ON SOME CONTINUITY PROPERTIES OF DERIVATIONS AND HOLOMORPHIC AUTOMORPHISMS OF JB*-TRIPLES

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To honour *her* memory
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0. Introduction and preliminaries

Some twenty years ago, W. Kaup [10], [11], introduced a "ternary-type" structure known as JB*-triple systems. This structure turned out to be the natural algebraic-metric setting for the study of bounded symmetric domains in complex Banach spaces, and has been intensively studied for the last ten years (see [9] for a survey). To some extent, JB*-triples behave as C*-algebras or as JB*-algebras, of which they are a generalization. In particular, in dual JB*-triples (called JBW*-triples), besides the norm topology, one can consider the weak*, the strong*, the Mackey and the weak topologies (denoted by n, w*, s* τ* and w, respectively). Automatic continuity properties of the triple product (and of derivations) with respect to the topologies n and s* have been recently investigated by Barton-Friedman in [2] and Rodríguez Palacios in [12]. A thourough discussion of the continuity properties of the triple product with respect to the toplogy w* has been made in [16]. In particular, this latter study has given a purely W*-algebra structure characterization of compact operators in complex Hilbert spaces [16, prop. 4.2].

However, the weak topology on a JB*-triple seems to have never been considered in this context. It is the purpose of this note to make a study of the weak-weak continuity properties of both triple product, derivations, and holomorphic automorphisms of a non necessarily dual JB*-triple E. To be precise, if τ is one of the above mentioned topologies, we prove:

1. Everywhere defined derivations of E are automatically τ-τ continuous if and only if all surjective linear isometries of E lying in the connected identity component are τ-τ continuous.The latter property holds for any of the topologies w*, s*, τ*, w, n.

2. Holomorphic automorphisms of B_E are automatically τ-τ continuous if and only if the following two conditions hold: (a) For all $a \in E$, the mapping Q_a: $x \to xa^*x$ is τ-τ continuous on B_E. (b) All surjective linear isometries of E lying in the connected identity component are τ-τ continuous. These two properties hold for s*, τ*, and n.

3. Counterexamples to the *joint* τ-τ continuity of the triple product, and to the automatic τ-τ continuity of holomorphic automorphisms of B_E, are given for τ=w* and τ=w. In particular, we discuss the joint w-w continuity of the triple product in the classical JB*-triples $C_0(\Omega)$, $\mathcal{L}(\mathcal{H})$ and $c_0(\mathcal{H})$, whre Ω is a locally compact space, $\mathcal{H}$ is a complex Hilbert space, and $c_0(\mathcal{H})$ is the ideal of compact operators on $\mathcal{H}$.

We take from [10] and [11] the notation and basic results.

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1. Continuity of derivations and automorphisms

We recall that a *JB*-triple* is a Banach space E with a mapping $E \times E \times E \to E$, called the *triple product* and denoted by $\{.,.,.\}$, such that the following conditions $J_1,.., J_4$ hold:

(J_1) $\{x, y, z\}$ is jointly continuous, linear and symmetric in the external variables *x, z*, and conjugate linear in the internal variable *y*.

For fixed a,b in E, and fixed A,B in $\mathcal{L}(E)$, the symbols $a\square b$ and $[A, B]$ represent the operators $x \to \{a, b, x\}$, x in E, and AB-BA, respectively. Then

(J_2) The *Jordan identity* holds: for all a,b,x,y in E,

$$[a\square b, x\square y] = \{a, b, x\}\square y - x\square\{y, a, b\}.$$

(J_3) For x in E, $x\square x$ is a hermitian positive element of the Banach algebra $\mathcal{L}(E)$.

(J_4) For x in E, one has $\|x\square x\| = \|x\|^2$.

Let E be a JB*triple. *Homomorphisms* and *isomorphisms* can be introduced in the obvious manner. The set of surjective linear isometries of E, denoted by Isom(E), coincides with the set of its *automorphisms*. E is said to be a *JBW*-triple* if E is a dual Banach space. In that case, it has a unique predual E_*, and we refer to $w^* =: \sigma(E, E_*)$ as the weak* topology on E. Automorphisms of a JBW*-triple are w*-w* continuous. The bidual E** of E is a JBW*-triple whose triple product extends that of E. A *derivation* of E is a linear mapping δ defined on a (non necessarily closed) *subtriple* $\mathcal{D}(\delta)$ of E such that

$$\delta\{x, y, z\} = \{\delta x, y, z\} + \{x, \delta y, z\} + \{x, y, \delta z\} \qquad (x,y,z \in \mathcal{D}(\delta)).$$

We write Der(E) for the set of everywhere defined derivations of E. Any $\delta \in$ Der(E) is bounded [2, cor. 2.2]. Isom(E) is an *algebraic subgroup* of the linear group $\mathcal{L}(E)$; therefore [6], it is a Banach-Lie group whose Lie algebra is Der(E). We set $\mathrm{Isom}_0(E)$ for the connected identity component in Isom(E).

Any C*-algebra, and any JB*-algebra, $\mathcal{A}$ can be considered as a JB*-triple E with the triple product given respectively by

$$2\{x, y, z\} =: xy^*z + zy^*x \qquad \{x, y, z\} = x\circ(y^*\circ z) - y^*\circ(z\circ x) + z\circ(x\circ y^*) \qquad (x,y,z \in \mathcal{A})$$

and any *-derivation of $\mathcal{A}$ induces a derivation of the associated JB*-triple.

1.1 Definition. A linear topology τ on a JB*-triple E is said to be admissible if τ is coarser than the normed topology n.

1.2 Proposition. Let E be a JB*-triple, and let τ be an admissible topology. If $\delta \in$ Der(E) is a derivation, then the following statements are equivalent:

(i). The mapping δ is τ-τ continuous on E.

(ii). The one parameter group $t \to \Delta_t =: \exp t\delta$, $t \in \mathbb{R}$, consists of τ-τ continuous automorphisms of E.

Proof: "i⇒ii" One has $[\exp t\delta](x)=\sum_0^\infty \frac{t^n}{n!}\delta^n(x)$, $t\in\mathbf{R}$, $x\in E$. By assumption each map $x\to\frac{t^n}{n!}\delta^n(x)$, $x\in E$, $n\in\mathbf{N}$, is τ-τ continuous on E, and the convergence of the series is uniform for $\|x\|\le 1$ because δ is bounded. Thus, the limit mapping $x\to[\exp t\delta](x)$ is τ-τ continuous on the unit ball $\|x\|\le 1$, hence also on E.

