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On linear isometries of Banach lattices in $\mathcal{C}_0(\Omega)$-spaces

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Abstract. Consider the space $\mathcal{C}_0(\Omega)$ endowed with a Banach lattice-norm $\|\cdot\|$ that is not assumed to be the usual spectral norm $\|\cdot\|_\infty$ of the supremum over Ω. A recent extension of the classical Banach-Stone theorem establishes that each surjective linear isometry U of the Banach lattice $(\mathcal{C}_0(\Omega), \|\cdot\|)$ induces a partition Π of Ω into a family of finite subsets $S \subset \Omega$ along with a bijection $T: \Pi \to \Pi$ which preserves cardinality, and a family $[\mathbf{u}(S): S \in \Pi]$ of surjective linear maps $\mathbf{u}(S): \mathcal{C}(T(S)) \to \mathcal{C}(S)$ of the finite-dimensional C*-algebras $\mathcal{C}(S)$ such that

$$(Uf)|_{T(S)} = \mathbf{u}(S)(f|_S) \qquad \forall f \in \mathcal{C}_0(\Omega) \qquad \forall S \in \Pi.$$

Here we endow the space Π of finite sets $S \subset \Omega$ with a topology for which the bijection T and the map $\mathbf{u}$ are continuous, thus completing the analogy with the classical result.

Keywords. Banach lattices; Banach–Stone theorem; linear isometries.

1. Introduction and preliminaries

In a recent article [3], the author has studied the Banach lattice $E := (\mathcal{C}_0(\Omega), \|\cdot\|)$, where Ω is a locally compact Hausdorff topological space and $\mathcal{C}_0(\Omega)$ stands for the space of all continous complex valued functions $f: \Omega \to \mathbb{C}$ that vanish at infinity, endowed with a Banach lattice norm $\|\cdot\|$ that is not assumed to be the usual spectral norm $\|\cdot\|_\infty$ of the supremum over Ω. It is proven that each $\|\cdot\|$-Hermitian operator A on $\mathcal{C}_0(\Omega)$ gives rise to a uniquely determined partition Π of the set Ω into pairwise disjoints subsets $S \subset \Omega$ such that

$$(Af)|_S = \mathbf{a}(S)(f|_S), \qquad \forall f \in \mathcal{C}_0(\Omega) \qquad \forall S \in \Pi \tag{1}$$

holds with a uniquely determined family of linear maps $\mathbf{a}^A(S): \mathcal{C}(S) \to \mathcal{C}(S)$, $S \in \Pi$. There is also a uniquely determined family $\langle \cdot, \cdot \rangle_S$ of inner products on the finite-dimensional function spaces $\mathcal{C}(S)$, $S \in \Pi$, such that

$$\{f|_S: \|f\| \leq 1\} = \{\phi \in \mathcal{C}(S): \langle \phi\ \phi \rangle_S \leq 1\}. \tag{2}$$

It is also proved that, for each surjective linear $\|\cdot\|$-isometry $U: \mathcal{C}_0(\Omega) \to \mathcal{C}_0(\Omega)$, there is a uniquely determined bijection $T: \Pi \to \Pi$ along with a family $[\mathbf{u}(S): S \in \Pi]$ of surjective linear $\langle \cdot, \cdot \rangle_S$-unitary operators $\mathbf{u}(S): \mathcal{C}(T(S)) \to \mathcal{C}(S)$ such that the sets S and $T(S)$ have the same cardinalities and

$$(Uf)|_S = \mathbf{u}(S)(f|_{T(S)}) \qquad \forall f \in \mathcal{C}_0(\Omega) \qquad \forall S \in \Pi. \tag{3}$$

In the classical case (when the lattice norm $\|\cdot\|$ coincides with the spectral norm $\|\cdot\|_\infty$), each element $S \in \Pi$ is a singleton $S = \{\omega\}$ for some $\omega \in \Omega$, the family $\mathbf{a}^A$ can be identified with a continuous real-valued function $\mathbf{a}\colon \Omega \to \mathbb{R}$, the inner products $\langle\cdot, \cdot\rangle_\omega$ are all equal to the usual inner product in $\mathbb{C}$, the family $\mathbf{u}(S)$ of unitary operators is identified with a continuous function $u\colon \Omega \to \mathbb{C}$, $|u(\omega)| = 1$, and the permutation $T\colon \Pi \to \Pi$ actually is a *homeomorphism* of Ω.

The aim of this note is to make a study of the topological properties of the bijection T and of the other elements **a** and **u** that appear in the above situation. To be more precise, we endow the carrier space (the space whose points are the subsets $S \in \Pi$) with a natural topology that makes T into a homeomorphism. However, one can not expect this task to be a straightforward generalization of the classical situation. Indeed, some of the objects involved in our considerations (the points in Π) now are *finite subsets* of Ω rather than *points* (subsets of one single element) in Ω. We also deal with functions $T\colon S \in \Pi \to T(S) \in \Pi$ for which both the variable S and the values $T(S)$ are finite subsets in Ω. A relevant fact here is that a finite set $S = \{\omega_1, \ldots, \omega_k\} \in \Pi$ and all sets S' obtained by permuting its elements are the same point in Π, and hence we must have $T(S) = T(S')$, which poses a real difficulty concerning the continuity of T since the action of T on those singletons $S = \{\omega\}$ that lie in Π has to be continuous. In particular, we have to consider topologies σ in the space of finite subsets of Ω and an *appropriate* notion of continuity. One candidate for σ is the classical Hausdorff extension of the topology τ in Ω to a topology σ in the space $\mathcal{K}$ of all compact subsets $K \subset \Omega$. Since the elements of Π are finite (hence compact) sets, we can endow Π with the topology induced on it by σ. However, it is known that then the cardinality function $\#\colon \Pi \to \mathbb{N}$ given by $S \mapsto \#(S)$, though upper semicontinous, in general, is not continuous. To overcome this trouble, we consider Π as the disjoint union

