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A simple proof for von Neumann's minimax theorem

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To the memory of F. Riesz (1880—1956)

1. The usual proofs of the von Neumann minimax theorem and its generalizations are based on deep results of Sperner or Brouwer (cf. [2], [4], [5]). Our proof is based on the simple lemma due to F. Riesz (cf. [3], p. 41) that if a system of compact subsets of a topological space has the finite intersection property (i.e. every finite set has non-empty intersection) then the whole system has non-empty intersection. This proof is a development of the ideas of the paper [1].

2. Theorem. *Let E and F be topological vector spaces, and let $K_1\subset E$, $K_2\subset F$ be convex compact sets. Let $f(x, y)$ be a real-valued continuous function on $K_1\times K_2$, which is concave in x for any fixed $y\in K_2$, and convex in y for any fixed $x\in K_1$. Then*

$$\min_{y\in K_2}\max_{x\in K_1} f(x, y) = \max_{x\in K_1}\min_{y\in K_2} f(x, y).$$

Proof. Let c be a (fixed) real number such that

$$H_y^{(c)} = H_y = \{x : f(x, y) \geqq c\} \neq \emptyset \quad \text{for every} \quad y\in K_2,$$

where $\emptyset$ denotes the empty set. The sets H_y are convex and compact. We assert that

$$\bigcap_{y\in K_2} H_y \neq \emptyset. \tag{1}$$

According to the lemma of Riesz it is enough to prove that for any finite set $\{y_1, \ldots, y_n\}\subset K_2$ we have

$$\bigcap_{i=1}^{n} H_{y_i} \neq \emptyset.$$

We prove this by induction on n.

Consider the case $n=2$. Suppose there exist $y_1, y_2\in K_2$ for which

$$H_{y_1}\cap H_{y_2} = \emptyset \tag{2}$$

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and set $H(\lambda)=H_{y_1+(1-\lambda)y_2}$ for $\lambda\in[0,1]$; $H(\lambda)\neq\emptyset$ by the convexity of $f(x,y)$ in y. Next we show that

(3) $$H(\lambda)\subset H_{y_1}\cup H_{y_2}.$$

For every $x\in K_1$ and $x\notin H_{y_1}\cup H_{y_2}$ we have

$$f(x,\lambda y_1+(1-\lambda)y_2)\leqq f(x,y_1)+(1-\lambda)f(x,y_2)<c$$

since f is convex in y. Thus $x\notin H(\lambda)$. Therefore, (3) follows because of the definitions of H_{y_1}, H_{y_2}.

Using (2) and (3) we show that for arbitrary $\lambda\in[0,1]$

(4) $$\text{either } H(\lambda)\subset H_{y_1}\quad\text{or}\quad H(\lambda)\subset H_{y_2}.$$

Suppose the contrary:

$$H(\lambda^*)\cap H_{y_1}\neq\emptyset\quad\text{and}\quad H(\lambda^*)\cap H_{y_2}\neq\emptyset$$

for some $\lambda^*\in[0,1]$. Let $y_1^*\in H(\lambda_1^*)\cap H_{y_1}$ and $y_2^*\in H(\lambda^*)\cap H_{y_2}$ be arbitrarily chosen. Consider the closed interval

$$[y_1^*,y_2^*]=\{\lambda y_1^*+(1-\lambda)y_2^*:\ 0\leqq\lambda\leqq 1\}.$$

By the convexity of the sets H_y we have

$$[y_1^*,y_2^*]\subset H(\lambda^*).$$

From (2) and the compactness of H_{y_1} and H_{y_2} we see that there exists $y^*\in[y_1^*,y_2^*]$ such that

$$y^*\notin([y_1^*,y_2^*]\cap H_{y_1})\cup([y_1^*,y_2^*]\cap H_{y_2}),$$

and hence $y^*\notin H_{y_1}\cup H_{y_2}$. On the other hand, $y^*\in H(\lambda^*)$ which contradicts (3). So (4) is proved.

To complete the proof of (3), we need the following statement: If $H(\lambda_1)\cap\cap H_{y_1}\neq\emptyset$ for $\lambda_1\in[0,1]$, then there exists $\varepsilon_1=\varepsilon_1(y_1,y_2,\lambda_1)>0$ such that

(5) $$H(\lambda)\cap H_{y_1}\neq\emptyset\quad\text{for}\quad|\lambda-\lambda_1|<\varepsilon_1.$$

[Similarly: if $H(\lambda_2)\cap H_{y_2}\neq\emptyset$ for $\lambda_2\in[0,1]$, then there exists $\varepsilon_2=\varepsilon_2(y_1,y_2,\lambda_2)>0$ such that

(6) $$H(\lambda)\cap H_{y_2}\neq\emptyset\quad\text{for}\quad|\lambda-\lambda_2|<\varepsilon_2.]$$

We prove (5). If $H(\lambda_1)\cap H_{y_1}\neq\emptyset$ then according to (4), $H(\lambda_1)\cap H_{y_2}=\emptyset$, that is

(7) $$f(x,\lambda_1y_1+(1-\lambda_1)y_2)<c\quad\text{for every}\quad x\in H_{y_2}.$$

