overline{z}j$ where $j = -j' \in \mathsf{U}(n)$ satisfies $j^2 = -\mathbf{1}$. This j makes H to an $\mathbb{H}$-Hilbert space P of dimension p in the following way: We may assume that $H = X \oplus X$ is the orthogonal sum of two identical copies of a complex Hilbert space X with involution $x \mapsto \overline{x}$ such that the conjugation on H is given by $(x, y) \mapsto (\overline{x}, \overline{y})$. Realizing every $z \in \mathcal{L}(H)$

by a 2×2-matrix with entries in $\mathcal{L}(X)$ we may assume that

$$j = \begin{pmatrix} 0 & \mathbf{1} \\ -\mathbf{1} & 0 \end{pmatrix} .$$

Defining the conjugate-unitary operator $a \mapsto j(\overline{a})$ on H as left multiplication by $\mathrm{j} \in \mathbb{H}$ the $\mathbb{C}$-vector space H becomes a left $\mathbb{H}$-vector space and actually a left $\mathbb{H}$-Hilbert space P with respect to the unique $\mathbb{H}$-valued extension of the inner product from H. Clearly, an $\mathbb{R}$-linear map $z: P \to P$ is $\mathbb{H}$-linear if and only if z is $\mathbb{C}$-linear and satisfies $jz = \overline{z}j$, i.e. $E = \mathcal{L}(P)$. Also, for every $z \in \mathcal{L}(P)$ the adjoints z^* with respect to H and P coincide. The group $\mathsf{Aut}(E)$ can be identified with the group $\{g \in \mathsf{Aut}(U) : \tau g \tau = g\}$. Suppose that $g \in \mathsf{Aut}(E)$ is of the form $g(z) = uzv$ with $u, v \in \mathsf{U}(n)$. Then $\overline{j} = j$ implies $j\overline{u}jzj\overline{v}j = uzv$ and hence $\overline{a}z = zb$ for $a := u'ju j$, $b := vjv'j$ and all $z \in U$. Lemma 3.1 gives a $\lambda \in \mathsf{U}(1)$ with $b = -\lambda^2 \mathbf{1}$. Replacing u, v by $\lambda u, \overline{\lambda} v$ we do not change g but may assume that $a = b = -\mathbf{1}$ or equivalently $uj = j\overline{u}$ and $vj = j\overline{v}$ which implies $u, v \in \mathsf{Sp}(p)$. This shows that $G = \{w \mapsto uwv : u, v \in \mathsf{Sp}(p)\}$ is the group of all $g \in \mathsf{Aut}(E)$ that extend to an automorphism in $\mathsf{Aut}^0(U)$ of U. Since $\mathsf{Aut}(U)$ has 2 connected components the group G has index ≤ 2 in $\mathsf{Aut}(E)$. The automorphism $z \mapsto -jz'j$ of U commutes with τ and induces on $E = \mathcal{L}(P)$ the automorphism $w \mapsto w^*$ which is not in G. This settles the special case (ii) for $p = q$. All other cases are similar. $\square$

5.2 Corollary *For every non-exceptional real Cartan factor E the Banach Lie group $\mathsf{Aut}(E)$ has at most 8 connected components.*

5.3 Remark We will see later (Theorem 5.18) that the group $\mathsf{Aut}(E)$ for all cases (i) – (viii) in Proposition 5.1 coincides with the group $\mathsf{Imt}(E)$ of all $\mathbb{R}$-linear surjective isometries of E if E is not a real Hilbert space.

As a consequence of a result in [1] every derivation of a real JB*-triple is automatically continuous. Therefore, Proposition 5.1 immediately gives an explicit description of the Lie algebra $\mathsf{Der}(E)$ of all triple derivations of the classical Cartan factor E.

Suppose, E is a real JB*-triple and $e \in E$ is a unit vector. Then $x \circ y := \{xey\}$ defines on E the structure of a real Banach Jordan algebra $E^{(e)}$ with $\|x \circ y\| \leq \|x\| \cdot \|y\|$ and the following conditions are known to be equivalent [15]

(i) *$E^{(e)}$ is a JB-algebra with unit e,*

(ii) *e is an isolated point in the extreme boundary of the unit ball in E,*

(iii) *e is a complete tripotent with $\delta(e) = 0$ for all $\delta \in \mathsf{Der}(E)$.*

On the other hand, every unital JB-algebra occurs as $E^{(e)}$ for some real JB*-triple E and $\mathrm{Aut}(E^{(e)}) = \{g \in \mathrm{Aut}(E) : g(e) = e\}$ for the algebra automorphism group. In the special case of a JB-algebra $E^{(e)}$ with E a real JB*-triple factor the two points $\pm e$ are the only isolated extreme boundary points of the unit ball in E, i.e. $g(e) = \pm e$ for all $g \in \mathrm{Aut}(E)$ and hence $\mathrm{Aut}(E^{(e)})$ has index 2 in $\mathrm{Aut}(E)$. The groups $\mathrm{Aut}(E^{(e)})$ and hence $\mathrm{Aut}(E)$ for $E = \{z \in \mathcal{L}(H) : z = z^*\}$ and H an $\mathbb{F}$-Hilbert space of dimension ≥ 3, have also been determined in [16] using another method as the group of all transformations $z \mapsto uzu^{-1}$ where $u: H \to H$ is a surjective τ-linear isometry and $\tau \in \mathrm{Aut}(\mathbb{F})$ is an $\mathbb{R}$-linear field automorphism. Comparing this result with Proposition 5.1 shows that in case $\mathbb{F} = \mathbb{H}$ every transformation in $\mathrm{Aut}(E^{(e)})$ can already be given as $z \mapsto uzu^{-1}$ with u an $\mathbb{H}$-linear isometry, i.e. $\tau = \mathbf{1}_{\mathbb{H}}$ suffices for the description of $\mathrm{Aut}(E^{(e)})$ in the case $\mathbb{F} = \mathbb{H}$.

