Now define the function f_+: $[0, 1)\to\mathbf{R}$

$$f_+(x) = \lim_{\delta\to 0} \operatorname*{ess\,sup}_{[x,\, x+\delta]} f = \lim \operatorname*{ess\,inf}_{[x,\, x+\delta]} f;$$

f_+ is continuous on the right in $[0, 1)$.

For $\varepsilon>0$ the level set $(|f-f_+|>\varepsilon)$ has no density points, (this follows immediately from the definition of f_+), therefore

$$f = f_+ \text{ a.e. in } [0, 1].$$

We prove now that $\|f_+\| = Vf_+$ (V denotes the classical total variation.) Indeed, for each $n\in\mathbf{N}$ $\tilde{f}_n\leq Vf_+$ a.e., hence $\|f_+\|\leq Vf_+$; $\|f_+\|\geq Vf_+$ follows from Proposition 3 and the fact that f_+ is continuous on the right at each point of its domain.

Finally we remark that if f: $[0, 1)\to\mathbf{R}$ is a function of bounded variation and continuous on the right, then $f\in B_0^1$ and $\|f\| = Vf$.

This completes the proof of our Theorem.

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ANSWER TO A PROBLEM OF M. HORVÁTH AND A. SÖVEGJÁRTÓ

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In [4] the authors gave a simple proof for a minimax theorem, developing the ideas of [3] and considered a new concept of convexity in topological spaces. The aim of the present note is to give negative answer for a problem raised in [4] further investigate this area of questions. The just mentioned negative answer is the

Theorem 1. *There exist $f_1, f_2\in C([0, 1]\times[0, 1])$ such that*

$$(1)\quad \forall x_1, x_2 \forall \alpha_1, \alpha_2\geq 0\ \alpha_1+\alpha_2 = 1\ \exists x\in X\ \forall y\in Y\ f_1(x, y)\geq \sum_{j=1}^{2} \alpha_j f_1(x_j, y),$$

$$(2)\quad \forall y_1, y_2\in Y \forall \beta_1, \beta_2\geq 0\ \beta_1+\beta_2 = 1\ \exists y\in Y\ \forall x\in X\ f_2(x, y)\geq \sum_{k=1}^{2} \beta_j f_2(x, y_k)$$

hold and there is no point $(x_{,0}\, y_0)\in X\times Y$ with the property

$$(3)\quad \begin{aligned} f_1(x_0, y_0)&\geq f_1(x, y_0) \quad \forall x\in X,\\ f_2(x_0, y_0)&\geq f_2(x_0, y) \quad \forall y\in Y. \end{aligned}$$

We remark that the usual concavity of the functions f_1, f_2 (in x resp. y) is necessary for (3) in some sense, namely

Theorem 2. *Let f_1, f_2 : $I_1\times I_2\to\mathbf{R}$ be continuous functions with the property: for any pair f_1', f_2' of continuous functions, where f_i' is partially concave in its i-th variable, the pair (f_1+f_1', f_2+f_2') satisfies (3). Then f_i is partially concave in its i-th variable too.*

However, the Nikaidó-Isoda theorem — i.e. the (3) for a pair of partially concave functions — is true using the convexity introduced in [8] for pairs of partially quasiconcave functions, nevertheless in this case the convex sets may be very different from the usual ones. This is the statement of

Theorem 3. *Consider I^n the cube of n dimensions endowed with the convexity structure introduced in* [8] *and two continuous functions $f(x, y)$, $g(x, y)$: $I^n\times I^n\to\mathbf{R}$ and suppose that f is quasiconcave in x i.e. the set*

$$(4)\quad \{x : f(x, y_0)> C\}\subset I^n,$$

is convex for any $y_0 \in I^n$ *and* $C \in \mathbf{R}$,

(5) $$g \text{ is quasiconcave in } y.$$

Then there exists a point $(x_0, y_0) \in I^n \times I^n$ *such that*

$$f(x_0, y_0) \geq f(x, y_0) \qquad \forall x \in I^n,$$
$$g(x_0, y_0) \geq g(x_0, y) \qquad \forall y \in I^n.$$

1. Proof of the Theorem 1. We need a sequence of remarks.

R.1. Suppose $h_1, h_2 : I_1 \to \mathbf{R}$ are continuous functions satisfying

(6) $$\forall x_1, x_2 \in I_1 \ \forall \lambda \in [0, 1] \exists x_0 \in I_1:$$
$$\lambda h_1(x_1) + (1-\lambda) h_1(x_2) \leq h_1(x_0),$$
$$\lambda h_2(x_1) + (1-\lambda) h_2(x_2) \leq h_2(x_0).$$