Since $f(x,\lambda y_1+(1-\lambda)y_2)$ is a continuous function in (x,λ), it follows from (7) that for every $x\in H_{y_2}$ there exists a neighborhood U_x of x and $\varepsilon(x)>0$ such that

$$f(x,\lambda y_1+(1-\lambda)y_2)<c\quad\text{for}\quad(x,\lambda)\in U_x\times(\lambda_1-\varepsilon(x),\lambda_1+\varepsilon(x)).$$

Therefore,

$$H_{y_2} \subset \bigcup_{x \in H_{y_2}} U_x.$$

Since H_{y_2} is compact we can choose a finite system $\{U_{x_i}\}_{i=1}^{n}$ such that

$$H_{y_2} \subset \bigcup_{i=1}^{n} U_{x_i}.$$

Then for $\varepsilon_1 = \min\{\varepsilon(x_i): i=1, \ldots, n\}$ we have (5). The proof of (6) is similar.

From (4), (5), (6) it follows that the set $\{\lambda \in [0, 1]: H(\lambda) \subset H_{y_1}\}$ is open in [0, 1]. Similarly, the set $\{\lambda \in [0, 1]: H(\lambda) \subset H_{y_2}\}$ is also open in [0, 1]. Taking (4) into consideration, we arrive at a decomposition of the interval [0, 1] into two disjoint non-empty relatively open sets, which is impossible. Thus we proved that

$$H_{y_1} \cap H_{y_2} \neq \emptyset.$$

Suppose we know that for any subset $\{y_1, \ldots, y_k\}$ of $K_2(\subset F)$ having at most n elements we have

$$\bigcap_{i=1}^{k} H_{y_i} \neq \emptyset$$

and then we prove the same for $n+1$ elements.

Suppose there exist $y_1, \ldots, y_{n+1}$ such that

$$\bigcap_{i=1}^{n+1} H_{y_i} = \emptyset \tag{8}$$

Then we have

$$(H_{y_1} \cap H_3) \cap (H_{y_2} \cap H_3) = \emptyset \quad \text{for} \quad H_3 = \bigcap_{i=3}^{n+1} H_{y_i}.$$

Now using the induction assumption and (8) we can apply the idea of the proof of $n=2$ for the sets

$$H_{y_i}^3 = H_{y_i} \cap H_3 \quad (i = 1, 2).$$

Thus we obtain

$$\bigcap_{i=1}^{n+1} H_{y_i} \neq \emptyset,$$

and so, according to the lemma of Riesz, (1) is proved.

Denote by $\mathscr{C}$ the set of real numbers c for which $H_y^{(c)} = H_y \neq \emptyset$ whenever $y \in K_2$. If $c_0 \in \mathscr{C}$, then $c \in \mathscr{C}$ for every $c \leqq c_0$. Since the function f is continuous, the set $\mathscr{C}$ is bounded from above. Denote by c^* its smallest upper bound. From the lemma of Riesz we deduce that $c^* \in \mathscr{C}$. We prove that

$$\min_{y \in K_2} \max_{x \in K_1} f(x, y) \leqq c^*. \tag{9}$$

Suppose

$$\min_{y\in K_2}\max_{x\in K_1} f(x, y) > c^*,$$

then there exists $\tilde{c} > c^*$ for which

$$\min_{y\in K_2}\max_{x\in K_1} f(x, y) \geqq \tilde{c} > c^*.$$

Therefore $\max_{x\in K_1} f(x, y) \geqq \tilde{c}$ for every $y\in K_2$, hence $\{x\colon f(x, y) \geqq \tilde{c}\} \neq \emptyset$ for every $y\in K_2$, but this contradicts the choice of c^*.

On the other hand, because of (1), we have

$$A \overset{\text{def}}{=} \bigcap_{y\in K_2} H_y^{(c^*)} \neq \emptyset.$$

Let $x^*\in A$. From the definition of H_y we obtain $f(x^*, y) \geqq c^*$ for every $y\in K_2$; thus

(10) $$\min_{y\in K_2} f(x^*, y) \geqq c^* \quad\text{and}\quad \max_{x\in K_1}\min_{y\in K_2} f(x, y) \geqq c^*.$$

From (9) and (10) we deduce

$$\min_{y\in K_2}\max_{x\in K_1} f(x, y) \leqq \max_{x\in K_1}\min_{y\in K_2} f(x, y).$$

Since

$$\min_{y\in K_2}\max_{x\in K_1} f(x, y) \geqq \max_{x\in K_1}\min_{y\in K_2} f(x, y)$$

is obvious, the theorem is proved.

References

[1] I. Joó—A. P. Sövegjártó, A fixed point theorem, *Ann. Univ. Sci. Budapest*, Sect. Math. (to appear).

[2] J. von Neumann, Zur Theorie der Gesellschaftsspiele, *Math. Ann.*, **100** (1928), 295—320.

[3] B. Sz.-Nagy, Introduction to real functions and orthogonal expansions, Akadémiai Kiadó—Oxford Univ. Press (Budapest and New York, 1964).

[4] H. Brézis—L. Nirenberg—G. Stampacchia, Remark on Ky Fan's minimax theorem, *Bull. Univ. Math. Ital.*, (4) **6** (1972), 293—300.

[5] Ky Fan, Fixed-point and minimax theorems in locally convex topological linear spaces, *Proc. Nat. Acad. Sci.*, **38** (1952), 121—126.

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