"ii⇒i" Since δ is bounded, one has [13, th. 13.36]

$$\lim_{t\to 0}\| \delta x - \frac{1}{t}(\Delta_t\text{-Id})x \| = 0 \qquad (x\in E) \tag{1}$$

uniformly for $\|x\|\le 1$. As τ is admissible, i.e. $\tau\le n$, (1) entails

$$\tau \lim_{t\to 0}[\delta x - \frac{1}{t}(\Delta_t\text{-Id})x] = 0 \qquad (x\in E) \tag{2}$$

uniformly for $\|x\|\le 1$. As each transformation $\frac{1}{t}(\Delta_t\text{-Id})$, $0\ne t\in\mathbf{R}$, is τ-τ continuous on E, its uniform τ–limit (which is δ), is τ-τ continuous on $\|x\|\le 1$ and also on E.

1.3 Corollary. For a JB*-triple E and an admissible topology τ, the following statements are equivalent:

(i). Each derivation $\delta\in\text{Der}(E)$ is τ-τ continuous

(ii). Each surjective linear isometry of E lying in $\text{Isom}_0(E)$ is τ-τ continuous.

Proof: Since Der(E) is the Lie algebra of Isom(E), there are a neighbourhood $\mathcal{N}$ of 0 in Der(E), and a neighbourhood $\mathcal{M}$ of Id in $\text{Isom}_0(E)$, such that $\exp:\mathcal{N}\to\mathcal{M}$ is a homeomorphism. Assume (i) holds. By proposition 1.2, $\mathcal{M}=\exp(\mathcal{N})$ consists of τ-τ continuous automorphisms of E. Thus, (ii) follows from the fact that the connected component $\text{Isom}_0(E)$ is generated by any neigbourhood of the identity. The converse is a consequence of 1.2.

To give examples, we recall the definition of some admissible topologies on E. An element $0\ne u\in E$ is said to be *tripotent* if $\{u, u, u\}=u$. Each tripotent u produces a topologically direct sum decomposition

$$E = E_0(u) \oplus E_{1/2}(u) \oplus E_1(u)$$

where $E_k(u)=\{x\in E;\ u\square u(x)=kx\}$, $k\in\{0, 1/2, 1\}$, is the k-eigenspace of $u\square u\in\mathcal{L}(E)$. Here, $E_1(u)$ is a JB*-algebra in the product and involution given by

$$x\circ y=:\{x, u, y\} \qquad x^{\#}=:\{u, x, u\}.$$

Let E be a JBW*-triple, and let $\phi\in E_*$ be a weak* continuous functional. Then there is a tripotent $u\in E$, called the *support* of ϕ, uniquely determined by the fact that $\phi|_{E_1(u)}$ is a faithful positive functional on the JB*-algebra $E_1(u)$ and $\phi(u)=\|\phi\|=1$.Under those conditions [1, prop. 1.2]

$$\|x\|_\phi^2 =: \phi\{x, x, u\} \qquad (x\in E)$$

is the square of a seminorm, and the *strong* topology* on E, denoted by $s^*(E, E_*)$, is defined by the set $\{\|.\|_\phi \,;\, \phi\in E_*\}$. If $w=:\sigma(E, E^*)$ and $\tau^*=:\tau(E, E_*)$ are the weak, and the Mackey topology associated to the duality $\langle E, E_*\rangle$, then by [2, th. 3.2], one has the diagram

$$\sigma(E, E_*) \le \begin{array}{c} \le s^*(E, E_*) \le \tau^*(E, E_*) \le \\ \\ \le \sigma(E, E^*) \le \end{array} \le n$$

1.4 Corollary. Let E be a JBW*-triple, and denote by τ be any of the topologies w^*, s^*, τ^*, w, n. Then

(i). Each surjective automorphism Ψ of E is τ-τ continuous

(ii). Each derivation $\delta\in \mathrm{Der}(E)$ is τ-τ continuous.

Proof: For $\tau\neq s^*$, (respectively, $\tau=s^*$), the definition of τ involves only the Banach space (respectively, the JB*-triple) structure of E. Surjective automorphisms of E are isometric, hence they preserve both the Banach space and the JB*triple structure of E. Thus, surjective automorphisms of E are τ-τ homeomorphisms for any topology $\tau\in\{w^*, s^*, \tau^*, w, n\}$, and (ii) follows from (i) by proposition 1.2.

The following result, which is more or less known, is now recovered in a unified manner.

1.5 Corollary. If $\mathcal{A}$ is a W*-algebra or a JBW*-algebra, and δ is a *-algebra derivation with $\mathcal{D}(\delta)=\mathcal{A}$, then δ is τ-τ continuous for any $\tau\in\{w^*, s^*, \tau^*, w, n\}$.

Proof: We shall distinguish between the W*-algebra, or the JBW*-algebra, $\mathcal{A}$ and its associated JB*-triple, denoted by $\mathcal{A}^\wedge$. For $\tau\neq s^*$ there is no distinction between the τ-topology on $\mathcal{A}$ and the τ-topology on $\mathcal{A}^\wedge$ as remarked before, and by [12, prop. 3] this is also true for $\tau=s^*$. Each *-algebra derivation with $\mathcal{D}(\delta)=\mathcal{A}$ induces a triple derivation of $\mathcal{A}^\wedge$, and the result follows from corollary 1.4.

2. Continuity of holomorphic automorphisms

Let E and B_E be an arbitrary JB*-triple and its unit open ball. We recall that a *holomorphic automorphism* of B_E is a bijection Φ of B_E onto itself such that both Φ and Φ^{-1} are holomorphic mappings.The set $\mathrm{Aut}(B_E) =: \{\Phi: B_E\to B_E\,;\,\Phi$ is an automorphism$\}$ is a topological group with the usual law of composition and the topology of uniform convergence on B_E [17, th.4.3].

A holomorphic vector field $X: x\to X(x)$, $x\in B_E$, is said to be *complete in B_E* if, for each $x\in B_E$, the maximal solution of the initial value problem

$$\frac{d}{dt} f(t, x) = X[\, f(t, x)\,] \qquad f(0, x)=x \tag{1}$$

is valid on the whole real line **R**.We set $\mathrm{aut}(B_E)=:\{X:B_E\to E\,;\,X \text{ complete in } B_E\}$. The solution of (1) is denoted by

$$f(t, x)=: [\exp tX](x) \qquad (x\in B_E,\ t\in \mathbf{R}).$$

For $X\in \mathrm{aut}(B_E)$ and $t\in \mathbf{R}$, the mapping $f(t,\cdot):x\to f(t, x)$ satisfies $f(t,\cdot)\in \mathrm{Aut}(B_E)$ and $t\to f(t,\cdot)$ is a continuous one-parameter subgroup of $\mathrm{Aut}(B_E)$ whose infinitesimal generator is

$$X(x) = \frac{d}{dt}\Big|_0 f(t, x) \qquad (x\in B_E).$$

For $a\in E$, we let $a\text{-}a^*$ denote the vector field $x\to a - \{x, a, x\}$, $x\in E$. By [10] and [11], $\mathrm{Aut}(B_E)$ is a Banach-Lie group whose Lie algebra is $\mathrm{aut}(B_E)$, and $\{a\text{-}a^*\,;\,a\in E\} \subset \mathrm{aut}(B_E)$. As usualy, we write $\mathrm{Aut}_0(B_E)$ for the connected identity component of this group.The following result is taken from [16, lem. 2.3]