Define $f_1 : I_1 \times I_2 \to \mathbf{R}$ as follows: for $\mu \in [0, 1] = I_2$ denote $f_\mu = (1-\mu) h_1 + \mu h_2$ and for $x \in I_1$ denote $f_1(x, \mu) = f_\mu(x)$. Then (1) holds for f_1 because

$$\lambda f_\mu(x_1) + (1-\lambda) f_\mu(x_2) = (1-\mu)(\lambda h_1(x_1) + (1-\lambda) h_1(x_2)) +$$
$$+ \mu(\lambda h_2(x_1) + (1-\lambda) h_2(x_2)) \leq (1-\mu) h_1(x_0) + \mu h_2(x_0) = f_\mu(x_0).$$

R.2. The maximum places of the functions $f_1(\cdot, \mu) = f_\mu$ are in the points (x, μ) for which the scalar product of $(h_1(x), h_2(x))$ and $(1-\mu, \mu)$ takes its maximum, i.e. which maximalize the projection of $(h_1(x), h_2(x))$ on the line of $(1-\mu, \mu)$.

Now we are in the position to give the counterexample. Define first the pair $(h_1, h_2) : I_1 \to \mathbf{R}^2$ as follows. Let

$$(h_1(0), h_2(0)) = (0, 1), \qquad \left(h_1\left(\frac{1}{4}\right), h_2\left(\frac{1}{4}\right)\right) = \left(\frac{1}{2}, \frac{1}{2}\right),$$
$$\left(h_1\left(\frac{1}{2}\right), h_2\left(\frac{1}{2}\right)\right) = \left(\frac{1}{4}, \frac{1}{4}\right), \qquad \left(h_1\left(\frac{3}{4}\right), h_2\left(\frac{3}{4}\right)\right) = \left(\frac{1}{2}, \frac{1}{2}\right),$$
$$(h_1(1), h_2(1)) = (1,0)$$

and let (h_1, h_2) be linear in the segments $\left[0, \frac{1}{4}\right], \left[\frac{1}{4}, \frac{1}{2}\right], \left[\frac{1}{2}, \frac{3}{4}\right], \left[\frac{3}{4}, 1\right]$.

It can be easily checked that (h_1, h_2) satisfies (6), hence (1) holds for its linear extension f_1 defined in R.1. It is obvious by R.2. that there exist a $\mu_0 \in I_2$ and an open interval $I_1' \subset I_1$ such that:

$$C_1 = \{(x_0, y_0) \in I_1 \times I_2 : f_1(x_0, y_0) = \max_x f_1(x, y_0)\} =$$
$$= (\{1\} \times [0, \mu_0]) \cup ((I_1 \setminus I_1') \times \{\mu_0\}) \cup (\{0\} \times [\mu_0, 1]).$$

We can define a continuous function $\varphi : [0, 1] \to I_1 \times I_2$ with $\varphi([0,1]) \cap C_1 = \emptyset$, $\varphi(0) \in \{0\} \times [0, \mu_0)$ and $\varphi(1) \in \{1\} \times (\mu_0, 1]$ such that the first coordinate function of φ is monotone increasing. In this case we can find an f_2 for which $C_1 \cap C_2 = \emptyset$ where

$$C_2 = \{(x_0, y_0) \in I_1 \times I_2 : f(x_0, y_0) = \max_y f(x_0, y)\} = \varphi([0, 1]).$$

Theorem 1 is proved.

2. Proof of the Theorem 2. One can prove by standard continuity arguments the

Lemma 1. A continuous function $f : [a, b] \to \mathbf{R}$ is concave if and only if the set of maximum points of $f + c \cdot id$ form a closed interval for any $c \in \mathbf{R}$.

