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Holomorphic automorphisms and collective compactness in J*-algebras of operators

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Abstract: Let G be the Banach-Lie group of all holomorphic automorphisms of the open unit ball $B_{\mathfrak{A}}$ in a J*-algebra $\mathfrak{A}$ of operators. Let $\mathfrak{F}$ be the family of all collectively compact subsets $\mathcal{W}$ contained in $B_{\mathfrak{A}}$. We show that the subgroup $F \subset G$ of all those $g \in G$ that preserve the family $\mathfrak{F}$ is a closed Lie subgroup of G and characterize its Banach-Lie algebra. We make a detailed study of F when $\mathfrak{A}$ is a Cartan factor.

Keywords: J-algebras, Cartan factors, holomorphic automorphisms, Banach-Lie groups, collective compactness.*
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1 Introduction

Let $\mathfrak{A}$ be a J*-algebra, that is, a norm-closed complex vector subspace of $\mathcal{L}(H, K)$ closed under the triple product operation $A \mapsto AA^*A$. Here H, K are complex Hilbert spaces, $\mathcal{L}(H, K)$ is the space of all bounded linear operators $T\colon H \to K$ endowed with the operator norm, and A^* denotes the usual adjoint of $A \in \mathcal{L}(H, K)$. It is known [2, 9], that the open unit ball $B_{\mathfrak{A}}$ of $\mathfrak{A}$ is homogeneous under the action of the group $G := \mathrm{Aut}(B_{\mathfrak{A}})$ of all holomorphic automorphisms of $B_{\mathfrak{A}}$. On the other hand, G is a real Banach-Lie group in the topology of uniform convergence over $B_{\mathfrak{A}}$ [14], and its Banach-Lie algebra $\mathfrak{g} := \mathrm{aut}(B_{\mathfrak{A}})$ consists of all complete holomorphic vector fields on $B_{\mathfrak{A}}$, endowed with the topology of uniform convergence on the ball.

In this paper, we are interested in the study of some naturally defined subgroups

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of G. For instance, in [6, 7, 12] the authors considered the ball $B_{\mathfrak{A}}$ of the J*-algebra $\mathfrak{A} := \mathcal{L}(H, K)$, endowed it with the topology τ induced by one of the various natural topologies on $\mathcal{L}(H, K)$ and studied the subgroup of G of those $g \in G$ that are continuous relative to τ. Here we consider a family $\mathfrak{F}$ of subsets $\mathcal{W} \subset B_{\mathfrak{A}}$ and study the subgroup $F \subset G$ of those $g \in G$ that preserve $\mathfrak{F}$,

$$F := \{g \in G : g(\mathcal{W}) \in \mathfrak{F} \ \ \forall \mathcal{W} \in \mathfrak{F}\}.$$

Since the elements of G are (holomorphic) homeomorphisms of $B_{\mathfrak{A}}$ they automatically preserve the family $\mathfrak{F}_1$ of all compact subsets $\mathcal{W} \subset B_{\mathfrak{A}}$, hence $F = G$ in this case. The same happens if we take $\mathfrak{F}_2$ to be the family of all subsets $\mathcal{W} \subset B_{\mathfrak{A}}$ such that $\operatorname{dist}(\mathcal{W}, \partial B_{\mathfrak{A}}) > 0$ since the elements of G are isometries for the Carathéodory distance in $B_{\mathfrak{A}}$. One can define many families $\mathfrak{F}$ between these two "extreme" cases. In general, F will not be closed in G nor will it be an algebraic subgroup of G, and there is no reason to expect that F is a Lie subgroup of G in the induced topology.

In our case $\mathfrak{F}$ will be the family of all *collectively compact* subsets $\mathcal{W} \subset B_{\mathfrak{A}}$. Recall that a subset $\mathcal{W} \subset \mathfrak{A}$ is said to be collectively compact if the *collective image* of the unit ball B_H of H, that is, the set $\mathcal{W}(B_H) := \bigcup_{T \in \mathcal{W}} T(B_H)$, is relatively compact in K. Notice that collective compactness of $\mathcal{W}$ is defined in terms of the action of the operators in $\mathcal{W}$, which does not involve either the J*-algebra structure of $\mathfrak{A}$ or the holomorphic structure of the ball $B_{\mathfrak{A}}$. It is known that collective compactness is not preserved even by surjective linear isometries, see ([1] p. 422 and example 2.6). We establish necessary and sufficient conditions for an automorphism $g \in G$ to preserve the family $\mathfrak{F}$ and prove that F is a closed Lie subgroup of G whose Lie algebra $\mathfrak{h}$ admits the Cartan decomposition $\mathfrak{h} = \mathfrak{L}_{\mathfrak{F}} \oplus \mathfrak{P}_{\mathfrak{F}}$. Here $\mathfrak{L}_{\mathfrak{F}}$, the Lie algebra of the isotropy group of the origin, consists of those derivations of $\mathfrak{A}$ which preserve $\mathfrak{F}$. Moreover, $\mathfrak{P}_{\mathfrak{F}}$ consists of the vector fields that, in the canonical coordinate system (the identity $\mathrm{Id}\colon B_{\mathfrak{A}} \to B_{\mathfrak{A}}$ as a global chart), have the form $\mathcal{X} = Q_A \frac{\partial}{\partial X}$ where $Q_A(X) = A - XA^*X$, $(X \in \mathfrak{A})$, and $A = Q_A(0)$ is a compact operator that belongs to $\mathfrak{A}$. A deeper analysis is made when $\mathfrak{A}$ is a special Cartan factor, in which case we prove that all surjective linear isometries in the identity connected component of the unitary group of $\mathfrak{A}$ preserve collective compactness and all derivations of $\mathfrak{A}$ preserve the family $\mathfrak{F}$.