2.1 Proposition. If E is a JB*-triple and $a\in E$ satisfies $4\|t a\|<\pi$, then the mapping $f(t, x)=[\exp t(a\text{-}a^*)](x)$ is the uniform norm-limit on $\bar{B}_E$ of the series $\sum_0^\infty t^n a_n(x)$ where

$$a_0(x)=:x, \qquad a_1(x)=:a-\{x, a, x\},$$

$$a_{n+1}=\frac{1}{n+1}\sum_{j+k=n}\{a_j(x), a, a_k(x)\}, \qquad (n\geq 1), \qquad (2).$$

Proof: The scalar power series $\alpha(t)=:\sum_0^\infty \alpha_n t^n$ with coefficients $\alpha_0=:1$, $\alpha_1=:2\|a\|$ and

$$\alpha_{n+1}=\frac{1}{n+1}\sum_{j+k=n}\alpha_j\|a\|\alpha_k \qquad (n\geq 1)$$

dominates $z_x(t)=:\sum_0^\infty t^n a_n(x)$. But $\alpha(t)$ satisfies $\frac{d}{dt}\alpha(t)= \|a\| + \|a\|\alpha^2(t)$, $\alpha(0)=1$, i.e., $\alpha(t)=\mathrm{tg}(\frac{\pi}{4}+t\|a\|)$. By Cauchy´s dominated convergence criterion, $z_x(t)$ is uniformly convergent and satisfies $\frac{d}{dt}z_x(t)= a- \{z_x(t), a, z_x(t)\}$, $z_x(0)=x$, i.e., $z_x(t)=f(t, x)$.

2.2 Corollary. Let a be an element of a JB*-triple E, and let τ be an admissible topology on E. If the mapping Q_a is τ-τ continuous on B_E, then for each $t\in \mathbf{R}$, the holomorphic automorphism $\Phi_t(x)=:[\exp t(a\text{-}a^*)](x)$ is τ-τ continuous on B_E.

Proof: Constant maps and the identity are τ-τ continuous, and an induction argument shows that the coefficients a_{n+1}, $n\in \mathbf{N}$, in (2) are τ-τ continuous on B_E. By proposition 2.1, for small values of t one has

$$\Phi_t(x)=\sum_0^\infty t^n a_n(x)$$

where the series is norm-convergent, hence also τ-convergent, uniformly for $\|x\|\leq 1$. Since the terms are τ-τ continuous functions, so is its uniform limit.

2.3 Theorem. Let E be a JB*-triple, and let τ denote an admissible topology on E.

Then the following statements are equivalent:

(i). All holomorphic automorphisms $\Phi\in Aut_0 B_E$ are τ-τ continuous on B_E.

(ii). These two conditions hold: (a) All surjective linear isometries $L\in Isom_0(E)$ are τ-τ continuous. (b): For all $a\in E$, the mapping Q_a is τ-τ continuous on B_E.

Proof: "i⇒ii". Since $aut(B_E)$ is the Lie algebra of $Aut(B_E)$, there are a neighbourhood $\mathcal{N}$ of 0 in $aut(B_E)$ and a neighbourhood $\mathcal{M}$ of Id in $Aut_0(B_E)$ such that $\exp:\mathcal{N}\to\mathcal{M}$ is a homeomorphism. Suppose i) holds. Then clearly condition (a) also holds. Let $a\in E$ be small enough to have $a\in\mathcal{N}$. Then $\Phi_t(x)=:[\exp t(a-a^*)](x)$, $x\in B_E$, $t\in\mathbf{R}$, is a one parameter group which, by assumption, consists of τ-τ continuous transformations. By proposition 2.1, for small values of t (say $|t|\leq T$), one has

$$\frac{1}{t}[\Phi_t(x) - Id(x)] - a_1(x) = \sum_2^\infty t^{n-1} a_n(x) \qquad (x\in B_E)$$

hence,

$$\left\| \frac{1}{t}[\Phi_t(x) - Id(x)] - a_1(x)\right\| \leq |t|.\sum_2^\infty \| a_n(x)\| . |t|^{n-2} \leq |t|.\sum_2^\infty \| a_n(x)\| T^{n-2} \leq |t|.K$$

where K does not depend on $x\in B_E$. Since each transformation $\frac{1}{t}[\Phi_t - \Phi_0]$, $t\neq 0$, is τ-τ continuous, so is its uniform limit $x\to a_1(x)=a-\{x, a, x\}$ on B_E, and condition (b) holds for all $a\in\mathcal{N}$ hence for all $a\in E$.

"ii⇒i". Let $\Phi\in Aut_0(B_E)$ be given. By [17, th.4.3], one has $\Phi=L\circ M$ where L is a surjective isometry of E with $L\in Isom_0(E)$ and $M=[\exp(a-a^*)]$ for some $a\in E$. By assumption, L is τ-τ continuous and, by corollary 2.2, so is M, whence the result follows.

2.4 Corollary. If E is a JBW*-triple and τ is any of the topologies s*, τ^*, then any holomorphic automorphism $\Phi\in Aut(B_E)$ is τ-τ continuous on B_E.

Proof: By theorem 2.3 and corollary 1.4, it suffices to show that, for all $a\in E$, the mapping Q_a is τ-τ continuous on B_E, which is a consequence of [12, th. and note added in proof].

The continuity properties of $\Phi\in Aut(B_E)$ with respect to $\tau=w^*$ or $\tau=w$ are completely different from what preceeds, as shown in the following section.