Now for the proof of the Theorem 2 suppose in contrary – taking into account the Lemma 1 – and replacing f_1 by $f_1 + c \cdot pr_1$ if it is necessary – that there is an $y_0 \in I_2$ for which $f_1(\cdot, y_0)$ takes its maximum in a_1, a_2 and there is no more maximum place between a_1 and a_2. Let for $y \in I_2$ $\varepsilon_y = \sup_{I_1} |f_1(\cdot, y) - f_1(\cdot, y_0)|$ – it is continuous in y. Consider the intervals $(a_1 - \delta, a_1 + \delta)$, $(a_2 - \delta, a_2 + \delta)$; with $|a_1 - a_2| < 2\delta$. Supposing $I_1 = I_2 = [0, 1]$, define a function $f : I_1 \to \mathbf{R}$ as follows: let $f(0) = f(1) = 0$, $f(a_1) = f(a_2) = 1$ and let f be linear in $[0, a_1]$, $[a_1, a_2]$, $[a_2, 1]$. Now let for $y < y_0$ $f_y = f - \frac{2\varepsilon_y}{\delta} |id I_1 - a_2|$ and for $y > y_0$ $f_y = f - \frac{2\varepsilon_y}{\delta} |id I_1 - a_1|$, further let $f = f_{y_0}$. The function $f_1'(x, y) = f_y(x)$ is continuous and partially concave in X, hence the function $f_1(\cdot, y_0) + f_1'(\cdot, y_0)$ takes its maximum only in a_1 and a_2. If $y < y_0$ and $|x - a_2| > \delta$, then $f_1(a_2, y) + 2\varepsilon_y \geq f_1(x, y)$, hence denoting $\tilde{f}_1 = f_1 + f_1'$ we have $\tilde{f}_1(a_2, y) > \tilde{f}_1(x, y)$ i.e. $\tilde{f}_1(\cdot, y)$ has no maximum places outside of $[a_2 - \delta, a_2 + \delta]$. It follows that we can give a continuous function $\varphi : I_1 \to I_2$ such that Gr φ does not contain any (x_0, y_0) for which $\tilde{f}_1(x_0, y_0) = \max_x f_1(x, y_0)$. Let $f_2'(x, y) = -c|\varphi(x) - y|$. If $c \in \mathbf{R}$ is large enough, then setting $\tilde{f}_2 = f_2 + f_2'$, the maximum places of $\tilde{f}_2(x, \cdot)$ will be uniformly close to $\varphi(x)$ and hence there is no point (x_0, y_0) satisfying the assertion of the Theorem 2 with $(f_1 + f_1', f_2 + f_2')$. This contradiction proves the Theorem 2.

3. Proof of the Theorem 3. In the following the word „convex" will be used in the sense introduced in [8]. We need

Lemma 2. Suppose $\Phi \subset \mathbf{R}^n$ is compact and convex in the sense of [8]. Then Φ is contractible to a point.

Proof. For $n = 1$ the statement is trivial. We shall prove it by induction in n. Denote by x_n the last coordinate function in $\mathbf{R}^n$ and let $x_n^0 = \max\{x_n : x \in \Phi\}$, $\mathbf{R}^{n-1} = \{x \in \mathbf{R}^n : x_n = x_n^0\}$. Finally denote by Φ' the orthogonal projection of Φ on $\mathbf{R}^{n-1}$. It is obvious from the definition of the convexity that $\Phi' = \Phi \cap \mathbf{R}^{n-1}$ and that Φ' is convex in $\mathbf{R}^{n-1}$ (in the sense of [8]). Further $x \in \Phi$ implies $[x, x'] \subset \Phi$, x' denote the projection of x. Therefore Φ' is a strong deformation retract of Φ by the map $H : \Phi \times I \to \Phi'$ $(x, t) \to tx + (1-t)x'$. By the induction hypothesis Φ' is contractible, hence the Lemma 2 is proved.

Corollary. Let $\Phi\cap\mathbf{R}^n$ be convex in the new sense and finitely generated (i.e. the convex hull of finitely many elements). Then Φ satisfies the Brouwer fixed point property.

Indeed, a finitely generated convex set is a polyhedron in $\mathbf{R}^n$, further by Lemma 1 we have for any map $f:\Phi\to\Phi$ the equality $\lambda(f)=1$. Hence we can apply the fixed point theorem of Lefschetz (see [6]). Lemma 2 is proved.

Lemma 3. Let $\Phi\subset\mathbf{R}^n$ be convex and compact set. Define $T:\Phi\to P(\Phi)$ with the properties:

(a) $x\in\Phi \Rightarrow T(x)$ is convex;

(b) T is continuous, that is for any $y\in\Phi$ the set $T^{-1}(y)=\{x\in\Phi: y\in T(x)\}$ is open in Φ.

Then there is an $x_0\in\Phi$ such that $x_0\in T(x_0)$.