In contrast to the complex case for every real JB*-triple E the group $\mathrm{Aut}(E)$ may be a proper subgroup of the group $\mathrm{Imt}(E)$ of all linear surjective isometries. For instance, the underlying Banach space of $W = \mathbf{I}^{\mathbb{H}}_{2,4}$ is Hilbert with group $\mathrm{Imt}(W) \cong \mathrm{O}(8)$ of dimension 28 whereas the group $\mathrm{Aut}(W) \cong (\mathrm{Sp}(1) \times \mathrm{Sp}(2))/\mathrm{O}(1)$ has only dimension 13 (compare the more general statement in Lemma 5.12). It is known that $\mathrm{Aut}(E) = \mathrm{Imt}(E)$ holds for every JB-algebra as well as for every real C*-algebra (compare [25], [5], [16]). It might be interesting to determine $\mathrm{Imt}(E)$ for all real Cartan factors E or more generally $\mathrm{Imt}(E, F)$ for any such pair of spaces. For complex Cartan factors U, V of rank > 1 Dang [6] proved: *Every surjective $\mathbb{R}$-linear isometry $U \to V$ is semi-linear and hence respects the triple product. In particular,* $\mathrm{Imt}(U_{\mathbb{R}}) = \mathrm{Aut}_s(U)$ *if* $\mathrm{r}(U) > 1$. It is clear that this last statement does not hold for any complex Hilbert space U of dimension > 1 and it is to be expected that also the case of rank 1 plays a special rôle in the real situation.

5.4 Proposition *The real JB*-triples E with $\mathrm{r}(E) = 1$ are precisely those whose underlying Banach space is a real Hilbert space. These are precisely the following real and (realifications of) complex Cartan factors:* $\mathbf{I}^{\mathbb{R}}_{1,n}$, $\mathbf{I}^{\mathbb{H}}_{2,2n}$, $\mathbf{IV}^{n,0}_n$, $\mathbf{V}^{\mathbb{O}}$ *and* $\mathbf{I}_{1,n}$.

Proof E is the real form of a Cartan factor of rank ≤ 2 or the realification of a Cartan factor of rank 1. Inspecting the list in [23] together with Theorem 4.1 gives the result. □

As a consequence, real JB*-triples E with $\mathrm{r}(E) = 1$ are isometrically isomorphic if and only if they have the same dimension d. We may therefore assume $\mathsf{r} := \mathsf{r}(E) > 1$ in the following.

	d	r	a	z	
$I^{\mathbb{R}}_{n,m}$	nm	n	$\{1\}$	1	$m \geq n \geq 2$
$I^{\mathbb{H}}_{2p,2q}$	$4pq$	p	$\{4\}$	4	$q \geq p \geq 2$
$I^{\mathbb{C}}_{n,n}$	nn	n	$\{0,2\}$	1	$n \geq 2$
$II^{\mathbb{R}}_{n}$	$n(n-1)/2$	$[n/2]$	$\{2\}$	1	$n \geq 4$
$II^{\mathbb{H}}_{2p}$	$p(2p-1)$	p	$\{0,4\}$	1	$p \geq 2$
$III^{\mathbb{R}}_{n}$	$n(n+1)/2$	n	$\{0,1\}$	1	$n \geq 2$
$III^{\mathbb{H}}_{2p}$	$p(2p+1)$	p	$\{4\}$	3	$p \geq 2$
$IV^{r,s}_{n}$	n	2	$\{r-1, s-1\}$	1	$n \geq 3,\ r \geq s \geq 1$
$V^{\mathbb{O}_0}$	16	2	$\{3\}$	1	
$VI^{\mathbb{O}_0}$	27	3	$\{4\}$	1	
$VI^{\mathbb{O}}$	27	3	$\{0,8\}$	1	
$I_{n,m}$	$2nm$	n	$\{2\}$	2	$m \geq n \geq 2$
II_{n}	$n(n-1)$	$[n/2]$	$\{4\}$	2	$n \geq 4$
III_{n}	$n(n+1)$	n	$\{1\}$	2	$n \geq 2$
IV_{n}	$2n$	2	$\{n-2\}$	2	$n \geq 3$
V	32	2	$\{6\}$	2	
VI	54	3	$\{8\}$	2	

TABLE 1: Isometric invariants of real and complex Cartan factors of rank $\mathsf{r} > 1$

For the rest of the section let E be a real or complex Cartan factor and F an arbitrary real JB*-triple. We are interested in finding invariants that are respected by $\mathbb{R}$-linear surjective isometries. An obvious one is the real dimension $\mathsf{d} = \mathsf{d}(E)$ which by definition is the minimal cardinality of a total subset of E over $\mathbb{R}$. Denote by $\mathsf{a} = \mathsf{a}(E)$ the set of all cardinal numbers $\big(\dim E^1(e) - 2\big)$ where e runs over all tripotents $e \in E$ with $\mathsf{r}(e) = 2$. Clearly, $\mathsf{a}(E) = \emptyset$ in case E has rank 1. The following result (that actually holds for arbitrary real JB*-triples E) immediately shows that $\mathsf{a}(E)$ as well as the rank $\mathsf{r}(E)$ are isometric invariants.

5.5 Proposition *Let $\lambda: E \to F$ be a surjective $\mathbb{R}$-linear isometry. Then*

(i) *λ commutes with the cube mappings $x \mapsto x^3 := \{xxx\}$ and in particular respects the orthogonality relation.*

(ii) *For every tripotent $e \in E$ the spaces $E^1(e)$, $E^{-1}(e) \oplus E_{1/2}(e)$ and $E_0(e)$ are mapped under λ into the corresponding spaces with respect to the tripotent $\lambda(e) \in F$.*

Proof Lemma 4.3, Proposition 3.8 and Corollary 4.10 in [15]. □

5.6 Lemma *$E_0(e)^\perp = E_1(e)$ for every non-complete tripotent $e \in E$.*

Proof We may assume that $e \neq 0$ and also that E is a complex Cartan factor, otherwise pass to the hermitification. $V := E_0(e)^\perp$ is a closed subtriple of E with $V = E_1(e) \oplus V_{1/2}(e)$. Fix $z \in V_{1/2}(e)$. We have to show $z = 0$. To begin with, assume that $E = \mathcal{L}(H, K)$ where H, K are complex Hilbert spaces. The projections e^*e and ee^* give orthogonal decompositions $H = H_1 \oplus H_2$ and $K = K_1 \oplus K_2$ such that the corresponding operator matrices for e and z have the form

$$e = \begin{pmatrix} u & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad z = \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix}$$

with $u \in \mathsf{Imt}(H_1, K_1)$. Then $z \in V$ implies $ub^*x = 0$ and $xa^*u = 0$ for all $x \in \mathcal{L}(H_2, K_2)$ and hence $z = 0$ since $H_2 \neq 0 \neq K_2$ by assumption. In the same way the exceptional type **VI** as well as the Cartan factors of type **II** and **III** are settled using for instance Lemma 3.1. For the remaining cases e is a minimal tripotent, a case covered by Lemma 2.1 in [6]. □

With Lemma 5.6 we get the following improvement of Proposition 5.5, see also Corollary 5.11.