3. Weak continuity of holomorphic automorphisms in spin factors

We recall that a *spin factor* is a Hilbert space $\mathcal{H}$ with a conjugation -, endowed with the triple product and the norm

$$\{a, b, x\} =: (a, b)x + (x, b)a - (a, \bar{x})\bar{b} \qquad ||| x |||^2 =: \|x\|^2 + \left(\|x\|^2 - |(x, \bar{x})|^2\right)^{\frac{1}{2}}$$

The norm in a JB*-triple is uniquely determined by the triple product, and it is essential to note that $|||.|||$ and $||.||$ are equivalent though they do not coincide [5, th.7.3].If $E=(\mathcal{H}, \{.,.,.\}, |||.|||)$ is a spin factor, one has $E_*=E^*=\mathcal{H}$ as vector spaces, and so there is no distinction between the weak* and the weak topologies on E.The equalities $s^*(E, E_*)= =\tau^*(E, E_*)=n$ also hold in this case.The conjugation on the Hilbert space $\mathcal{H}$ underlying to E is a surjective **R**-linear isometry, hence in order to study the w*-w*continuity of Q_a we may assume that $a=\bar{a}$. Fix any non null $a=\bar{a}\in\mathcal{H}$. Then the orthogonal complement of **C**a in $\mathcal{H}$ is a selfconjugate space, and due to $\dim\mathcal{H}=\infty$, one can choose an orthogonal sequence $\{x_n, n\in \mathbf{N}\}\subset\mathcal{H}$ with

$$\| x_n \| = 1, \qquad x_n=\bar{x}_n, \qquad x_n \perp a, \qquad (n\in \mathbf{N}).$$

Since the norms $|||.|||$ and $||.||$ are equivalent on E, $(x_n)_{n\in \mathbf{N}}$ is a bounded w*-null sequence. If Q_a is w*-w* continuous, then $\{x_n, a, x_n\}= -a$, $(n\in \mathbf{N})$, is also w*-null, and so a=0. We have proved [16, prop. 4.3]:

3.1 Proposition. Let E be any spin factor with $\dim E=\infty$. Then the triple product of E is *not jointly continuous* with respect to the weak* (or the weak) topology on E. The only weak*-weak* (or weak-weak) continuous holomorphic automorphisms of B_E are surjective isometries.

Note that, by [3, th. 2.1], the triple product is *separately* weak*-weak* continuous on E since any spin factor is a JBW*-triple.

4 Weak continuity of holomorphic automorphisms in the spaces $\mathcal{C}_0(\Omega)$

In this section, we establish that the multiplication $(f, g)\to f\cdot g$ is jointly weak-weak continuous on bounded subsets of $\mathcal{C}_0(\Omega)$. We apply this result to prove that all holomorphic automorphisms of the unit ball $B_{\mathcal{C}_0(\Omega)}$ are weak-weak continuous. Here, Ω is a locally compact σ-compact Hausdorff space, and $\mathcal{C}_0(\Omega)$ is the Banach algebra of continuous complex valued functions on Ω that vanish at infinity,with the norm of the supremum. We denote by $\mathcal{B}(\Omega)$ the σ-algebra of Borel subsets of Ω, and by $\mathcal{M}(\Omega)$ the space of complex valued Borel measures on Ω. If $\mu\in\mathcal{M}(\Omega)$ and $S\in\mathcal{B}(\Omega)$, $|\mu|(S)$ denotes the variation of μ on S. Then $\mathcal{M}(\Omega)$ with the norm $\|\mu\|=:|\mu|(\Omega)$ is a Banach space which is isometricaly isomorphic to the dual $\mathcal{C}_0(\Omega)^*$ of $\mathcal{C}_0(\Omega)$ in the representation $\mu\to\langle\mu, .\rangle$, where $\langle\mu, f\rangle =: \int_\Omega f(\omega)d\mu(\omega)$ for $f\in \mathcal{C}_0(\Omega)$ and $\mu\in\mathcal{M}(\Omega)$. The space of measures is also a module over the ring $\mathcal{C}_0(\Omega)$ in the product

$$g\mu(S)=: \int_S g(\omega)d\mu(\omega) \qquad (g\in \mathcal{C}_0(\Omega),\ \mu\in\mathcal{M}(\Omega),\ S\in\mathcal{B}(\Omega)).$$

4.1 Theorem. If Ω is a locally compact σ-compact Hausdorff space, then the multiplication $(f, g)\to f\cdot g$ is jointly weak-weak continuous on bounded subsets of $\mathcal{C}_0(\Omega)$.

Proof: Since only bounded subsets of $\mathcal{C}_0(\Omega)$ are involved, we can restrict our considerations to the unit ball $B_{C_0(\Omega)}$. We have to show that, if $(u_i)_{i\in I}$ and $(v_i)_{i\in I}$ are nets in $B_{C_0(\Omega)}$ wealky convergent to u and v respectively, $(u,v\in B_{C_0(\Omega)})$, then $(u_i\cdot v_i)_{i\in I}$ is weakly convergent to u·v, that is, $(u_i\cdot v_i - u\cdot v)_{i\in I}$ is a weakly null net. Due to the identity

$$u_i\cdot v_i - u\cdot v = (u_i - u)\cdot(v_i-v) + (u_i - u)\cdot v + u\cdot(v_i-v),$$

it suffices to prove that the three nets $(u_i-u)\cdot v$, $u\cdot(v_i-v)$, and $(u_i - u)\cdot(v_i-v)$, $i\in I$, are weakly null. We divide the proof into two steps.

Step 1. Since $\mathcal{M}(\Omega)$ is a module over the ring $C_0(\Omega)$, we have $\alpha=:v\cdot\mu\in\mathcal{M}(\Omega)$, and as $(u_i - u)_{i\in I}$ is weakly null,

$$\lim_{i\in I} \langle\mu, (u_i-u)\cdot v\rangle = \lim_{i\in I} \int_\Omega (u_i-u)\cdot v\, d\mu = \int_\Omega (u_i-u)\, d\alpha = \lim_{i\in I} \langle\alpha, u_i-u\rangle = 0.$$

Similarly,

$$\lim_{i\in I} \langle\mu, u\cdot(v_i-v)\rangle = 0.$$

Step 2. Let us write $f_i=:u_i-u$ and $g=:v_i-v$, $i\in I$, and assume that $(f_i.g_i)_{i\in I}$ is not weakly null. Then there exists a $\mu\in\mathcal{M}(\Omega)$ such that the net of complex numbers $(\langle\mu, f_i\cdot g_i\rangle)_{i\in I}$ is not convergent to zero. Hence there exists a number $\varepsilon_0>0$ and there exists a sequence of indices $(i_n)_{n\in N}\subset I$ such that (by writing f_n and g_n instead of f_{i_n}, g_{i_n})

$$|\langle\mu, f_n\cdot g_n\rangle| \geq \varepsilon_0 \qquad (n\in N) \qquad (1).$$

As Ω is σ-compact, there is a sequence $(K_n)_{n\in N}$ of compact subsets of Ω such that $K_n \subset K_{n+1}$ for $n\in N$, and $\bigcup_n K_n=\Omega$. In particular, there is a compact set $L\subset\Omega$ such that

$$|\mu|(\Omega\setminus L)\leq \frac{1}{6}\varepsilon_0 \qquad (2).$$

To each point $\omega\in L$, we associate the Dirac measure on ω, $\delta_\omega\in\mathcal{M}(\Omega)$; since the subnet $(f_n)_{n\in N}\subset(f_i)_{i\in I}$ is weakly null,

$$\lim_{n\to\infty} \langle\delta_\omega, f_n\rangle = \lim_{n\to\infty} f_n(\omega) = 0 \qquad (\omega\in L) \qquad (3).$$