$$\Pi = \bigcup_n \mathbf{\Omega}_n, \qquad \mathbf{\Omega}_n := \{S \in \Pi\colon \#S = n\}$$

where each $\mathbf{\Omega}_n$ is equipped with the topology induced by σ and Π is considered as the disjoint topological direct sum of the $\mathbf{\Omega}_n$, a topology that we denote by κ. In this way, the $\mathbf{\Omega}_n$ are open and closed subsets in (Π, κ) and, in order to study the continuity of a function $f\colon \Pi \to X$, where X is a topological space, we only need to analyse its restriction to the $\mathbf{\Omega}_n$. The fact that a net (S_i) in $\mathbf{\Omega}_n$ with $S_i = \{\omega_i^1, \ldots, \omega_i^n\}$, $(i \in I)$, converges to $S_0 = \{\eta^1, \ldots, \eta^n\} \in \mathbf{\Omega}_n$ relative to κ only provides us with the following information on the components: There is a subnet (S_j), $(J \subset I)$, along with a reordering of $S_0 = \{\eta^1, \ldots, \eta^n\}$ (that is, a permutation π of the indices $\{1, \ldots, n\}$) such that $\lim_{j\in J} \omega_j^k = \eta^{\pi(k)}$ holds in Ω for $1 \le k \le n$. Thus, we have had to weaken the notion of continuity, though, of course, the new notion agrees with the classical one when restricted to singletons $S = \{\omega\} \in \Pi$. For details, see §§2, 3 and 4 below.

In what follows $\mathcal{C}_0(\Omega)$ is endowed with a complete complex *lattice norm*, denoted by $\|\cdot\|$, whose open unit ball is

$$D := \{f \in \mathcal{C}_0(\Omega)\colon \|f\| < 1\}.$$

Notice that we *do not* assume that $\|\cdot\|$ coincides with $\|\cdot\|_\infty$. We let $\mathfrak{M}(\Omega) := (\mathfrak{M}(\Omega), \|\cdot\|^*)$ be the toplogical dual of $\mathcal{C}_0(\Omega)$, that is, the space of all Radon measures on Ω, endowed with the corresponding dual norm $\|\cdot\|^*$, whose open unit ball is

$$D^* = \{\mu \in \mathfrak{M}(\Omega)\colon \|\mu\|^* < 1\}.$$

Notice that, in general $\|\cdot\|^*$ *does not* coincide with the usual norm of total variation on Ω. We recall that both $\mathcal{C}_0(\Omega)$ and $\mathfrak{M}(\Omega)$ are Banach lattices when endowed with their respective usual order.

2. Preliminaries on the space of measures $\mathfrak{M}_\Pi(\Omega)$

Denote by $\mathfrak{M}_\Pi(\Omega) := \{\mu \in \mathfrak{M}(\Omega)\colon \operatorname{supp}\mu \in \Pi\}$ the set of the Radon measures on Ω whose support $S := \operatorname{supp}\mu$ is an element of the partition Π of Ω. Remark that $\mathfrak{M}_\Pi(\Omega)$ is not a vector subspace of $\mathfrak{M}(\Omega)$ and that whenever μ and ν are measures in $\mathfrak{M}_\Pi(\Omega)$ with $\mu \neq \nu$ we have $\operatorname{supp}\mu \cap \operatorname{supp}\nu = \emptyset$. Define an equivalence on $\mathfrak{M}_\Pi(\Omega)$ by setting $\mu \sim \nu$ if and only if $\operatorname{supp}\mu = \operatorname{supp}\nu$. Clearly we can identify the quotient set $\mathfrak{M}_\Pi(\Omega)/_\sim$ and the partition Π by the map $\operatorname{supp}\colon \mathfrak{M}_\Pi(\Omega/_\sim \leftrightarrow \Pi$ taking each class of measures $[\mu]$ to their common support. If $S := \operatorname{supp}\mu$ for some $\mu \in \mathfrak{M}_\Pi(\Omega)$, then $S \subset \Omega$ is a finite subset $S = \{s_1, \ldots, s_r\}$ for certain pairwise distinct points $s_j \in \Omega$ and we have

$$[\mu] = \left\{ \sum_{k=1}^{r} \alpha_k \delta_{s_k}\colon\ \alpha_k \in \mathbb{C}\backslash\{0\},\ 1 \le k \le r \right\},$$

where δ_s denotes the Dirac measure at the point $s \in \Omega$ and none of the coefficients α_k can vanish in order to ensure $\operatorname{supp}\mu = S$. Thus the class $[\mu]$ is not a vector space. Let $S \in \Pi$ be given and, instead of the condition $\operatorname{supp}\nu = S$, consider the weaker one $\operatorname{supp}\nu \subset S$; then the set

$$\mathfrak{N}(S, \Omega) := \{\nu \in \mathfrak{M}(\Omega)\colon \operatorname{supp}\nu \subset S\}$$

is a vector subspace of $\mathfrak{M}(\Omega)$ that is linearly spanned by the elements in the class $[\mu]$, that is $\mathfrak{N}(S, \Omega) = \operatorname{span}[\mu]$. Notice, however, that $\mathfrak{N}(S, \Omega)$ fails to be contained in $\mathfrak{M}_\Pi(\Omega)$ since there are measures $\nu \in \mathfrak{N}(S, \Omega)$ whose support $S' := \operatorname{supp}\nu$ is a proper subset $S' \subset S$ and therefore $S' \notin \Pi$. According to the proof of Theorem 1.4 of [3], for every surjective linear isometry $U\colon \mathcal{C}_0(\Omega) \to \mathcal{C}_0(\Omega)$, the family of vector spaces