5.7 Proposition *Suppose that E, F are real or complex Cartan factors of rank > 1. Then every surjective $\mathbb{R}$-linear isometry $\lambda: E \to F$ also maps the spaces $E^{-1}(e)$ and $E_1(e)$ into the corresponding subspaces of F for every tripotent $e \in E$.*

Proof Because of Proposition 5.5 and Lemma 5.6 we may assume that the tripotent e is complete. By assumption E has rank > 1, therefore there exists a representation $e = e_1 + e_2$ with $e_1 \perp e_2$ non-complete tripotents. By Lemma 5.6 both $E_1(e_i)$ are respected by λ, hence also $E^{-1}(e_i)$,

$$E^{-1}(e) = E^{-1}(e_1) \oplus E^{-1}(e_2) \oplus \bigcap_{i=1,2} \left(E^{-1}(e_i) \oplus E_{1/2}(e_i)\right) \quad \text{and}$$

$$E_1(e) = E^1(e) \oplus E^{-1}(e)\,. \qquad \square$$

For every orthogonal family $(e_i)_{i\in I}$ of minimal tripotents in E the w^*-convergent sum $e := \sum_i e_i$ is a tripotent and we call $(e_i)_{i\in I}$ a *frame* in E if e is a maximal tripotent in E. Every frame is a maximal orthogonal family of minimal tripotents, the opposite is not true in general.

5.8 Proposition *Let E be a real or complex Cartan factor. Then*

(i) *Every finite orthogonal family of minimal tripotents in E can be extended to a frame.*

(ii) *Every frame in E has cardinality $\mathsf{r}(E)$.*

(iii) *For every pair of frames $(e_i)_{i\in I}$, $(\widetilde{e}_i)_{i\in I}$ in E there exists an automorphism $g \in \mathsf{Aut}(E)$ with $g(e_i) = \pm\widetilde{e}_i$ for all $i \in I$.*

Proof The finite dimensional case is contained in [23] and the spin factor case is almost trivial. So only Cartan factors of types **I** – **III** and their real forms have to be considered. Fix $\mathbb{F}$-Hilbert spaces H, K with $n := \dim_{\mathbb{F}} H \leq m := \dim_{\mathbb{F}} K$. In case $n = m$ we assume $H = K$ and in case $\mathbb{F} = \mathbb{C}$ the spaces H, K are endowed with a conjugation.

Case 1: $E = \mathcal{L}(H, K)$. Suppose $e \in E$ is a minimal tripotent. Then there are unit vectors $h \in H$ and $k = e(h) \in K$ such that the projections $e^*e \in \mathcal{L}(H)$, $ee^* \in \mathcal{L}(K)$ have images $\mathbb{F}h$, $\mathbb{F}k$. Therefore, if $(e_i)_{i\in I}$ is an orthogonal family of minimal tripotents in E we get for every $i \in I$ unit vectors h_i, k_i with the above properties and it is easily verified that $(h_i)_{i\in I}$ and $(k_i)_{i\in I}$ are orthonormal families in H and K. Furthermore, $(e_i)_{i\in I}$ is a frame in E if and only if: *$(h_i)_{i\in I}$ is an orthonormal basis of H and in case $n = m$ also $(k_i)_{i\in I}$ is an orthonormal basis of K*. Therefore (i) and (ii) are obvious in this case. Furthermore, if the unit vectors $\widetilde{h}_i$, $\widetilde{k}_i$ belong to the frame $(\widetilde{e}_i)$ there exist operators $u \in \mathsf{Imt}(K)$ and $v \in \mathsf{Imt}(H)$ with $v(\widetilde{h}_i) = h_i$ and $u(k_i) = \widetilde{k}_i$ for all $i \in I$. Then $g(e_i) = \widetilde{e}_i$ for all $i \in I$ with $g \in \mathsf{Aut}(E)$ defined by $g(z) = uzv$.

Case 2: $E = \{z \in \mathcal{L}(H) : z^* = \varepsilon z\}$ with $\varepsilon = \pm 1$ a square in $\mathbb{F}$. Every frame $(e_i)_{i\in I}$ in E is also a frame in $\mathcal{L}(H)$ and hence has cardinality n. Therefore, if $(\widetilde{e}_i)_{i\in I}$ is another frame in E we get with the notations of the first case the same as before but with the additional conditions $k_i = \pm h_i$ and $\widetilde{k}_i = \pm\widetilde{h}_i$ for all i. Define again $u \in \mathsf{Imt}(H)$ and $g \in \mathsf{Aut}(E)$ by $u(k_i) = \widetilde{k}_i$ and $g(z) = uzu^*$.

Case 3: $E = \{z \in \mathcal{L}(H) : z' = z\}$ and $\mathbb{F} = \mathbb{C}$. Same as case 2 with the conditions $k_i = \lambda_i \overline{h}_i$, $\lambda_i \in \mathsf{U}(1)$, for the first frame and the same for the second frame.

Case 4: $E = \{z \in \mathcal{L}(H) : z' = -z\}$ with n even and $\mathbb{F} \neq \mathbb{H}$. As tripotent in $\mathcal{L}(H)$ every e_i has rank 2 and there is a minimal tripotent $f_i \in \mathcal{L}(H)$ such that $f_i \perp f_i'$ and $e_i = f_i - f_i'$. Since n is even by assumption (which covers the case

n infinite) the union of the families (f_i) and (f_i') is a frame in $\mathcal{L}(H)$ and the proof proceeds as above. □

Proposition 5.8 in particular implies that $\mathsf{Aut}(E)$ acts transitively on the minimal tripotents of E since $x \mapsto -x$ always is in $\mathsf{Aut}(E)$. Therefore, if we fix a minimal tripotent $e \in E$ the real dimension $\mathsf{z}(E) := \mathsf{d}(E_0(e)^{\perp})$ does not depend on e and is an isometric invariant of E by proposition 5.5. A case-by-case computation gives TABLE 1 where elementary arithmetic of cardinal numbers is used – so that for instance $n(n-1)/2 = [n/2] = n$ for every infinite cardinal n. Inspecting the table gives immediately the following result.