As $|\mu|(L)<\infty$, the Egoroff theorem applies [7, th. 11.32]; hence there exists a partition A, B of L such that

$$|\mu|(B)\leq\frac{1}{6}\varepsilon_0 \quad \text{and} \quad \lim_{n\to\infty} f_n(\omega) = 0 \ \text{ uniformly for } \omega\in A. \qquad (4).$$

In particular, there exists an index $n_0\in N$ such that (note that the case $|\mu|(A)=0$ may be disregarded)

$$\sup_{\omega\in A} |f_n(\omega)| \leq \frac{1}{6|\mu|(A)}\varepsilon_0, \qquad (n\geq n_0) \qquad (5).$$

Since $(g_i)_{i\in I}$ is contained in $B_{C_0(\Omega)}$, $\|g_n\|\leq 1$ for $n\in N$, and from (5)

$$\left|\int_A f_n\cdot g_n d\mu\right| \leq \int_A |f_n|\,|g_n|\, d|\mu| \leq \|g_n\| \int_A |f_n|\, d|\mu| \leq$$

$$\leq \left(\sup_{\omega\in A} |f_n(\omega)|\right)\cdot |\mu|(A) \leq \frac{1}{6}\varepsilon_0 \qquad (n\geq n_0) \qquad (6).$$

From $|\mu|(B)\leq \frac{1}{6}\varepsilon_0$ and the boundedness of $(f_n)_{n\in N}$ and $(g_n)_{n\in N}$

$$\left|\int_B f_n\cdot g_n d\mu\right| \leq \int_B |f_n|\,|g_n|\,d|\mu| \leq \|f_n\|\cdot\|g_n\|\cdot|\mu|(B) \leq \frac{1}{6}\varepsilon_0 \qquad (7).$$

From (2) by a similar argument

$$\left|\int_{\Omega-L} f_n\cdot g_n d\mu\right| \leq \int_{\Omega-L} |f_n|\,|g_n|\,d|\mu| \leq \|f_n\|\cdot\|g_n\|\cdot|\mu|(\Omega\setminus L)\leq \frac{1}{6}\varepsilon_0 \qquad (n\in \mathbf{N}) \quad (8).$$

Finally, by (6), (7) and (8), we have for $n\geq n_0$

$$|\langle\mu, f_n\cdot g_n\rangle| = \left|\int_\Omega f_n\cdot g_n d\mu\right| \leq \left|\int_A\right| + \left|\int_B\right| + \left|\int_{\Omega-L}\right| \leq 3\,\frac{1}{6}\varepsilon_0 = \frac{1}{2}\varepsilon_0 \qquad (n\geq n_0)$$

which contradicts (1) and completes the proof.

4.2 Corollary. If $\mathcal{A}$ is a commutative unital complex C*-algebra, then the multiplication $(x, y)\rightarrow x\cdot y$ is jointly weak-weak continuous on bounded subsets of $\mathcal{A}$.

Proof: Use Gelfand's representation and theorem 4.1.

4.3 Corollary. If Ω is a locally compact σ-compact space, then all holomorphic automorphisms $\Phi\in \mathrm{Aut}(B_{\mathcal{C}_0(\Omega)})$ are weak-weak continuous in $B_{\mathcal{C}_0(\Omega)}$.

Proof: By theorem 4.1, the triple product of $\mathcal{C}_0(\Omega)$ is jointly weak-weak continuous on bounded sets. Surjective linear isometries of any Banach space are weak-weak continuous; thus the result follows by theorem 2.3.

4.4 Example. If $\mathcal{A}$ is the classical algebra l_∞ of all bounded complex valued sequences, and we apply corollaries 4.2 and 4.3, we get:

All holomorphic automorphisms $\Phi\in \mathrm{Aut}(B_{l_\infty})$ of B_{l_∞} are weak-weak continuous.

5. Weak continuity properties of the triple product in the algebra $c_0(\mathcal{H})$

We would like to prove an analogous to theorem 4.1 for non abelian C*- algebras, i.e., essentially for the algebra $\mathcal{L}(\mathcal{H})$ of bounded linear operators in a Hilbert space $\mathcal{H}$. Unfortunately, no representation of the dual $\mathcal{L}(\mathcal{H})^*$ of $\mathcal{L}(\mathcal{H})$ is known. Thus, we consider the algebra $c_0(\mathcal{H})$ of compact operators in $\mathcal{H}$, whose dual space is well known.

We recall [14, § 1.15] that $a\in c_0(\mathcal{H})$ is said to be a *trace operator* if there is an orthonormal basis $(\xi_\alpha)_{\alpha\in A}$ of $\mathcal{H}$ such that $\sum_{\alpha\in A} \|a\xi_\alpha\|<\infty$. In that case, the sum $\|a\|_1 =: = \sum_{\alpha\in A} \|a\xi_\alpha\|$ does not depend on the basis $(\xi_\alpha)_{\alpha\in A}$ we consider in $\mathcal{H}$. We write $l_1(\mathcal{H})$ for the set of all trace operators on $\mathcal{H}$. Since for $a\in l_1(\mathcal{H})$, the sum $\sum_{\alpha\in A} \|a\xi_\alpha\|$ is finite, the family $\{\alpha\in A;\ a(\xi_k)\neq 0\}$ is countable and we order it into a sequence $(\xi_k)_{k\in N}$; then any extension of $(\xi_k)_{k\in N}$ to an orthonormal basis of $\mathcal{H}$ is said to be *associated* to a. For $a\in l_1(\mathcal{H})$, the series $\mathrm{trace}(a)=: \sum_{\alpha\in A} (a\xi_\alpha \mid \xi_\alpha)$ has a well defined sum which does not depend on the basis $(\xi_\alpha)_{\alpha\in A}$ and is called the *trace of* a. Also $(l_1(\mathcal{H}), \|\cdot\|_1)$ is a

Banach space and a two-sided ideal over the ring $\mathcal{L}(\mathcal{H})$. Thus, for $x\in c_0(\mathcal{H})$ and $a\in l_1(\mathcal{H})$, the series

$$\langle a, x\rangle =: \operatorname{trace}(x\cdot a) = \sum_{\alpha\in A} (xa\xi_\alpha \mid \xi_\alpha)$$

is well defined, and $\langle a, \cdot\rangle$ is a continuous linear form on $c_0(\mathcal{H})$. Finally, the mapping $a\to\langle a, \cdot\rangle$ is an isometric isomorphism of $(l_1(\mathcal{H}), \|\cdot\|_1)$ onto the dual $c_0(\mathcal{H})^*$ of $c_0(\mathcal{H})$ and $l_1(\mathcal{H})$ is a module over the ring $c_0(\mathcal{H})$. We shall need the following lemmas.