$$\mathfrak{N}(S, \Omega), \qquad S \in \Pi \tag{4}$$

is invariant under the operator U^*. Therefore, U^* takes each $\mathfrak{N}(S, \Omega)$ with $S \in \Pi$ into another element of the family (4)

$$U^*(\mathfrak{N}(S, \Omega)) = \mathfrak{N}(S^*, \Omega), \qquad S \in \Pi$$

for some $S^* \in \Pi$ which depends on the operator U^* and satisfies $\#S^* = \#S$. Besides, $[\mu]$ contains a maximal free set $\{\delta_s\colon s \in S\}$ which spans $\mathfrak{N}(S, \Omega)$. Since U^* is invertible, it must transform the maximal free set S into a maximal free set S^* which spans $\mathfrak{N}(\Omega, S^*)$, and hence U^* takes the class $[\mu]$ into a class $[\mu^*]$ with $\operatorname{supp}\mu^* = S^*$.

Recall that the transposed $U^*\colon \mathfrak{M}(\Omega) \to \mathfrak{M}(\Omega)$ is a surjective linear $\|\cdot\|^*$-isometry, and U^* is weak*–weak*-continuous, hence U^* is a homeomorphism of $(\mathfrak{M}(\Omega), \|\cdot\|^*)$ and of $(\mathfrak{M}(\Omega), w^*)$. By the preceding discussion, the set $\mathfrak{M}_\Pi(\Omega) \subset \mathfrak{M}(\Omega)$ is invariant under U^* and, in the terminology introduced above, U^* is compatible with the equivalence $\sim$. Moreover, U^* induces a bijection $T\colon \Pi \to \Pi$ as suggested by the commutative diagram

$$\begin{array}{ccc}
\mathfrak{N}(S, \Omega) & \xrightarrow{\ U^*\ } & \mathfrak{N}(S^*, \Omega) \\
\operatorname{supp}^{-1}\big\uparrow & & \big\downarrow\operatorname{supp} \\
S \in \Pi & \xrightarrow{\ T\ } & S^* = T(S) \in \Pi
\end{array}$$

in which the left hand side vertical arrow $S \to \mathrm{supp}^{-1} S$ takes each $S \in \Pi$ to the vector space $\mathfrak{N}(S, \Omega)$ of the Radon measures whose support is contained in S, and the right hand side vertical arrow $\mathfrak{N}(S^*, \Omega) \to S^*$ takes the vector space $\mathfrak{N}(S^*, \Omega)$ to its *joint support*. Here, by joint support of a vector space $M \subset \mathfrak{M}(\Omega)$ of measures we mean the set $\bigcup_{\mu \in M} \mathrm{supp}\,\mu$. Clearly its complement $\Omega \setminus \bigcup_{\mu \in M} \mathrm{supp}\,\mu$ can be characterized as the largest open subset $U \subset \Omega$ with the following property:

$$\phi \in \mathcal{C}_0(\Omega),\ \ \mathrm{supp}\,\phi \subset U \Longrightarrow \langle \mu\ \phi \rangle = 0, \quad \forall \mu \in M.$$

In our case the spaces M under consideration are finite-dimensional, and the joint support is nothing but the union of the supports of the elements in a maximal free set in M, and it does not depend on the spanning set we choose in M. Thus

$$T := \mathrm{supp} \circ U^* \circ \mathrm{supp}^{-1}. \tag{5}$$

Remark that in the classical case, all classes $S \in \Pi$ are of the form $S = \{\omega\}$ for a unique $\omega \in \Omega$ and we reobtain the homeomorphism $\Omega \to \Omega$ provided by the Banach-Stone representation theorem for surjective linear isometries of $\mathcal{C}_0(\Omega)$.

3. Convergence of nets in $\mathfrak{M}_\Pi(\Omega)$

By Proposition 4.3 of [3], we have $N := \sup_{S \in \Pi} \#S < \infty$ where N is a characteristic of the Banach lattice $\mathcal{C}_0(\Omega)$. Remark that $\emptyset \notin \Pi$, hence $0 \notin \mathfrak{M}_\Pi(\Omega)$. Thus $\mathfrak{M}_\Pi(\Omega)$ is a finite union of pairwise disjoint subsets

$$\mathfrak{M}_\Pi(\Omega) = \bigcup_1^N \mathfrak{M}_k(\Omega), \quad \mathfrak{M}_k(\Omega) := \{\mu \in \mathfrak{M}_\Pi(\Omega) \colon \#\mathrm{supp}\,\mu = k\}, \quad 1 \le k \le N.$$

PROPOSITION 3.1

Let $n \in \mathbb{N}$ be given. Let $\mu_i = \sum_{k=1}^n \alpha_i^k \delta_{\omega_i^k}$, $(i \in I)$, where $\omega_i^k \in \Omega$ and $\alpha_i^k \in \mathbb{C}$ for $1 \le k \le n$ and $i \in I$, be a net in $\mathfrak{M}_n(\Omega)$, and assume that

(i) *μ_i is weak*-convergent to a point $\nu = \sum_{k=1}^n \alpha^k \delta_{\eta^k}$ that belongs to $\mathfrak{M}_n(\Omega)$.*
(ii) *None of the nets (ω_i^1), $(\omega_i^2) \cdots (\omega_i^{n-1})$ contains a subnet convergent to η^n in Ω.*

Then $\omega_k^n \to \eta^n$ in Ω and $\alpha_k^n \to \alpha^n$ in $\mathbb{C}$.