Our main references are [1, 11] for background on collective compactness in the spaces of operators $\mathcal{L}(X, Y)$ with X, Y Banach spaces, [2, 3] for the study of J*-algebras of operators and [2, 4, 9] for the study of their groups of holomorphic automorphisms. See also [12] and [7] for related problems.

2 Notation and Preliminaries

The open unit ball of a Banach space Z is denoted by B_Z. By $\mathcal{L}(X, Y)$ and $\mathcal{K}(X, Y)$ we denote the space of bounded operators and the closed subspace of compact operators, respectively, endowed with the operator norm. For $X = Y$ we write $\mathcal{L}(X)$ instead of $\mathcal{L}(X, X)$, and X^* is the dual space of X. For $T \in \mathcal{L}(X, Y)$, we let $T^* \in \mathcal{L}(Y^*, X^*)$ be the transpose of X. By $\mathrm{Isom}(X) \subset \mathcal{L}(X)$ we denote the group of all surjective linear isometries of X, endowed with the topology of the operator norm. Whenever G is a topological group, G_0 denotes the connected component of the identity in G.

Definition 2.1. A subset $\mathcal{W} \subset \mathcal{L}(X,Y)$ is said to be collectively compact if the set $\mathcal{W}(B_Z) := \bigcup_{T\in\mathcal{W}} T(B_X)$ is relatively compact in Y.

If $\mathcal{W} \subset \mathcal{L}(X,Y)$ is collectively compact then $\mathcal{W} \subset \mathcal{K}(X,Y)$. A subset $\mathcal{W} \subset \mathcal{K}(X,Y)$ is *relatively* compact in $\mathcal{K}(X,Y)$ if and only if both $\mathcal{W}$ and $\mathcal{W}^* := \{T^* : T \in \mathcal{W}\}$ are *collectively* compact, and hence relative compactness implies collective compactness but the converse is not true. Surjective linear isometries of the space $\mathcal{L}(X,Y)$ preserve *relative* compactness since they are homeomorphisms, but in general they do not preserve *collective* compactness. For later reference we state the following lemmas, the proofs of which can be found in [1]

Lemma 2.2. *Let $K \subset \mathbb{C}$ be a compact set and let $\mathcal{U}$, $\mathcal{V} \subset \mathcal{K}(X,Y)$ be collectively compact sets of operators. Then: $K \cdot \mathcal{U}$, $\mathcal{U} + \mathcal{V}$ and $\overline{\mathcal{U}}$ are collectively compact sets.*

Lemma 2.3. *Let X, Y, Z be Banach spaces and assume that $\mathcal{U}$ and $\mathcal{V}$ are subsets of $\mathcal{L}(X,Y)$ and $\mathcal{L}(Y,Z)$, respectively.*

i) If $\mathcal{U}$ is a bounded subset of bounded operators and $\mathcal{V}$ is a collectively compact set of operators, then $\mathcal{V}\mathcal{U}$ is collectively compact.

ii) If $\mathcal{U}$ is a collectively compact set of operators and $\mathcal{V}$ is a compact set of bounded operators, then $\mathcal{V}\mathcal{U}$ is collectively compact.

We refer to [10] for the functional calculus in J*-algebras (odd functional calculus) used below.