5.9 Theorem *Let $\mathcal{CF}$ be the class of all real and (realifications of) complex Cartan factors. Then the invariants $\mathsf{d}, \mathsf{r}, \mathsf{a}, \mathsf{z}$ form a complete system of invariants in $\mathcal{CF}$ with respect to isometric equivalence, more precisely, two triples E, F in $\mathcal{CF}$ allow a surjective $\mathbb{R}$-linear isometry if and only if both have the same invariants. For the subclass of all triples of rank > 1 isometric equivalence is the same as equivalence with respect to triple isomorphisms.*

To get a complete system of invariants for $\mathcal{CF}$ with respect to triple isomorphy one may take for instance $\mathsf{r}, \mathsf{a}, \mathsf{z}, \mathsf{b}, \mathsf{h}$ where $\mathsf{b}(E) := \dim E^{-1}(e)$ and $\mathsf{h}(E) := \dim E_{1/2}(e)$ for $e \in E$ a minimal tripotent.

The result of Theorem 5.9 can be extended to other real JB*-triples. As an example, suppose E is a JB*-triple in the class $\mathcal{CF}$. Denote by I the closed linear span of all minimal tripotents in E (the subtriple of all compact operators in case of the types **I** – **III**). Then I is the unique minimal closed ideal of E and the bidual of I can be identified with E. Therefore every $\lambda \in \mathsf{Imt}(I)$ has a unique extension to an isometry of E and hence $\mathsf{Imt}(I) = \mathsf{Imt}(E)$ as well as $\mathsf{Aut}(I) = \mathsf{Aut}(E)$ in this sense. Suppose furthermore that $F \subset E$ is a closed real subtriple containing the ideal I. For instance, if H, K are $\mathbb{F}$-Hilbert spaces and $L \subset H$ is a closed linear subspace then $\{z \in \mathcal{L}(H, K) : z|L \text{ compact}\}$ is such a subtriple which in general is not an ideal of E. Now suppose, $\widetilde{E}$ is another object in $\mathcal{CF}$ and $\widetilde{I} \subset \widetilde{F} \subset \widetilde{E}$ are closed subtriples with $\widetilde{I}$ the minimal closed ideal of $\widetilde{E}$. Then every surjective linear isometry $\lambda: F \to \widetilde{F}$ maps I onto $\widetilde{I}$ and hence induces a surjective linear isometry $E \to \widetilde{E}$.

TABLE 1 reveals that E in $\mathcal{CF}$ has the structure of a JB-algebra of dimension > 1 if and only if $0 \in \mathsf{a}(E)$. Also, the following can be seen: *The automorphism group $\mathsf{Aut}(E)$ acts transitively on frames in E if and only if $\mathsf{a}(E)$ contains at most one element.*

5.10 Lemma *Let E be a real or complex JB*-triple of rank > 1 and let*

$e \in E$ be a minimal tripotent. Then every element in the unit sphere $S := \{c \in E_1(e) : \|c\| = 1\}$ of the Peirce space $E_1(e)$ also is a minimal tripotent in E and satisfies $e\Box e = c\Box c$. In addition $\bigcap_{c\in S} E^{-1}(c) = 0$ holds.

Proof Fix $c \in S$. Then c is a minimal tripotent in E since $E_1(e)$ has rank 1. In case $\mathsf{a}(E) = 1$ we have $E^{-1}(e) = 0$ and nothing more has to be shown. For the remaining cases, see TABLE 1, $e\Box e = c\Box c$ is easily checked and $c \notin E^{-1}(c)$ gives the last statement. □

5.11 Corollary *Let E be a real or complex Cartan factor of rank > 1. Then $\lambda(E_{1/2}(e)) = E_{1/2}(\lambda e)$ for every minimal tripotent $e \in E$ and every $\lambda \in \mathsf{Imt}(E)$.*

Proof Follows from Propositions 5.5 and 5.7 as well as from $E_k(e) = E_k(c)$ for $k = 1, 1/2$ and all $c \in S$ together with

$$E_{1/2}(e) = \bigcap_{c\in S} \left(E^{-1}(c) \oplus E_{1/2}(c)\right) . \qquad \Box$$

Corollary 5.11 implies that surjective ℝ-linear isometries in case of rank > 1 respect the collinearity relation between minimal tripotents. By definition, two tripotents $e, c \in E$ are called *collinear* and we write $e\top c$ if $e \in E_{1/2}(c)$ and $c \in E_{1/2}(e)$. From Lemma 5.10 it is easily derived that the triple product of any three pairwise collinear minimal tripotents in E vanishes.

The question arises whether in $\mathcal{CF}$ isometric equivalence and triple isomorphy are the same – or equivalently as a consequence of Theorem 5.9 – whether $\mathsf{Imt}(E) = \mathsf{Aut}(E)$ holds for every real Cartan factor E. Therefore denote by $\mathcal{IS}$ the class of all real JB*-triples E with $\mathsf{Imt}(E) = \mathsf{Aut}(E)$. As already has been pointed out (compare [25], [5], [6]) every JB-algebra, every real C*-algebra and every complex JB*-triple is in $\mathcal{IS}$ – more precisely the underlying structure of a real JB*-triple for every such object. On the other hand, not every real Cartan factor is in the class $\mathcal{IS}$, as the following characterization in the rank-1-case shows – compare also Proposition 5.4 and notice that $\mathbf{I}^{\mathbb{H}}_{2,2} \approx \mathbf{IV}^{4,0}_4$ and $\mathbb{C}_{\mathbb{R}} \approx \mathbf{IV}^{2,0}_2$.