Proof.

Step 1. First we show that $\omega_i^n \to \eta^n$, for which we proceed by contradiction. Thus, let us assume that ω_i^n does not converge to η^n in Ω. Hence there are an open neighbourhood U of η^n in Ω and a subnet $J \subset I$ such that

$$\omega_j^n \notin U, \qquad \forall j \in J.$$

Now (ω_j^1) $(j \in J)$ is a subnet of (ω_i^1) which by (ii) does not converge to η^n. Hence there are an open neighbourhood $V_1 \subset U$ of η^n in Ω and a subnet $J_1 \subset J \subset I$ such that

$$\omega_j^1 \notin V_1, \qquad \forall j \in J_1.$$

Again (ω_j^2), $(j \in J_1)$, is a subnet of (ω_i^2) which by (ii) does not converge to η^n, and we can argue as before. After a finite number of steps we get a neighbourhood V of η^n in Ω and a subnet $K \subset I$ such that

$$\omega_k^1, \omega_k^2, \dots, \omega_k^n \notin V, \qquad \forall k \in K.$$

Since the points in $\{\eta^1, \dots, \eta^n\}$ are pairwise distinct, by shrinking V if needed we can assume that $\eta_1, \dots, \eta^{n-1}$ do not lie in V. Take any function $\phi \in \mathcal{C}_0(\Omega)$ with $\phi\colon \Omega \to [0, 1]$, $\phi(\eta^n) = 1$ and $\operatorname{supp}\phi \subset V$. By construction, we have

$$\langle \mu_k\ \phi\rangle = 0, \qquad \forall k \in K \quad \text{whereas} \quad \langle \nu\ \phi\rangle = \alpha^n \neq 0$$

which contradicts the assumption $w^* \lim \mu_i = \nu$.

Step 2. We claim that the net of coefficients α_i^n satisfies $\alpha_i^n \to \alpha^n$ in $\mathbb{C}$. Otherwise there would exist a subnet $J \subset I$ and some $\varepsilon > 0$ such that

$$|\alpha_j^n - \alpha^n| \geq \varepsilon, \qquad \forall j \in J.$$

By (ii) the subnets (ω_j^r) $(j \in J)$ for $1 \leq r \leq n-1$ do not converge to η^n, hence there are a subnet $K \subset J \subset I$ and a neighbourhood V of η^n in Ω such that

$$\omega_k^1, \dots, \omega_k^{n-1} \notin V, \qquad \forall j \in J.$$

Since $\omega_j^n \to \eta^n$, we have $\omega_k^n \in V$ for large enough $k \geq k_0$. We may assume that V does not contain any of the points $\eta^1, \dots, \eta^{n-1}$. Take any function $\psi \in \mathcal{C}_0(\Omega)$ with $\psi\colon \Omega \to [0, 1]$, $\psi(\eta^n) = 1$ and $\operatorname{supp}\psi \subset V$. Then by construction

$$\langle \mu_k\ \psi\rangle = \alpha_k^n, \qquad \langle \nu\ \psi\rangle = \alpha^n, \qquad \forall k \in K$$

hence $|\langle \mu_k - \nu\ \psi\rangle| = |\alpha_k^n - \alpha^n| \geq \varepsilon$ which contradicts $w^* \lim \mu_k = \nu$. □

COROLLARY 3.2

Let $n \in \mathbb{N}$ be given. Let $\mu_i = \sum_{k=1}^n \alpha_i^k \delta_{\omega_i^k}$, $(i \in I)$, be a net in $\mathfrak{M}_n(\Omega)$, and assume that μ_i is weak-convergent to a point $\nu = \sum_{k=1}^n \alpha^k \delta_{\eta^k}$ that belongs to $\mathfrak{M}_n(\Omega)$. Then there is a reordering of $(\eta^1, \dots, \eta^n)$ such that*

$$\omega_i^r \to \eta^r \quad \textit{in} \quad \Omega \quad \textit{and} \quad \alpha_i^r \to \alpha^r \quad \textit{in} \quad \mathbb{C} \quad (1 \leq r \leq n).$$

Proof. By (3.1), there is an index k $(1 \leq k \leq n)$ such that

$$\omega_i^k \to \eta^k \quad \text{in} \quad \Omega \quad \text{and} \quad \alpha_i^k \to \alpha^k \quad \text{in} \quad \mathbb{C}. \tag{6}$$

After reordering the n-tuple $(\eta^1, \dots, \eta^n)$ if needed, we may assume that the index k is precisely $k = n$. Clearly $w^* \lim \alpha_i^n \delta_{\omega_i^n} = \alpha^n \delta_{\eta^n}$ by (6). Thus from

$$\mu_i = \sum_{k=1}^n \alpha_i^k \delta_{\omega_i^k} \to \nu = \sum_{k=1}^n \alpha^k \delta_{\eta^k} \quad \text{and} \quad \alpha_i^n \delta_{\omega_i^n} \to \alpha^n \delta_{\eta^n}$$

we derive

$$\tilde{\mu}_i := \sum_{k=1}^{n-1} \alpha_i^k \delta_{\omega_j^k} \to \tilde{\nu} := \sum_{k=1}^{n-1} \alpha^k \delta_{\eta^k}.$$

A new application of (3.1), now to the net $(\tilde{\mu_i})$ $(i \in I)$ and the measure $\tilde{\nu}$ in the space $\mathfrak{M}_{n-1}(\Omega)$, and an induction argument completes the proof. □