Proposition 2.4. *Let $f(z) = \sum_0^\infty c_{2n+1} z^{2n+1}$ be an odd analytic function defined by a power series with radius of convergence $R > 1$, and let $\mathfrak{A}$ be a J*-algebra of operators. For each $T \in \mathfrak{A}$ let $f(T) \in \mathfrak{A}$ be the operator defined by the odd functional calculus. If $\mathcal{W} \subset B_{\mathfrak{A}}$ is a collectively compact set, then so is $f(\mathcal{W}) := \{f(T) : T \in \mathcal{W}\}$.*

Proof. Since $R > 1$, the series $f(z) = \sum c_{2n+1} z^{2n+1}$ is absolutely convergent at the point $z = 1$, that is $\rho := \sum |c_{2n+1}| < \infty$. Since f is odd, we have $f(z) = zg(z)$ where the radius of convergence of $g(z) := \sum_0^\infty c_{2n+1} z^{2n}$ is $R > 1$. Thus, for each $T \in B_{\mathfrak{A}}$ we have

$$\sum \|c_{2n+1}(T^*T)^n\| \leq \sum |c_{2n+1}| \|T\|^{2n} \leq \sum |c_{2n+1}| = \rho \tag{1}$$

and $g(T^*T) := \sum_0^\infty c_{2n+1}(T^*T)^n$ is a well-defined element in $\mathcal{L}(H)$. Moreover, (1) shows that $\mathcal{U} := \{g(T^*T) : T \in \mathcal{W}\}$ is a bounded (and therefore equicontinuous) set of operators in $\mathcal{L}(H)$. The result then follows from (2.3) and $f(\mathcal{W}) \subset \mathcal{W}\mathcal{U}$. □

3 Collective compactness and holomorphic automorphisms

In what follows $\mathfrak{A} \subset \mathcal{L}(H,K)$ is a J*-algebra of operators and Isom($\mathfrak{A}$) is the Lie group of all surjective linear isometries of $\mathfrak{A}$ whereas Der($\mathfrak{A}$) is the Lie algebra of all J*-algebra derivations of $\mathfrak{A}$. See [4] for background on this topic. We let $G := \text{Aut}(B_{\mathfrak{A}})$ be the group of all holomorphic automorphisms of the open unit ball $B_{\mathfrak{A}}$, endowed with the topology of local uniform convergence over $B_{\mathfrak{A}}$. Recall [14] that on G, the topology of local uniform convergence is the same as the topology of uniform convergence on the unit ball $B_{\mathfrak{A}}$, which (unless $\dim(\mathfrak{A}) < \infty$) does not coincide with the topology of uniform convergence on compact subsets of $B_{\mathfrak{A}}$. Recall also ([2] th. 2 and 3) that every $g \in G$ can be represented in a unique way in the form $g = M_A L$, where $L \in \text{Isom}(\mathfrak{A})$ and M_A, a Möbius transformation of $B_{\mathfrak{A}}$, is given by

$$M_A(T) := (I - AA^*)^{-1/2}(T + A)(I + A^*T)^{-1}(I - A^*A)^{1/2}, \qquad T \in B_{\mathfrak{A}} \tag{2}$$

where $A = g(0) \in B_{\mathfrak{A}}$. Here positive and negative square roots are defined by the usual series expansions and I at each occurrence denotes the identity mapping on the appropriate underlying Hilbert space. We let $\mathfrak{F}$ and $\mathfrak{F}_{B_{\mathfrak{A}}}$ be the families of all collectively compact subsets $\mathcal{W}$ in $\mathfrak{A}$ and in the ball $B_{\mathfrak{A}}$, respectively

Theorem 3.1. *For an operator $A \in B_{\mathfrak{A}}$ the following conditions are equivalent*

i) The operator A is compact,

ii) The quadratic map $Q_A(T) := A - TA^*T$, ($T \in \mathfrak{A}$), preserves the family $\mathfrak{F}$.

iii) M_A preserves the family $\mathfrak{F}_{B_{\mathfrak{A}}}$.

Proof. i)⇔ ii). Assume that A is compact. Let $\mathcal{W} \in \mathfrak{F}$ be collectively compact. Since A^* is bounded, $\mathcal{U} := \mathcal{W}A^*\mathcal{W}$ is collectively compact by (2.3 ii)). Clearly the set $\mathcal{V} := \{A\} \in \mathfrak{F}$ is collectively compact and so is $Q_A(\mathcal{W}) \subset \mathcal{V} + \mathcal{U}$ by (2.2 ii). For the converse, notice that $\mathcal{W} := \{0\} \in \mathfrak{F}$ is collectively compact, hence so is $A = Q_A(0)$.

i) ⇔ iii). Assume A is compact. Let $\mathcal{W} \in \mathfrak{F}_{B_{\mathfrak{A}}}$ be a collectively compact subset of $B_{\mathfrak{A}}$. Then $\|T\| \leq 1$ for all $T \in \mathcal{W}$ and, as $\|A\| < 1$, we have

$$f(T) := T(I + A^*T)^{-1} = T\sum_0^\infty (-1)^n (A^*T)^n = Tg(T)$$

where $g(T) := \sum_0^\infty (-1)^n (A^*T)^n$ is norm-convergent since

$$\|g(T)\| \leq \sum_0^\infty \|A^*T\|^n \leq \sum_0^\infty \|A\|^n \leq \frac{1}{1 - \|A\|} \qquad T \in B_{\mathfrak{A}}.$$