5.12 Lemma *Let E be a real JB*-triple of rank 1. Then E is in the class $\mathcal{IS}$ if and only if $E^{-1}(e) = 0$ or $E_{1/2}(e) = 0$ for some (and hence every) minimal tripotent $e \in E$, that is, if and only if E is isomorphic to $\mathbf{I}^{\mathbb{R}}_{1,m}$ for some $m \geq 1$ or to $\mathbf{IV}^{n,0}_n$ for some $n \geq 2$.*

Proof E is a real Hilbert space. Therefore, if $E^{-1}(e) \neq 0$ and $E_{1/2}(e) \neq 0$ there is a $\lambda \in \mathsf{Imt}(E)$ and a vector $a \in E_{1/2}(e)$ with $\lambda(e) = e$ and $\lambda(a) \notin E_{1/2}(e)$. But such a λ cannot be in $\mathsf{Aut}(E)$. Therefore only those factors of rank 1 can be in $\mathcal{IS}$ that are listed in the Lemma. That actually $\mathbf{I}_{1,m}^{\mathbb{R}}$ lies in $\mathcal{IS}$ is obvious and the spin case is covered by the next Lemma. □

5.13 Lemma *Every real spin factor E is in the class $\mathcal{IS}$.*
Proof Write $E = \mathbf{IV}_n^{r,s}$ as in Theorem 4.1. Then $\mathsf{Imt}(E) = \mathsf{O}(n) = \mathsf{Aut}(E)$ in case $s = 0$. In case $s > 0$ the extreme boundary of the unit ball in E has two connected components – the two unit spheres in X_1 and X_2. Since surjective isometries respect the extreme boundary the statement follows easily from the description of $\mathsf{Aut}(E)$ in Proposition 5.1. □

As a generalization of Theorem 5.1 in [5] we have

5.14 Proposition *Let H, K be $\mathbb{F}$-Hilbert spaces of dimension > 1 over the field $\mathbb{F}$ which is either $\mathbb{R}$ or $\mathbb{H}$. Then $E := \mathcal{L}(H, K)$ is in the class $\mathcal{IS}$.*
Proof Choose orthonormal bases $\{h_\beta : \beta \in \mathcal{B}\}$ of H and $\{k_\alpha : \alpha \in \mathcal{A}\}$ of K. Define the tripotent $e_{\alpha\beta} \in E$ by $e_{\alpha\beta}(h) = (h|\, h_\beta)k_\alpha$ for all $h \in H$ and put $E_{\alpha\beta} := E_1(e_{\alpha\beta})$. Then $e^*_{\alpha\beta}(k) = (k|\, k_\alpha)h_\beta$ implies

$$E_{\alpha\gamma}E^*_{\delta\gamma}E_{\delta\beta} \subset E_{\alpha\beta}$$

and every other associative triple product of the $E_{\alpha\beta}$'s vanishes that is not of this form. The $\mathbb{R}$-linear span D of $R := \bigcup_{\alpha,\beta} E_{\alpha\beta}$ is a (not necessarily closed) subtriple of E. Fix $\lambda \in \mathsf{Imt}(E)$ and assume that there are unit vectors $a, b, c \in R$ with $\lambda\{abc\} \neq \{(\lambda a)(\lambda b)(\lambda c)\}$. But a, b, c are minimal tripotents in E and the three cases '$a \perp b$' or '$b \perp c$' or 'a, b, c pairwise collinear' cannot happen since then both sides of the above inequality would vanish. Therefore there exist orthogonal tripotents $e_{\alpha\beta}$, $e_{\gamma\delta}$ in R with $a, b, c \in E_1(e)$ for $e := e_{\alpha\beta} + e_{\gamma\delta}$. By Proposition 5.7 λ is an isometry from $E_1(e)$ onto $E_1(\lambda e)$. But both Peirce spaces are in the class $\mathcal{IS}$ by [5] (in the real case they are even real spin factors), hence the restriction $\lambda|E_1(e)$ is a triple homomorphism by Lemma 5.13, a contradiction. Therefore the restriction $\lambda|D$ is a triple homomorphism. Since λ is w^*-continuous, the triple product is separately w^*-continuous and D is w^*-dense in E the statement follows. □

As a consequence of Proposition 5.1 we may reformulate Proposition 5.14 also as:

5.15 Proposition *Let H, K, X, Y be real Hilbert spaces. Then*

$$\begin{aligned}\mathsf{Imt}(\mathcal{L}(H,K), \mathcal{L}(X,Y)) &= \{z \mapsto uzv : u \in \mathsf{Imt}(K,Y), v \in \mathsf{Imt}(X,H)\} \\ &\quad \cup\ \{z \mapsto uz'v : u \in \mathsf{Imt}(H,Y), v \in \mathsf{Imt}(X,K)\}\,.\end{aligned}$$

The same is true for quaternionic Hilbert spaces provided H, K have dimension > 1 over $\mathbb{H}$ (notice that in this case Imt on the left hand side of the equation means $\mathbb{R}$-linear isometries in contrast to $\mathbb{H}$-linear isometries on the other side).

5.16 Proposition *$E = \{z \in \mathcal{L}(H) : z' = -z\}$ is in the class $\mathcal{IS}$ for every real Hilbert space H of dimension > 1.*

Proof Fix $\lambda \in \mathrm{Imt}(E)$ and choose an orthonormal basis $\{h_\alpha : \alpha \in \mathcal{A}\}$ of H. Define for every $\alpha \neq \beta$ the minimal tripotent $e_{\alpha\beta} \in E$ by $e_{\alpha\beta}(h) = (h|\,h_\alpha)h_\beta - (h|\,h_\beta)h_\alpha$ for all $h \in H$ and put $E_{\alpha\beta} := E_1(e_{\alpha\beta})$. Then $E_{\alpha\beta} = E_{\beta\alpha}$ and as above the $\mathbb{R}$-linear span D of $R := \bigcup_{\alpha,\beta} E_{\alpha\beta}$ is a w^*-dense subtriple of E. Therefore we only have to show that $\lambda: D \to E$ is a triple morphism. Now $\{E_{\alpha\beta}E_{\gamma\delta}E_{\mu\nu}\} \neq 0$ implies that there are at most 4 different indices involved, that is, as a consequence of Proposition 5.5 we may assume without loss of generality that H has dimension ≤ 4. But this implies that E is isomorphic to one of the triples $\mathbf{I}^{\mathbb{R}}_{1,1}$, $\mathbf{I}^{\mathbb{R}}_{1,3}$ or $\mathbf{IV}^{3,3}_6$ in $\mathcal{IS}$. □