4. Convergence in the carrier space $\boldsymbol{\Omega}$

Now we analyse the set $\boldsymbol{\Omega} = \Omega/_\sim$ of the classes defined by the partition Π of Ω. Clearly it is contained in the space $\mathcal{K}$ whose points are the compact subsets of Ω and it would be reasonable to consider the classical Hausdorff extension of the topology τ in Ω to a topology σ in $\mathcal{K}$ and view $\boldsymbol{\Omega}$ as a topological subspace of it. However, the cardinality function $S \mapsto \#(S)$ is not continuous, which is a trouble for our purpose. The discussion in the previous section and the definition of the bijection T induced by U^* suggest the consideration of the carrier space $\Omega/_\sim$ as the finite union of the pairwise disjoint subsets

$$\boldsymbol{\Omega} = \bigcup_1^N \boldsymbol{\Omega}_n, \qquad \boldsymbol{\Omega}_n := \{S \in \Pi \colon \#S = k\}, \qquad 1 \le k \le N, \tag{7}$$

where $N := \sup_{S \in \Pi} \#(S) < \infty$ is a characteristic of the Banach lattice $\mathcal{C}_0(\Omega)$. Notice that $\boldsymbol{\Omega}_n$ is a subset of Π, but not a subset of Ω, and that $\mathrm{supp}^{-1}\boldsymbol{\Omega}_n = \mathfrak{M}(\Omega)_n$.

We endow each $\boldsymbol{\Omega}_n$ with the topology that σ induces on it and view $\boldsymbol{\Omega}$ as the disjoint direct topological sum of these spaces $\boldsymbol{\Omega}_n$. In this way we get a topological space $(\boldsymbol{\Omega}, \kappa)$ in which each of the $\boldsymbol{\Omega}_n$ is an open and closed subspace. Hence, in order to study the continuity properties of a function $f \colon \boldsymbol{\Omega} \to X$, where X is an arbitrary topological space, we need only to analyse the restrictions of f to the $\boldsymbol{\Omega}_n$.

Recall that for $S \in \boldsymbol{\Omega}$ we have $S = \mathrm{supp}\,\mu = \{\omega_1, \dots, \omega_r\}$ for some pairwise distinct points ω_k in Ω. For $\eta \in \Omega$, let S_η denote the unique element of the partition Π such that $\eta \in S_\eta$, and take open disjoint neighbourhoods V_k of ω_k in Ω. Then the family

$$\mathcal{E}_{(V_1,\dots,V_r)}(S) := \{S_\eta \colon \eta \in V_1 \cup \dots \cup V_r\}, \tag{8}$$

where $(V_1, \dots, V_r)$ ranges over the r-tuples of open disjoint neighbourhoods of the points ω_k in S, form a basis of neighbourhoods of S for the restriction to Π of the Hausdorff topology σ on $\mathcal{K}$. First we consider the subspace (Ω_n, κ) with n fixed and study convergence of nets. Recall that each point $S \in \boldsymbol{\Omega}_n$ is a finite subset of Ω, $S \in \Pi$. In what follows we assume S to have been ordered $S = \{s_1, \dots, s_n\}$, however the particular order given to S is not relevant.

PROPOSITION 4.1

Let $S_i = \{\omega_i^1, \dots, \omega_i^n\}$, $(i \in L)$, and $S = \{\eta^1, \dots, \eta^n\}$ be a net and a point in $\boldsymbol{\Omega}_n$ such that $\kappa \lim_{i \in I} S_i = S$. Then there are a subnet S_j, $(j \in J)$ and an index l with $1 \le l \le n$ such that the net (ω_j^l) converges to η^l in Ω.

Proof. Let $V_1, \dots, V_n$ open pairwise disjoint neighbourhoods of the points $\eta^1, \dots, \eta^n$ in Ω. Then

$$\mathcal{E}_{(V_1,\dots,V_n)} := \{\{\zeta^1, \dots, \zeta^n\} \in \boldsymbol{\Omega}_n \colon \zeta^k \in V_k,\ \ 1 \le k \le n\}$$

is a κ-neighborhood of S in $\boldsymbol{\Omega}_n$. By assumption we have $\kappa \lim_{i \in I} S_i = S$, hence there is an index $i_0 \in I$ such that

$$S_i \subset V_1 \cup \dots \cup V_n, \qquad \forall i \ge i_0. \tag{9}$$

Now we proceed by contradiction. Assume that for every subnet S_j $(J \subset I)$, the net (ω_j^1) does not converge in Ω to any of the points $\eta^1, \ldots, \eta^n$. Thus there exists a subnet $J \subset I$ such that

$$\omega_j^1 \notin V_1, \qquad \forall j \in J.$$

Consider now the net S_j $(J \in J)$, which satisfies $\kappa \lim_{j \in J} S_j = S$. By assumption the net (ω_j^1) does not converge to any of the points $\eta^2, \ldots, \eta^n$, hence there exists a subnet $J' \subset J \subset I$ such that

$$\omega_j^1 \notin V_2, \qquad \forall j \in J'.$$

After repeating this argument n times we get a subnet $J^n \subset J^{n-1} \subset \cdots \subset I$ such that

$$\omega_j^1 \notin V_1 \cup \cdots \cup V_n, \qquad \forall j \in J^n$$

which contradicts (9) and completes the proof. □

COROLLARY 4.2

Let $S_i = \{\omega_i^1, \ldots, \omega_i^n\}$, $(i \in L)$, and $S = \{\eta^1, \ldots, \eta^n\}$ be a net and a point in $\mathbf{\Omega}_n$ such that $\kappa \lim_{i \in I} S_i = S$. Then, after reordering S if needed, there is a subnet (S_k), $(K \subset I)$, such that $\lim_{i \in I} \omega_i^l = \eta^l$ for all l with $1 \leq l \leq n$.