5.1 Lemma. If $(x_i)_{i\in I}\subset c_0(\mathcal{H})$ be a weakly null net, then:

(i). For any pair of vectors $\xi, \eta\in\mathcal{H}$, one has $\lim_{i\in I}(x_i\xi \mid \eta)=0$.

(ii). For any $a\in c_0(\mathcal{H})$, the nets $(a\cdot x_i)_{i\in I}$ and $(x_i\cdot a)_{i\in I}$ are weakly null.

Proof: (i). Clearly $a=:(\cdot \mid \xi)\xi\in l_1(\mathcal{H})$, and we may assume $\|\xi\|=1$. Let $(\xi_\alpha)_{\alpha\in A}$ extend the singleton $\{\xi\}$ to an orthonormal basis of $\mathcal{H}$; since$(x_i)_{i\in I}$ is weakly null,

$$0 = \lim_{i\in I}\langle a, x_i\rangle = \lim_{i\in I} \operatorname{trace}(x_i\cdot a) = \lim_{i\in I}\sum_{\alpha\in A} (x_i\cdot a\, \xi_\alpha \mid \xi_\alpha) = \lim_{i\in I}(x_i\, \xi \mid \xi).$$

By polarizing we get $\lim_{i\in I}(x_i\xi \mid \eta)=0$ for all $\xi,\eta\in\mathcal{H}$.

(ii). Let $b\in l_1(\mathcal{H})$. Since $(x_i)_{i\in I}$ is weakly null and $l_1(\mathcal{H})$ is an ideal over $c_0(\mathcal{H})$,

$$\lim_{i\in I}\langle b, x_i\cdot a\rangle = \lim_{i\in I} \operatorname{trace}[(x_i\cdot a)\cdot b] = \lim_{i\in I} \operatorname{trace}[(x_i\cdot(a\cdot b)] = 0$$

which shows that $(x_i\cdot a)_{i\in I}$ is weakly null. The other half follows from $\operatorname{trace}(x\cdot y) = \operatorname{trace}(y\cdot x)$ for $x,y\in l_1(\mathcal{H})$.

5.2 Lemma. Let $b\in l_1(\mathcal{H})$, and $(\xi_\alpha)_{\alpha\in A}$be a basis associated to b. If $A\subset c_0(\mathcal{H})$ is a bounded subset, then for each $\varepsilon>0$ there is an index $N\in \mathbb{N}$ such that

$$\Big|\sum_{k=N+1}^{\infty}(ab\xi_k \mid \xi_k)\Big| \leq \varepsilon \qquad (a\in A).$$

Proof: We have $a\cdot b\in l_1(\mathcal{H})$ for all $a\in A$. From the boundedness of A, if $M=: \sup_{a\in A}\|a\|$,

$$\sum_{k=1}^{\infty}|(a\cdot b\xi_k \mid \xi_k)| \leq \sum_{k=1}^{\infty}\|a\|\|b\xi_k\| \leq M\sum_{k=1}^{\infty}\|b\xi_k\| \qquad (a\in A).$$

Since $\sum_{k=1}^{\infty}\|b\xi_k\|$ is finite, we can choose $N\in\mathbb{N}$ so that $\sum_{k=N+1}^{\infty}\|b\xi_k\| \leq \frac{1}{M}\varepsilon$.

If E is an arbitrary JB*-triple and τ is an admissible topology on E, then we write $\mathrm{Cont}_\tau(E)$ for the set of $a\in E$ such that Q_a is τ-τ continuous on B_E. One can prove, as in [16, lem. 2.2], that

5.3 Lemma. If E is an arbitrary JB*-triple, then $\mathrm{Cont}_\tau(E)$ is a norm closed subtriple of E; actually it is a quadratic ideal, i.e.,

$$\{\mathrm{Cont}_\tau(E), E, \mathrm{Cont}_\tau(E)\} \subset \mathrm{Cont}_\tau(E).$$

5.4 Theorem. Each mapping Q_a, $a\in c_0(\mathcal{H})$, is weak-weak continuous on bounded subsets.

Proof: We restrict our considerations to the unit ball $B_{c_0(\mathcal{H})}$ of $c_0(\mathcal{H})$. By the identity

$$\{x_i, a, x_i\} - \{x, a, x\} = \{(x_i - x), a, (x_i - x)\} + \{x, a, (x_i - x)\}$$

where x_i, x, $a \in B_{c_0(\mathcal{H})}$, $i \in I$, it suffices to prove the statements (i) and (ii) below:

(i). If $(x_i)_{i\in I} \subset B_{c_0(\mathcal{H})}$ is a weakly null net and $u, v \in c_0(\mathcal{H})$, then $(\{u, v, x_i\})_{i\in I}$ is weakly null.

(ii). If $(x_i)_{i\in I} \subset B_{c_0(\mathcal{H})}$ is a weakly null net, then $(\{x_i, a, x_i\})_{i\in I}$ is weakly null.

Proof:(i). One has $2\{u, v, x_i\} = uv^*x_i + x_iv^*u$, $i \in I$, and the statement is an immediate consequence of lemma 5.1.

(ii). We shall prove the statement in the special case in which a can be written in the form $a=(\cdot\,|\xi)\eta$ for some $\xi, \eta \in \mathcal{H}$. As a consequence, (ii) holds for all finite rank operators $a \in FR(\mathcal{H})$. The result follows then by the norm density of $FR(\mathcal{H})$ in $c_0(\mathcal{H})$ and lemma 5.3. Thus, let $a=(\cdot\,|\xi)\eta$. We have to show that, for each $b \in l_1(\mathcal{H})$, one has

$$\lim_{i\in I} \langle b, \{x_i, a, x_i\}\rangle = 0,$$

which is equivalent to

$$\lim_{i\in I} \text{trace}\,(\{x_i, a, x_i\}b) = 0 \qquad (1).$$

If $(\xi_\alpha)_{\alpha\in A}$ is a basis associated to b, the latter is equivalent to

$$\lim_{i\in I} \sum_{k=1}^{\infty} (\{x_i, a, x_i\}b\xi_k \,|\, \xi_k) = 0.$$

Let $\varepsilon>0$ be given. Since $(x_i)_{i\in I}$ is bounded, so is $(\{x_i, a, x_i\})_{i\in I}$, whence one can apply lemma 5.2. Let us fix $N \in \mathbf{N}$ in such a way that

$$\Big| \sum_{k=N+1}^{\infty} (y_i b\xi_k \,|\, \xi_k) \Big| \le \frac{1}{2}\varepsilon \qquad (i \in I) \qquad (2)$$

where $y_i = \{x_i, a, x_i\}$ for $i \in I$. From the expression $a = (\cdot\,|\xi)\eta$, it follows that

$$\{x_i, a, x_i\}b = (\cdot\,|\, x_i^*\xi)bx_i\eta \qquad (i \in I).$$

Hence, to each fixed pair $\phi, \varphi \in \mathcal{H}$,

$$(\{x_i, a, x_i\}b\phi \,|\, \varphi) = (x_i\phi \,|\, \xi)\,(bx_i\eta \,|\, \varphi), \qquad (i \in I).$$