5.17 Proposition *$E = \{z \in \mathcal{L}(H) : z^* = -z\}$ is in $\mathcal{IS}$ for every $\mathbb{H}$-Hilbert space H.*

Proof Fix $\lambda \in \mathrm{Imt}(E)$ and choose an orthonormal basis $(h_\alpha)_{\alpha\in\mathcal{A}}$ of H. Since $\mathbf{III}^{\mathbb{H}}_2 \approx \mathbf{IV}^{3,0}_3$ is in $\mathcal{IS}$ we may assume $\dim_{\mathbb{H}} H > 1$. For every $\alpha \in \mathcal{A}$ define $e_\alpha \in \mathcal{L}(H)$ by $e_\alpha(h) = (h|\,h_\alpha)h_\alpha$. Then re_α is a minimal tripotent of E for every $r \in \mathbb{H}$ with $r^2 = -1$ and the Peirce spaces

$$E_{\alpha\alpha} := E_1(re_\alpha) = \{se_\alpha : s \in \mathbb{H} \quad \text{with} \quad \overline{s} = -s\}$$

$$E_{\alpha\beta} := E_{1/2}(re_\alpha) \cap E_{1/2}(re_\beta) \qquad \text{for} \qquad \alpha \neq \beta$$

do not depend on r. Clearly, $(\mathrm{i}e_\alpha)_{\alpha\in\mathcal{A}}$ is a frame in E and because of Proposition 5.8.iii we may assume that $\lambda(E_{\alpha\beta}) = E_{\alpha\beta}$ for all $\alpha, \beta \in \mathcal{A}$. The restriction of λ to the real Hilbert space $E_{\alpha\alpha}$ is an orthogonal transformation, denote by $\delta_\alpha = \pm 1$ its determinant. We may assume without loss of generality that not all δ_α are negative (otherwise replace λ by $-\lambda$). For every family $(g_\alpha)_{\alpha\in\mathcal{A}}$ of isometries $g_\alpha \in \mathrm{Imt}(E_{\alpha\alpha})$ with positive determinant there is a $g \in \mathrm{Aut}(E)$ with $g_\alpha = g|E_{\alpha\alpha}$ – therefore we may assume furthermore that for all $\alpha \in \mathcal{A}$

$$\lambda(\mathrm{i}e_\alpha) = \mathrm{i}e_\alpha, \qquad \lambda(\mathrm{j}e_\alpha) = \mathrm{j}e_\alpha, \qquad \lambda(\mathrm{ji}e_\alpha) = \delta_\alpha \mathrm{ji}e_\alpha\,.$$

Claim 1: *For every α, β there exists $\varepsilon_{\alpha\beta} = \pm 1$ such that $\varepsilon_{\alpha\beta}\lambda$ is the identity on $E_{\alpha\beta}$. In particular, $\delta_\alpha = 1$ for all $\alpha \in \mathcal{A}$.*

For the proof of this claim we may assume that $\mathcal{A} = \{1, 2\}$ and we may identify

E with the matrix space $\{z \in \mathbb{H}^{2\times 2} : z^* = -z\}$ in such a way that $\delta_1 = 1$ and

$$re_1 = \begin{pmatrix} r & 0 \\ 0 & 0 \end{pmatrix}, \qquad re_2 = \begin{pmatrix} 0 & 0 \\ 0 & r \end{pmatrix}$$

for all r. Then λ leaves invariant the subtriple

$$E^{-1}(c) = E \cap \mathrm{j}\mathbb{C}^{2\times 2} \approx V := \{z \in \mathbb{C}^{2\times 2} : z' = z\} \qquad \text{for} \qquad c := \begin{pmatrix} \mathrm{i} & 0 \\ 0 & \mathrm{i} \end{pmatrix}$$

and hence induces an $\mathbb{R}$-linear isometry σ of the complex Cartan factor V by $\lambda(\mathrm{j}v) = \mathrm{j}\sigma(v)$ for all $v \in V$. By [6] σ is either complex-linear or conjugate-linear. The latter case cannot occur since σ is complex-linear on the complex line generated by e_1, i.e. $\delta_2 = 1$. More generally, for every $u \in \mathbb{H}$ with $u\overline{u} = 1$ one has

$$E^{-1}(c_u) = \begin{pmatrix} u & 0 \\ 0 & -\mathrm{i} \end{pmatrix} E^{-1}(c) \begin{pmatrix} \overline{u} & 0 \\ 0 & \mathrm{i} \end{pmatrix} \approx V \qquad \text{for} \qquad c_u := \begin{pmatrix} u\mathrm{i}\overline{u} & 0 \\ 0 & \mathrm{i} \end{pmatrix}$$

and hence

$$\lambda \begin{pmatrix} 0 & u\mathrm{j} \\ \mathrm{j}\overline{u} & 0 \end{pmatrix} = \pm \begin{pmatrix} 0 & u\mathrm{j} \\ \mathrm{j}\overline{u} & 0 \end{pmatrix} \in E^{-1}(c_u) \cap E_{12}\,.$$

But then $\mathbb{H} = \mathbb{H}\mathrm{j}$ proves the claim.

Claim 2: *$\varepsilon_{\alpha\gamma}\varepsilon_{\gamma\beta} = \varepsilon_{\alpha\beta}$ holds for all $\alpha, \beta, \gamma \in \mathcal{A}$.*

For the proof of this claim only the case $\dim_{\mathbb{H}} H = 3$ and $\mathcal{A} = \{\alpha, \beta, \gamma\}$ has to be considered. Then for $c := \mathrm{i}e_\alpha + \mathrm{i}e_\beta + \mathrm{i}e_\gamma \in E$ again

$$E^{-1}(c) \approx W := \{z \in \mathbb{C}^{3\times 3} : z' = z\}$$

and the claim follows from the explicit description of $\mathrm{Aut}(W)$ in section 3.