Proof. By (4.1) we have $\lim_{j \in J} \omega_j = \eta^l$ for a suitable subnet $J \subset I$ and some l, $1 \leq l \leq n$. Reordering S we may assume that $l = 1$. A repetition of the argument gives the result. □

The above results clearly suggest the following weakened notion of continuity.

DEFINITION 4.3

Let $n \in \mathbb{N}$ be fixed, and let X be a topological space. A function $T\colon \mathbf{\Omega}_n \to X$ is said to be κ-continuous at a point $S_0 = \{\eta^1, \ldots, \eta^n\}$ if, for each net $S_i = \{\omega_i^1, \ldots . \omega_i^n\}$ in $\mathbf{\Omega}_n$ with $\kappa \lim_{i \in I} S_i = S_0$, for each permutation π of the indices $\{1, \ldots, n\}$ and for each subnet (S_j) $(J \subset I)$ such that

$$\lim_{j \in J} \omega_j^k = \eta^{\pi(k)}, \qquad 1 \leq k \leq n \ \ (\text{convergence in } \Omega),$$

we have $\lim_{j \in J} T(S_j) = T(S_0)$ in the space X.

And T is said to be κ-continuous in $\mathbf{\Omega}_n$ if it is κ-continuous at each point of $\mathbf{\Omega}_n$. Remark that X is an arbitrary topological space, hence the case $X = \mathbf{\Omega}_n$ is not excluded. Notice also that for $n = 1$, the above definition clearly coincides with the usual notion of continuity.

PROPOSITION 4.4

For each $n \in \mathbb{N}$, the restriction to $\mathbf{\Omega}_n$ of the map $T\colon \Pi \to \Pi$ defined by (4.3) is κ-continuous on $\mathbf{\Omega}_n$.

Proof. Let $S_i = \{\omega_i^1, \ldots, \omega_i^n\}$ and $S_0 = \{\eta^1, \ldots, \eta^n\}$ be a net and a point in $\mathbf{\Omega}_n$ and assume that $\kappa \lim S_i = S_0$. By the definition, after a reordering of S_0 if needed, we may assume that

$$\omega_i^1 \to \eta^1, \ldots, \omega_i^n \to \eta^n \quad \text{(convergence in } \Omega\text{)}.$$

Hence $w^* \lim_i \delta_{\omega_i^k} = \delta_{\eta^k}$ for $1 \leq k \leq n$ and therefore for the Radon measures $\mu_i := \sum_{k=1}^n \delta_{\omega_i^k}$ and $\nu := \sum_{k=1}^n \delta_{\eta^k}$ we have

$$w^* \lim_i \mu_i = \nu. \tag{10}$$

Now apply the operator $U^*\colon \mathfrak{M}(\Omega) \to \mathfrak{M}(\Omega)$ and remark these two facts:

(1) The measures μ_i belong to the space $\mathfrak{M}_n(\Omega)$ which is invariant under U^*, therefore we have

$$\mu_i^* := U^*\mu_i = \sum_{k=1}^n \alpha_i^k \delta_{\omega_i^{*k}}, \qquad \nu^* := U^*\nu = \sum_{k=1}^n \alpha^k \delta_{\eta^{*k}}$$

for some $S^* = \{\omega_i^{*1}, \ldots, \omega_i^{*n}\}$ and $S_0* = \{\eta^{*1}, \ldots, \eta^{*n}\}$ where S_i^* and S_0^* are elements of Π.

(2) The operator U^* is w^*–w^*-continuous, hence (10) gives $w^* \lim_i \mu_i^* = \nu^*$, that is

$$w^* \lim_i \sum_{k=1}^n \alpha_i^k \delta_{\omega_i^{*k}} = \sum_{k=1}^n \alpha^k \delta_{\eta^{*k}}.$$

By (3.1), after a reordering of $S_0^* = \{\eta^{*1}, \ldots, \eta^{^n}\}$ if needed, we may assume that

$$\omega_i^{*1} \to \eta^{1*}, \ldots, \omega^{*n} \to \eta^{*n} \quad \text{(convergence in } \Omega\text{)},$$

which means that the net of finite sets $S_i^* = \{\omega_i^{*1}, \ldots, \omega_i^{*n}\}$ is κ-convergent to the finite set $S_0^* = \{\eta^{*1}, \ldots, \eta^{*n}\}$. Since $S_i^* = \operatorname{supp} \mu_i * = T(S_i)$ and $S_0^* = \operatorname{supp} \nu^* = T(S_0)$, we have proven

$$\kappa \lim S_i = S_0 \Longrightarrow \kappa \lim T(S_i) = T(S_0)$$

that is, the map $T = \operatorname{supp} \circ U^* \circ \operatorname{supp}^{-1}$ is κ-continuous. □

5. Continuity of Hermitian operators

As stated in the introduction, each hermitian operator $A\colon \mathcal{C}_0(\Omega) \to \mathcal{C}_0(\Omega)$ on the Banach lattice $E := (\mathcal{C}_0(\Omega), \|\cdot\|)$ gives rise to a uniquely determined family of maps $\mathbf{a}^A\colon S \to \mathbf{a}^A(S)$, where for each $S \in \Pi$, $\mathbf{a}^A(S)$ is a linear transformation of the finite-dimensional $\mathbf{C}^*$-algebra $\mathcal{C}(S)$ (that is, $\mathbf{a}^A(S) \in \mathcal{L}(\mathcal{C}(S))$), such that

$$(Af)|_S = \mathbf{a}^A(S)(f|_S), \qquad \forall S \in \Pi, \qquad \forall f \in \mathcal{C}_0(\Omega). \tag{11}$$

We discuss the continuity properties of the map $S \mapsto \mathbf{a}^A(S)$, $S \in \Pi$, for which we first analyse its restriction to $\mathbf{\Omega}_n$ with n fixed.