Proof. We follow the proof of the Brouwer fixed point theorem (see in [7] in a slightly generalized form). The family $\{T^{-1}(y): y\in\Phi\}$ forms an open covering of Φ, hence there exists $y_1, \ldots, y_s$ such that $\bigcup_{i=1}^{s} T^{-1}(y_i)=\Phi$. Let $\langle y_1, \ldots, y_s\rangle$ be the convex hull of these points (in the new sense), and $A_i=\langle y_1, \ldots, y_s\rangle\cap T^{-1}(y_i)$; $\langle y_1, \ldots, y_s\rangle$ is compact, therefore there is a continuous function $g=(g_1, \ldots, g_s):\langle y_1, \ldots, y_s\rangle\to\Delta_s$ (Δ_s denotes the standard simplex, i.e. $g_i\geq 0\ \forall i$ and $\sum g_i\equiv 1$) such that $\{g_i>0\}\subset A_i\ \forall i$. Now define $h:\Delta_s\to\langle y_1, \ldots, y_s\rangle$ as follows: Let $\alpha=(\omega_1, \ldots, \alpha_s)\in\Delta_s$. The „segment" $\langle y_1, y_2\rangle$ is a line from at last n classical segments, which starts at y_1, ends at y_2 and don't intersect. Let $x_2\in\langle y_1, y_2\rangle$ such a point for which $\alpha_2\cdot\text{length}\,\langle y_1, x_2\rangle=\alpha_1\cdot\text{length}\,\langle x_2, y_2\rangle$[1],. Suppose that we have found the point x_i for a $2\leq i\leq S-1$. Let $x_{i+1}\in\langle x_i, y_{i+1}\rangle$ the point for which $\alpha_{i+1}\cdot\text{length}\,\langle x_{i+1}, y_{i+1}\rangle=(\alpha_1+\ldots+\alpha_i)\cdot\text{length}\,\langle x_i, x_{i+1}\rangle$. Finally, let $h(\alpha)=x_s\in\langle y_1, \ldots, \ldots, y_s\rangle$. Since h is continuous, $h\circ g:\langle y_1, \ldots, y_s\rangle\to\langle y_1, \ldots, y_s\rangle$ has a fixed point x_0 by the Corollary. Let $\alpha=(\alpha_1, \ldots, \alpha_s)=g(x_0)$, suppose $\alpha_{i_1}, \ldots, \alpha_{i_k}\neq 0$, and the other α_i – s are zeros. It is seen from the construction of h that $x_0=h(\alpha)\in\langle y_{i_1}, \ldots, y_{i_k}\rangle$. On the other hand $g_{ij}(x_0)=\alpha_{ij}>0$ implies $x_0\in A_{ij}$ ($j=1, \ldots, k$), that is $y_{ij}\in Tx_0\ \forall j$. Tx_0 is convex, hence $Tx_0\supset\langle y_{i_1}, \ldots, y_{i_k}\rangle\ni x_0$. Lemma 3 is proved.

Lemma 4. If $\Phi_1\subset\mathbf{R}^n$ and $\Phi_2\subset\mathbf{R}^k$ are convex sets, then $\Phi=\Phi_1\times\Phi_2\subset\subset\mathbf{R}^{n+k}$ is also convex.

Proof. Given $(x_1, x_2), (y_1, y_2)\in\Phi_1\times\Phi_2$, we have to prove that $\langle x_1, y_1\rangle_{\mathbf{R}^n}\times\times\langle x_2, y_2\rangle_{\mathbf{R}^k}\supset\langle(x_1, x_2), (y_1, y_2)\rangle_{\mathbf{R}^{n+k}}$ We apply induction in k. The case $k=0$ is trivial. Let $x_2, y_2\in\Phi_2\subset\mathbf{R}^{k+1}$, and suppose that the $k+1$ – th coordinate of x_2 isn't greater than that of y_2. Then $\langle(x_1, x_2), (y_1, y_2)\rangle$ starts from (x_1, x_2) and its first segment ends at (x_1, x_2^*), where the $k+1$-th coordinate of x_2 is changed in x_2^* by that of y_2. Obviously, $[(x_1, x_2), (x_1, x_2^*)]\subset\langle x_1, y_1\rangle\times\times\langle x_2, y_2\rangle$, $\langle(x_1, x_2^*), (y_1, y_2)\rangle$ goes in $\mathbf{R}^{n+k}\times\{y_2^{(k+1)}\}$. By induction hypothesis we have

$$\langle(x_1, (x_2^{(1)}, \ldots, x_2^{(k)}, y_2^{(k+1)})), (y_1, (y_2^{(1)}, \ldots, y_2^{(k)}))\rangle\subset\langle x_1, y_1\rangle\times$$

$$\times\langle(x_2^{(1)}, \ldots, x_2^{(k)}), (y_2^{(1)}, \ldots, y_2^{(k)})\rangle\times\{y_2^{(k+1)}\}.$$

Lemma 4 is proved.

[1] If $\alpha_1=0$ and α_j is the first non-vanishing coordinate, we start with $\langle y_j; y_{j+1}\rangle$ instead of $\langle y_1, y_2\rangle$.

Now we are in the position to prove Theorem 3. Define the function

$$T_n: I^n\times I^n\to P(I^n\times I^n)$$

$$(x', y')\mapsto\left\{x'': f(x'', y')>\max_x f(x, y')-\frac{1}{n}\right\}\times$$

$$\times\left\{y'': g(x', y'')>\max_y g(x', y)-\frac{1}{n}\right\}.$$

By the compactness of I^n, T_n is continuous (see Lemma 2). Lemma 3 shows that $T_n(x, y)$ is convex, hence there is an (x^n, y^n) such that $(x^n, y^n)\in T_n(x^n, y^n)$. We choose a condensation point (x^0, y^0) of the set $\{(x,^n y^n): n\in\mathbf{N}\}$. Using once more the compactness of $I^n\times I^n$, we get by a standard argument that (x^0, y^0) has the required properties. The Theorem 3 is proved.

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