We now proceed the main proof with H arbitrary. Fix $\gamma \in \mathcal{A}$ and define $u \in \mathcal{L}(H)$ by

$$u(h_\alpha) = \varepsilon_{\alpha\gamma} h_\alpha$$

for all $\alpha \in \mathcal{A}$. Then $g(w) = uwu^*$ defines an element $g \in \mathrm{Aut}(E)$ that coincides with $\varepsilon_{\alpha\gamma}\varepsilon_{\gamma\beta}\,\mathrm{id}$ on every $E_{\alpha\beta}$. As a consequence of claim 2 λ and g coincide on every $E_{\alpha\beta}$. Since λ and g are w^*-continuous and the linear span of all $E_{\alpha\beta}$ is w^*-dense in E we conclude that $\lambda = g \in \mathrm{Aut}E)$. □

We leave open the question whether the two exceptional factors $\mathbf{V}^{\mathbf{O}_0}$ and $\mathbf{VI}^{\mathbf{O}_0}$ belong to the class $\mathcal{IS}$. But we know that all other real or complex Cartan factors of rank > 1 do. This allows us to state our main result:

5.18 Theorem *Let E be a non-exceptional real or complex Cartan factor of rank > 1. Let furthermore F be a real JBW*-triple. Then a bijective $\mathbb{R}$-linear map $\lambda: E \to F$ is an isometry if and only if it respects the triple products.*

Proof Suppose that λ is a triple isomorphism. Then λ is isometric by [15]. Conversely, suppose that λ is an isometry. By assumption, $F = U^\tau$ for a JBW*-triple U. Since λ respects the orthogonality relation (see Proposition 5.5) U cannot contain any non-trivial τ-invariant w^*-closed ideal. Also, for every minimal tripotent $e \in E$ the image $\lambda(e)$ is a minimal tripotent in F and hence the sum of at most 2 orthogonal minimal tripotents in U. Therefore only the following two cases can occur: (i) U is a Cartan factor and hence F is a real Cartan factor, or (ii) $U = V \oplus V$ for a Cartan factor V and hence $F = V_{\mathbb{R}}$. We may therefore assume $E = F$. The proof then follows by combining Theorem 5.9, Lemma 4.5 and Propositions 5.14 – 5.17 together with the known fact that every JB-algebra and also every complex Cartan factor of rank > 1 is in the class $\mathcal{IS}$. □

6. Some remarks

6.1 Remark Suppose E is a real (or complex) JB*-triple and $I, J, K \subset E$ are closed ideals. Then

(i) $I \cap J = \{IJE\} = \{IEJ\}$,

(ii) $I \perp J \iff I \cap J = 0$,

(iii) $K = (K \cap I) \oplus (K \cap J) \quad \text{if} \quad E = I \oplus J$.

Proof Use that to every $a \in E$ there is an element $c \in E$ with $\{ccc\} = a$. □

Consider JB*-triples of the form $U = C(S, E)$ where S is a compact topological space. Then every closed subset $A \subset S$ determines the closed ideal $I := \{f \in U : f|A = 0\}$ of U. Clearly, I is complemented by another ideal in U if and only if A is also open in S. Hence the following statement may be considered as a first step in generalizing Proposition 4.8 to arbitrary JBW*-triples.

6.2 Lemma *Suppose U is a (complex) JBW*-triple and σ is a conjugation of U. Then there are w^*-closed ideals $I, J, K \subset U$ such that*

(i) $U = I \oplus J \oplus K$,

(ii) *$K = \sigma(J)$ and every w^*-closed ideal of I is σ-invariant.*

Proof Let J be an ideal of U that is maximal with respect to the property $J \perp \sigma(J)$. Then J, $K := \sigma(J)$ and $I := (J + K)^\perp$ are w^*-closed ideals of U satisfying (i) and $\sigma(I) = I$. Suppose $A \subset I$ is a w^*-closed ideal. Then there is a decomposition $A = T \oplus B$ into w^*-closed ideals where $T = A \cap \sigma(A)$. But then $\sigma(A) = T \oplus \sigma(B)$ implies $B \cap \sigma(B) = 0$ and hence $B \perp \sigma(B)$ by (6.1). For $J' = (J + B)$ also $J' \perp \sigma(J')$ holds, i.e. $B = 0$ by the maximality of J. □

6.3 Corollary *Every real JBW*-triple E has a unique decomposition $E = R \oplus C$ into w^*-closed ideals R, C where C is complexifiable and R has the following property: Let I with conjugation τ be the hermitification of R. Then every w^*-closed ideal of I is τ-invariant.*

Proof Let U with conjugation σ be the hermitification of E. Then $E \cong I^\tau \oplus J_{\mathbb{R}}$ with the notation of Lemma 6.2. □

In [15] the notion of real type for a real JB*-triple E has been introduced, that is, E has the following property:

(P_1) *There exists a maximal abelian subtriple of E that is flat.*

We do not know whether the subtriple R in 6.3 is of real type in this sense. Other properties of a real JB*-triple that might be interesting in this context are as follows

(P_2) *Every maximal flat subtriple of E is maximal abelian.*

(P_3) *Every closed ideal of U is σ-invariant where U with conjugation σ is the hermitification of E.*

We do not know which of the conditions (P_1) – (P_3) are equivalent. They are equivalent for real JB*-triples of finite dimension.

As an example let E be the real JB*-triple of all symmetric real 2×2-matrices. As a Banach space E can be identified with $\mathbb{C} \oplus \mathbb{R}$ endowed with norm $\|(z,t)\| = |z| + |t|$ – just identify (z,t) with the matrix

$$\begin{pmatrix} t+a & b \\ b & t-a \end{pmatrix} \qquad \text{where} \qquad z = a + ib\,.$$

The unit ball B of E is the double cone over the unit disc of $\mathbb{C}$ and t is the normalized trace of E. It is easily verified that the maximal flat subtriples of E are precisely the linear 2-planes passing through the unit matrix and hence are also maximal abelian (since E itself is not abelian). On the other hand, there is exactly one more maximal abelian subtriple A of E – it is given by $t = 0$ or equivalently as the linear span of the matrices

$$e = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \text{and} \qquad j = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

A is isomorphic to $\mathbb{C}_{\mathbb{R}}$ and is not flat. A is the image of the operator $\lambda := j \square e$ and the restriction of λ to A is a complex structure on A.