Let us assume that Ω is compact, hence $\mathcal{C}_0(\Omega)$ is a unital $\mathbf{C}^*$-algebra with unit element the constant function 1_Ω. When the relation (11) is applied to $f := 1_\Omega$, we get

$$(A1_\Omega)|_S = \mathbf{a}^A(S)(1|_S), \qquad \forall S \in \mathbf{\Omega}_n. \tag{12}$$

Here 1_S is the unit of the C*-algebra $\mathcal{C}(S)$ (hence defined in an intrinsic way) and neither 1_S nor its image by $\mathbf{a}^A(S)$ (which is $(A1_\Omega)|_S$ in accordance with (6.1)) depend on the particular order we choose for the set S. We refer to $S \mapsto \mathbf{a}^A(S)(1_S)$ as the *action* of the family $\mathbf{a}^A$ on the units 1_S, and we shall prove that this action $\mathbf{a}^A\colon \mathbf{\Omega}_n \to \mathbb{C}^n$ is continuous (in the sense of Definition (4.3)). Remark that in the classical case, when all sets S are singletons $S = \{\omega\}$, this action can be identified with the multiplication by a continuous complex-valued function.

PROPOSITION 5.1

Let Ω be a compact topological space and let A be a hermitian operator on the Banach lattice $E = (\mathcal{C}_0(\Omega), \|\cdot\|)$. Then for each $n \in \mathbb{N}$, the action of the family $\mathbf{a}^A\colon S \to \mathbf{a}^A(S)$, $(S \in \mathbf{\Omega}_n)$, associated to A by (6.1) *is κ-continuous on $\mathbf{\Omega}_n$.*

Proof. Let $S_i = \{\omega_i^1, \ldots, \omega_i^n\}$ $(i \in I)$ and $S_0 = \{\eta^1, \ldots, \eta^n\}$ be a net and a point in $\mathbf{\Omega}_n$ such that $\lim_{i\in I} S_i = S_0$ in the κ topology of the space $\mathbf{\Omega}_n$. We have to show that, for any reordering of S_0 and for every subnet (S_j) $(J \subset I)$ such that

$$\lim_{j\in J} \omega_j^1 = \eta^{\pi(1)}, \ldots, \lim_{j\in J} \omega^n = \eta^{\pi(n)} \ \text{(convergence in } \Omega),$$

we have $\lim_{j\in J} \mathbf{a}^A(S_j) = \mathbf{a}^A(S_0)$ (convergence in $\mathbb{C}^n$).

Let a reordering of S_0 and a subnet (S_j) $(J \subset I)$ of (S_i) be given in the above conditions. Without loss of generality we may assume that the reodering of S_0 corresponds with the natural permutation $\pi = \mathrm{Id}$ of the indices $1 \le k \le n$. Since $1_\Omega \in \mathcal{C}_0(\Omega)$, its image $A1_\Omega \in \mathcal{C}_0(\Omega)$ is uniformly continuous on the compact set $K := \{\eta^1, \ldots, \eta^n\}$. Thus, given any $\varepsilon > 0$ there are pairwise disjoint open neighbourhoods $V_1, \ldots, V_n$ of the points $\eta^1, \ldots, \eta^n$ in Ω such that

$$|(A1_\Omega)(s) - (A1_\Omega)(t)| \le \varepsilon \tag{13}$$

whenever s and t lie in one of the V_k for some k with $1 \le k \le n$. Now

$$\mathcal{E}_{(V_1,\ldots,V_n)} := \{\{\zeta^1, \ldots, \zeta^n\} \in \mathbf{\Omega}_n\colon \zeta^k \in V_k,\ 1 \le k \le n\}$$

is a κ-neighbourhood of S_0 in $\mathbf{\Omega}_n$ and by assumption the subnet (S_j) satisfies

$$\lim_{j\in J} \omega_j^1 = \eta^1, \ldots, \lim_{j\in J} \omega_j^n = \eta^n \ \text{(convergence in } \Omega).$$

Therefore, if $j \ge j_0$ is large enough we have $\omega_j^k \in V_k$ $(1 \le k \le n)$, and by (13)

$$|(A1_\Omega)(\omega_j^k) - (A1_\Omega)(\eta^k)| \le \varepsilon, \qquad 1 \le k \le n, \qquad \forall j \le j_0.$$

Since the action $\mathbf{a}^A$ is given by $\mathbf{a}^A(S) := \mathbf{a}^A(S)(1_S) = (A1_\Omega)|_S$ for all $S \in \mathbf{\Omega}_n$, the above can be written in the form

$$|\mathbf{a}^A(S_j)(1_{S_j}) - \mathbf{a}^A(S_0)(1_{S_0})| \le \varepsilon, \qquad \forall j \ge j_0$$

which completes the proof. □

6. Continuity of the isometry-valued map $S \mapsto \mathbf{u}(S)$

As stated in the Introduction, each surjective linear isometry $U\colon \mathcal{C}_0(\Omega) \to \mathcal{C}_0(\Omega)$ of the Banach lattice $E := (\mathcal{C}_0(\Omega), \|\cdot\|)$ gives rise to the following intertwined elements:

(1) A uniquely determined partition Π of the set Ω into pairwise disjoints subsets $S \subset \Omega$ and a uniquely determined bijection $T\colon \Pi \to \Pi$ such that $\#T(S) = \#S$ for all $S \in \Pi$.
(2) A family $\langle \cdot, \cdot \rangle_S$ of inner products on the finite-dimensional function spaces $\mathcal{C}(S)$, $S \in \Pi$, such that

$$\{f|_S\colon \|f\| \le 1\} = \{\phi \in \mathcal{C}(S)\colon \langle \phi\, \phi\rangle_S \le 1\}.$$

(3) A family $[\mathbf{u}(S)\colon S \in \Pi]$ of surjective linear $\langle \cdot, \cdot \rangle_S$-unitary operators $\mathbf{u}(S)\colon \mathcal{C}(T(S)) \to \mathcal{C}(S)$ such that

$$(Uf)|_S = \mathbf{u}(S)(f|_{T(S)}), \qquad \forall f \in \mathcal{C}_0(\Omega), \qquad \forall S \in \Pi.$$

If Ω is compact and we apply the above relation to the unit $\mathbf{1}_\Omega$ of the C*-algebra $\mathcal{C}_0(\Omega)$ we get

$$(U\mathbf{1}_\Omega)|_S = \mathbf{u}(S)(\mathbf{1}_{T(S)}), \qquad \forall S \in \Pi. \tag{14}$$

In the classical case both $\mathcal{C}(T(S))$ and $\mathcal{C}(S)$ are canonically isomorphic to $\mathbb{C}$, and the isometry $\mathbf{u}(S)$ can be identified with its value $\mathbf{u}(T(S))$ at the unit $\mathbf{1}_{T(S)}$ of $\mathcal{C}(T(S))$. Thus, in general we are led to consider the *action* of the isometry $\mathbf{u}$ at the unit $\mathbf{1}_{T(S)}$, that is, the map $\boldsymbol{\Omega}_n \to \mathbb{C}^n$ given by $S \mapsto \mathbf{u}(\mathbf{1}_{T(S)})$ and we have

PROPOSITION 6.1

Let Ω be a compact topological space and let U be a surjective linear isometry of the Banach lattice $E = (\mathcal{C}_0(\Omega), \|\cdot\|)$. Then for each $n \in \mathbb{N}$, the function $\mathbf{u}\colon \boldsymbol{\Omega}_n \to \mathbb{C}^n$ given by $S \mapsto \mathbf{u}(\mathbf{1}_{T(S)})$ is κ-continuous on $\boldsymbol{\Omega}_n$.

Proof. Let $S_i = \{\omega_i^1, \ldots, \omega_i^n\}$ $(i \in I)$ and $S_0 = \{\eta^1, \ldots, \eta^n\}$ be a net and a point in $\boldsymbol{\Omega}_n$ such that $\lim_{i\in I} S_i = S_0$ in the κ topology of the space $\boldsymbol{\Omega}_n$. We have to show that, for any reordering π of S_0 and for every subnet (S_j) $(J \subset I)$ such that

$$\lim_{j\in J} \omega_j^1 = \eta^{\pi(1)}, \ldots, \lim_{j\in J} \omega^n = \eta^{\pi(n)} \quad \text{(convergence in } \Omega),$$

we have $\lim_{j\in J} \mathbf{u}(S_j)(\mathbf{1}_{T(S_j)}) = \mathbf{u}(S_0)(\mathbf{1}_{T(S_0)})$, where the limit is taken in $\mathbb{C}^n$. By (14) the latter is equivalent to

$$\lim_{j\in J} (U\mathbf{1}_\Omega)|_{S_j} = (U\mathbf{1}_\Omega)|_{S_0}. \tag{15}$$

Let a reordering of S_0 and a subnet (S_j) $(J \subset I)$ of (S_i) be given in the above mentioned conditions. Without loss of generality, we may assume that the reodering of S_0 corresponds with the natural permutation $\pi = \mathrm{Id}$ of the indices $1 \le k \le n$. Since $\mathbf{1}_\Omega \in \mathcal{C}_0(\Omega)$, its image $U\mathbf{1}_\Omega \in \mathcal{C}_0(\Omega)$ is uniformly continuous on the compact set $K := \{\eta^1, \ldots, \eta^n\}$. Thus, given any $\varepsilon > 0$ there are pairwise disjoint open neighbourhoods $V_1, \ldots, V_n$ of the points $\eta^1, \ldots, \eta^n$ in Ω such that

$$|(U\mathbf{1}_\Omega)(s) - (U\mathbf{1}_\Omega)(t)| \le \varepsilon \tag{16}$$

whenever s and t lie in one of the V_k for some k with $1 \leq k \leq n$. Now

$$\mathcal{E}_{(V_1,\ldots,V_n)} := \{\{\zeta^1, \ldots, \zeta^n\} \in \mathbf{\Omega}_n \colon \zeta^k \in V_k,\ 1 \leq k \leq n\}$$

is a κ-neighbourhood of S_0 in $\mathbf{\Omega}_n$ and by assumption the subnet (S_j) satisfies

$$\lim_{j\in J} \omega_j^1 = \eta^1, \ldots, \lim_{j\in J} \omega_j^n = \eta^n \quad \text{(convergence in } \Omega).$$

Therefore for $j \geq j_0$ large enough, we have

$$\omega_j^k \in V_k, \qquad 1 \leq k \leq n \qquad \forall j \geq j_0$$

and so from (16) we obtain

$$|(U\mathbf{1}_\Omega)(\omega_j^k) - (U\mathbf{1}_\Omega)(\eta^k)| \leq \varepsilon, \qquad 1 \leq k \leq n, \qquad \forall j \leq j_0$$

which is (15) and completes the proof. □

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