As $(x_i)_{i\in I}$ is weakly null, by lemma 5.1 the latter shows that $\lim_{i\in I}(\{x_i, a, x_i\}b\phi \,|\, \varphi) = 0$. Thus, in particular, for $\phi = \varphi = \xi_k$,

$$\lim_{i\in I} (\{x_i, a, x_i\}b\xi_k \,|\, \xi_k) = 0 \qquad (1 \le k \le N),$$

and so, there exists an index $i_0 \in I$ such that

$$\Big| \sum_{k=1}^{N} (\{x_i, a, x_i\}b\xi_k \,|\, \xi_k) \Big| \le \frac{1}{2}\varepsilon \qquad (i \ge i_0) \qquad (3).$$

Finally, from (2) and (3),

$$|\text{trace}\,(\{x_i, a, x_i\}b)| = \Big| \sum_{k=1}^{\infty} (\{x_i, a, x_i\}b\xi_k \,|\, \xi_k) \Big| \le$$

$$\le \Big| \sum_{k=1}^{N} \Big| + \Big| \sum_{k=N+1}^{\infty} \Big| \le \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon \qquad (i \ge i_0)$$

which completes the proof.

5.5 Corollary. All holomorphic automorphisms $\Phi \in \mathrm{Aut}_0(B_{C_0(\mathcal{H})})$ are weak-weak continuous in $B_{C_0(\mathcal{H})}$.

Proof: It follows immediately from theorems 5.4 and 2.3.

6. Weak continuity properties of the triple product in the space $\mathcal{L}(\mathcal{H})$

In this section, we investigate the set $\mathrm{Cont}_w(\mathcal{L}(\mathcal{H}))$ of the operators $a \in \mathcal{L}(\mathcal{H})$ for which Q_a is weak-weak continuous in $B_{\mathcal{L}(\mathcal{H})}$. We recall [8, th. 4.2] that $\mathcal{L}(\mathcal{H})$ is a dual JB*-triple, and that any weak*-closed ideal M in a JBW*-triple E has an orthogonal complement $M^{\perp}$ which is an ideal,

$$E = M \oplus M^{\perp} \qquad M \square M^{\perp} = M^{\perp} \square M = \{0\}.$$

The canonical factor projection $\pi_M : \mathcal{L}(\mathcal{H}) \to M$ is a JB*-homomorphism, hence continuous. If $(\xi_\alpha)_{\alpha \in A} \subset \mathcal{H}$ is an orthonormal basis in $\mathcal{H}$, then we write

$$a_{\alpha\beta} =: (\cdot \mid \xi_\alpha)\xi_\beta \qquad (\alpha, \beta \in A).$$

Any operator $a \in \mathcal{L}(\mathcal{H})$ can be represented uniquely in the form

$$a = \sum_{\alpha, \beta \in A} \lambda_{\alpha\beta} (\cdot \mid \xi_\alpha)\xi_\beta$$

for some bounded family of scalars $(\lambda_{\alpha\beta})_{\alpha\beta \in A} \subset \mathbb{C}$. Here, the series is to be understood in the weak-operator topology of $\mathcal{L}(\mathcal{H})$.

6.1 Lemma. Let $a \in \mathrm{Cont}_w(\mathcal{L}(\mathcal{H}))$, and let M be a weak*-closed ideal in $\mathcal{L}(\mathcal{H})$. Then $b =: \pi_M(a) \in \mathrm{Cont}_w(M)$.

Proof: Let $(x_i)_{i \in I} \subset M$ be a bounded weakly null net in M. We have to show that

$$\lim_{i \in I} \langle \mu, \{x_i, b, x_i\} \rangle = 0$$

whenever $\mu \in M^*$. By the preceding remarks, we have

$$\{x_i, b, x_i\} = \{x_i, \pi_M a, x_i\} = \{\pi_M x_i, \pi_M a, \pi_M x_i\} = \pi_M\{x_i, a, x_i\}.$$

But $(x_i)_{i \in I}$ is also a bounded weakly null net in $\mathcal{L}(\mathcal{H})$, and clearly $\mu \circ \pi_M \in \mathcal{L}(\mathcal{H})^*$; thus, by the assumption $a \in \mathrm{Cont}_w(\mathcal{L}(\mathcal{H}))$,

$$\lim_{i \in I} \langle \mu, \{x_i, b, x_i\} \rangle = \lim_{i \in I} \langle \mu, \pi_M\{x_i, a, x_i\} \rangle = \lim_{i \in I} \langle \mu \circ \pi_M, \{x_i, a, x_i\} \rangle = 0$$

as we wanted to show.

6.2 Proposition. We have $c_0(\mathcal{H}) \subset \mathrm{Cont}_w(\mathcal{L}(\mathcal{H}))$.

Proof: By lemma 5.3, $\mathrm{Cont}_w(\mathcal{L}(\mathcal{H}))$ is a norm closed quadratic ideal of $\mathcal{L}(\mathcal{H})$; hence it suffices to prove that, whenever $\xi \in \mathcal{H}$, we have $a =: (\cdot \mid \xi)\xi \in \mathrm{Cont}_w(\mathcal{L}(\mathcal{H}))$. Let $(x_i)_{i \in I}$ be a bounded weakly null net in $\mathcal{L}(\mathcal{H})$, and let $\mu \in \mathcal{L}(\mathcal{H})^*$ be given. We have to prove that

$$\lim_{i \in I} \langle \mu, \{x_i, a, x_i\} \rangle = 0.$$

By Dixmier's theorem [15, §IV.3, th.5], μ admits a unique representation of the form

$$\mu = \phi + \varphi \qquad \phi \in l_1(\mathcal{H}), \qquad \varphi \in c_0(\mathcal{H})^{\perp}$$

i.e., ϕ can be identifyed to a trace operator $b \in l_1(\mathcal{H})$ and $c_0(\mathcal{H}) \subset \ker(\varphi)$. From $a \in c_0(\mathcal{H})$ we get $\{x_i, a, x_i\} \in c_0(\mathcal{H})$, $i \in I$, and so

$$<\mu, \{x_i, a, x_i\}> = < \phi + \varphi, \{x_i, a, x_i\}> = < \phi, \{x_i, a, x_i\}> = \text{trace}(\{x_i, a, x_i\} \cdot b)$$

whence we can draw $\lim_{i \in I} <\mu, \{x_i, a, x_i\}> = 0$ as we did in the proof of theorem 5.4.