The idea of this example may be generalized in the following way: Let E be an arbitrary real JB*-triple and suppose $e \neq 0$ is a tripotent in E. Call an element $j \in E$ an *imaginary unit* (with respect to e) if

$$(6.4) \qquad Q(j)e = -e \quad \text{and} \quad Q(e)j = -j\,.$$

This implies $j \in E_1(e)$ and from $\{jej\}\Box e - j\Box\{eje\} = [j\Box e, j\Box e] = 0$ we derive $e\Box e = j\Box j$ and hence that j is a tripotent with the same Peirce spaces in E as e. Obviously $\mathbb{R}e \oplus \mathbb{R}j$ is a subtriple of E isomorphic to $\mathbb{C}_{\mathbb{R}}$.

6.5 Lemma *$j\Box e = -e\Box j$.*

Proof Suppose $x \in E_k(e)$ for $k = 1, 1/2, 0$. Then

$$\begin{aligned} k\{jex\} = \{je\{eex\}\} &= \{\{jee\}ex\} - \{e\{eje\}x\} + \{ee\{jex\}\} \\ &= \{jex\} + \{ejx\} + k\{jex\} \end{aligned}$$

implies the statement. □

6.6 Lemma *The operator $\lambda := j\Box e$ is a derivation of the triple product and satisfies the algebraic equation $\lambda(\lambda^2+1)(4\lambda^2+1) = 0$.*

Proof The derivation property follows immediately from 6.5 and the Jordan triple identity. E becomes a real Jordan algebra with respect to the product $xy = \{xey\}$. Denote for every $u \in E$ by $L(u) := u\Box e$ the corresponding left multiplication operator on E. Then $2L(u)^3 = 3L(u^2)L(u) - L(u^3)$ holds by [3] p. 154 which gives $2\lambda^3 + (3e\Box e - 1)\lambda = 0$ for $u := j$. Since λ commutes with $e\Box e$ we may consider the Peirce spaces $E_k(e)$ separately. λ vanishes on $E_0(e)$ by the Peirce multiplication rules and for $k = 1, 1/2$ the restriction of λ to $E_k(e)$ annihilates the polynomial $2\theta^3 + (3k-1)\theta$ which is a factor of $\theta(\theta^2+1)(4\theta^2+1) \in \mathbb{R}[\theta]$. □

6.7 Corollary *E splits into the direct sum $E = A \oplus B \oplus C$ of λ-invariant subtriples where $A := \ker(\lambda^2+1) \subset E_1(e)$, $B := \ker(4\lambda^2+1) \subset E_{1/2}(e)$ and $C := \ker(\lambda) \supset E_0(e)$.*

Proof We only have to show that A, B, C are subtriples. Let $U := E \oplus \mathrm{i}E$ be the hermitification of E and denote by U_r the r-eigenspace of λ when considered as a complex-linear operator on U. Then

$$\{U_r U_s U_t\} \subset U_{r+\bar{s}+t}$$

for all $r, s, t \in \mathbb{C}$ since λ is a derivation of U. The result now follows from $A = E \cap (U_{\mathrm{i}} \oplus U_{-\mathrm{i}})$, $B = E \cap (U_{\mathrm{i}/2} \oplus U_{-\mathrm{i}/2})$ and $C = E \cap U_0$. □

It is clear that on A (on B resp.) a complex structure is given by λ (by 2λ resp.). Actually we have

6.8 Lemma *A is a complex JB*-triple with respect to the complex structure λ.*

Proof We may assume without loss of generality that $E = A$. Then E is a unital real Jordan algebra with product $xy = \{xey\}$ and involution $x \mapsto \overline{x} := Q(e)x$. The conditions (6.4) are equivalent to $j^2 = -e$ and $\overline{j} = -j$. For every $u, v \in E$ the identity

$$L(u^2v) = 2L(uv)L(u) + L(u^2)L(v) - 2L(u)L(v)L(u)$$

holds by [3] p. 145. For $u = j$ this implies $L(jv)L(j) = L(j)L(v)L(j)$ and hence $L(jv) = L(j)L(v)$ since $L(j)$ is invertible. Therefore λ commutes with every $L(v)$ and hence with every operator $x \square y = [L(x), L(\overline{y})] + L(x\overline{y})$. But then [15] proposition 2.6 gives the result. □

6.9 Corollary *Let E be a real JBW*-triple and suppose that $e_1, e_2 \in E$ are orthogonal tripotents with $E_{12} := E_{1/2}(e_1) \cap E_{1/2}(e_2) \neq 0$. Then E contains an abelian subtriple that is not flat.*

Proof Put $e := e_1 + e_2$. The Peirce space $E_1(e)$ is a Jordan algebra with product $xy = \{xey\}$ and involution $x \mapsto \overline{x} := Q(e)x$. Then $\{x\overline{y}z\} = (xy)z - (zy)x + (xz)y$ is an easy consequence of the Jordan triple identity. Replacing e_2 by $-e_2$ if necessary we may assume without loss of generality that $V := \{v \in E_{12} : Q(e)v = -v\} \neq 0$. Since V is a real JBW*-triple we may choose a tripotent $c \neq 0$ in V. Then $\overline{c} = -c = c^3$ and also $\overline{j} = -j$ holds for $j := c^2 \in E_1(e_1) \oplus E_1(e_2)$. Now $Q(c)j = -j$ and $Q(j)c = -c^5 = -c$ implies that $A = \mathbb{R}c + \mathbb{R}j$ is a subtriple isomorphic to $\mathbb{C}_{\mathbb{R}}$. □

6.10 Corollary *Let E be a real JBW*-triple factor. Then the following conditions are equivalent*

(i) *Every maximal abelian subtriple of E is flat,*

(ii) *E is a real Cartan factor of rank 1, i.e. the underlying Banach space of E is Hilbert.*

Proof Let $e \in E$ be a complete tripotent and choose a maximal flat subtriple $F \subset E$ containing e. Then $F \subset E_1(e)$. Suppose that (i) and $\dim F > 1$ holds. Since F is w^*-closed in E there are orthogonal tripotents $e_1, e_2 \in F$ with $e = e_1 + e_2$. E cannot contain a non-flat abelian subtriple by assumption, i.e. $E_{12} = 0$ as a consequence of 6.9. But this is a contradiction to our assumption that E is a factor. Therefore (i) implies $F = \mathbb{R}e$ and E is the real form of a Cartan factor of rank ≤ 2. □

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