6.3 Proposition. Let $a \in \text{Cont}_W(\mathcal{L}(\mathcal{H}))$ admit a representation

$$a = \sum_{\alpha, \beta \in A} \lambda_{\alpha\beta} (. \mid \xi_\alpha) \xi_\beta \tag{1}$$

(weak operator convergence) for some bounded family $(\lambda_{\alpha\beta})_{\alpha\beta \in A} \subset \mathbf{C}$ and minimal pairwise orthogonal tripontents $(a_{\alpha\beta})_{\alpha\beta \in A}$. Then $a \in c_0(\mathcal{H})$.

Proof: By the pairwise orthogonality of $(a_{\alpha\beta})_{\alpha\beta \in A}$, the net of partial sums in (1) is norm bounded in $\mathcal{L}(\mathcal{H})$. The weak-operator topology agrees with the weak* topology on bounded sets, hence we may assume that (1) is w*-convergent to a. We can suppose that there is an infinity of coefficients $\lambda_{\alpha\beta} \neq 0$ (otherwise, we would obviously have $a \in c_0(\mathcal{H})$). Since $(\lambda_{\alpha\beta})_{\alpha\beta \in A}$ is bounded, it has at least a cluster point, and we claim that $\lambda = 0$ is its only cluster point. Indeed, let λ be a limit point of $(\lambda_{\alpha\beta})_{\alpha\beta \in A}$, and choose a sequence $(\lambda_{nm})_{nm \in \mathbf{N}} \subset (\lambda_{\alpha\beta})_{\alpha\beta \in A}$ such that $\lim_{nm \to \infty} \lambda_{nm} = \lambda$. Let M denote the weak* closed ideal generated by $\{(. \mid \xi_n)\xi_m;\ n, m \in \mathbf{N}\}$ in $\mathcal{L}(\mathcal{H})$. Clearly the sequence $\{(. \mid \xi_n)\xi_m;\ n, m \in \mathbf{N}\}$ is weak* summable in $\mathcal{L}(\mathcal{H})$, and the projection $b = \pi_M a$ of a onto M is given by

$$b =: w^* \sum_{nm} \lambda_{nm} (. \mid \xi_n) \xi_m \tag{2}$$

We define a sequence $(x_{rs})_{rs \in \mathbf{N}} \subset M$ by

$$x_{rs} =: (. \mid \xi_r)\xi_1 + (. \mid \xi_1)\xi_s = a_{r1} + a_{1s} \qquad (r, s \in \mathbf{N}).$$

Clearly $(x_{rs})_{rs \in \mathbf{N}}$ is bounded and weakly null

$$(x_{rs})_{rs \in \mathbf{N}} \subset 2B_{\mathcal{L}(\mathcal{H})} \qquad w \lim_{r, s \to \infty} x_{rs} = 0.$$

Since the triple product in $\mathcal{L}(\mathcal{H})$ is separately weak* continuous, by (1) we have

$$\{x_{rs}, b, x_{rs}\} = \{x_{rs}, \sum_{nm} \lambda_{nm} a_{nm}, x_{rs}\} = w^* \sum_{nm} \overline{\lambda}_{nm} \{x_{rs}, a_{nm}, x_{rs}\} \qquad (r, s \in \mathbf{N}).$$

The only non-zero summands above are

$$\lambda_{nm}\{x_{rs}, a_{nm}, x_{rs}\} = \lambda_{r1} a_{r1} + (\lambda_{rs} a_{11} + \lambda_{11} a_{rs}) + \lambda_{1s} a_{1s} \tag{3}$$

By assumption $a \in \text{Cont}_W(\mathcal{L}(\mathcal{H}))$, hence by lemma 6.1, $b \in \text{Cont}_W(M)$, and as $(x_{rs})_{rs \in \mathbf{N}}$ is bounded and weakly null in M,

$$\lim_{r, s \to \infty} <\mu, \{x_{rs}, b, x_{rs}\}> = 0 \tag{4}$$

whenever $\mu \in M^*$. If μ is the functional associated to the trace operator $c =: (. \mid \xi_1)\xi_1$, we have (for r, s >1), $a_{rs} \cdot c = a_{r1} \cdot c = 0$ and $a_{1s} \cdot c = a_{1s}$, $a_{11} \cdot c = a_{11}$. Thus, by (3)

$$<\mu, \{x_{rs}, b, x_{rs}\}> = \mathrm{trace}[\{x_{rs}, b, x_{rs}\}.c] =$$

$$= \mathrm{trace}\,(\bar{\lambda}_{1s}a_{1s} + \bar{\lambda}_{rs}a_{11}).c = \bar{\lambda}_{rs}$$

and so, by (4)

$$0 = \lim_{r,s \to \infty} <\mu, \{x_{rs}, b, x_{rs}\}> = \bar{\lambda}_{rs}$$

Since $(\lambda_{nm})_{nm \in N}$ was convergent to λ, we have $\lambda=0$. As the origin is the only limit point of $(\lambda_{\alpha\beta})_{\alpha\beta \in A}$, the set of indices $\alpha\beta \in A$ such that $\lambda_{\alpha\beta} \neq 0$ is countable, and can be arranged into a decreasing sequence $|\lambda_1| \geq |\lambda_2| \geq \ldots \downarrow 0$. By the orthogonality of the tripotents $(. \mid \xi_n)\xi_m$, $(n, m \in N)$, the weak* closed subtriple E they generate in $\mathcal{L}(\mathcal{H})$ is commutative, hence isomorphic to an abelian von Neumann algebra, which in turn is isomorphic to l_∞, and clearly

$$a = w^* \sum_{nm} \lambda_{nm} (. \mid \xi_n)\xi_m \in E.$$

Thus, we have norm convergence and this series defines a compact operator. This completes the proof.

6.4 Corollary. One has $\mathrm{Cont}_W(\mathcal{L}(\mathcal{H})) = \mathcal{L}(\mathcal{H})$ if and only if $\dim(\mathcal{H}) < \infty$.

Proof: The identy operator has the representation $\mathrm{Id} = \sum_{\alpha \in A} (. \mid \xi_\alpha)\xi_\alpha$. If we had $\mathrm{Id} \in \mathrm{Cont}_W(\mathcal{L}(\mathcal{H}))$, then by proposition 6.3, $\mathrm{Id} \in c_0(\mathcal{H}))$ and so $\dim(\mathcal{H}) < \infty$.

6.5 Corollary. The multiplication in $\mathcal{L}(\mathcal{H})$ is jointly weak-weak continuous if and only if $\dim(\mathcal{H}) < \infty$.

Proof: If $(x, y) \to x.y$ is jointly weak-weak continuous, then so is $x \to x^2 = \{x, 1, x\}$, $x \in \mathcal{L}(\mathcal{H})$, whence $1 \in \mathrm{Cont}_W(\mathcal{L}(\mathcal{H}))$, which, by corollary 6.4 gives $\dim(\mathcal{H}) < \infty$. The converse is known.

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