Therefore the set $g(\mathcal{W}) = \{g(T) : T \in \mathcal{W}\}$ is bounded and hence equicontinuous in $\mathcal{L}(H,K)$. Hence by (2.3) i) $f(\mathcal{W}) = \{Tg(T) : T \in \mathcal{W}\}$ is a collectively compact subset in $\mathfrak{A}$. Moreover by (2.3),

$$\mathcal{V} := \{(A + T)(I + A^*T)^{-1} : T \in \mathcal{W}\} \subset Ag(\mathcal{W}) + f(\mathcal{W})$$

is a collectively compact subset in $\mathfrak{A}$ and it is quite simple to show that so is

$$M_A(\mathcal{W}) = (I - AA^*)^{-1/2}\mathcal{V}(I - A^*A)^{1/2}.$$

Since by assumption $\mathcal{W} \subset B_{\mathfrak{A}}$, we actually have $M_A(\mathcal{W}) \subset B_{\mathfrak{A}}$ and so $M_A(\mathcal{W}) \in \mathfrak{F}_{B_{\mathfrak{A}}}$. For the converse, notice that $A = M_A(0)$. □

Corollary 3.2. *For every $g = M_A L \in G$ the following conditions are equivalent*

i) g preserves the family $\mathfrak{F}_{\mathfrak{A}}$.

ii) The operator $A = g(0)$ is compact and L preserves the family $\mathfrak{F}_{\mathfrak{A}}$.

Proof. i)⇒ii) Clearly $\mathcal{W} := \{0\}$ is a collectively compact set in the ball, hence $-A = -g(0)$ is compact. Hence by (3.1), $M_{-A} = M_A^{-1}$ preserves collective compactness in $B_{\mathfrak{A}}$ and therefore $L = M_A^{-1}g$ preserves collective compactness in the ball and also in $\mathfrak{A}$ due to the linearity of L. The argument for the converse is similar. □

Proposition 3.3. *The set $F := \{g \in G : g(\mathfrak{F}) \subset \mathfrak{F}\}$ is a closed subgroup of G.*

Proof. It is clear that F is a subgroup of G. By [14], the topology of G is that of uniform convergence on $B_{\mathfrak{A}}$. Let $g_n = M_{A_n}L_n$, $(n \in \mathbb{N})$, be a sequence in F and assume that $\lim_{n\to\infty} g_n = g$ where $g = M_A L \in G$. Since $\mathcal{W} := \{0\}$ is a collectively compact set in $B_{\mathfrak{A}}$ and $g_n \in F$, the operators $A_n = g_n(0)$ are compact. From $g_n \to g$ we get that $A = \lim_{n\to\infty} A_n$ is compact and $\|A\| < 1$. By (3.1), the Möbius transformation M_A preserves the family $\mathfrak{F}_{B_{\mathfrak{A}}}$. By [14] the relation $g_n \to g$ entails $L_n \to L$, convergence in the operator norm of $\mathcal{L}(\mathfrak{A})$. To complete the proof we have to check that L preserves $\mathfrak{F}$, and due to linearity, it suffices to show that it preserves the family $\mathfrak{F}_{B_{\mathfrak{A}}}$. Let $\mathcal{W}$ be a collectively compact set in $B_{\mathfrak{A}}$. We have to show that $L(\mathcal{W}) = \{L(T) : T \in \mathcal{W}\}$ is collectively compact, that is, the set $\bigcup_{T\in\mathcal{W}} \big(L(T)\big)B_H$ is totally bounded in K. Let $\varepsilon > 0$ be given. Since $L_n \to L$ we can fix an index $n \in \mathbb{N}$ such that

$$\|L - L_n\| \leq \varepsilon,$$

where the norm is the operator norm in $\mathcal{L}(\mathfrak{A})$. Since by (3.2), L_n preserves the family $\mathfrak{F}$, the set $\bigcup_{T\in W} \big(L_n(T)\big)B_H$ is totally bounded in K, hence there are finite sets

$$\{T_1, \cdots, T_r\} \subset \mathcal{W} \qquad \{h_1, \cdots, h_s\} \subset B_H$$

with the following property: given any $T \in \mathcal{W}$ and any $h \in B_H$ there is a pair (T_i, h_j) for which

$$\|\big(L_n(T)\big)h - \big(L_n(T_i)\big)h_j\| \leq \varepsilon.$$

But then, for $T \in \mathcal{W}$ and $h \in B_H$ we have

$$\|\big(L(T)\big)h - \big(L(T_i)\big)h_j\| \leq \|\big(L(T)\big)h - \big(L_n(T)\big)h\| + \\ +\|\big(L_nT)\big)h - \big(L_n(T_i)\big)h_j\| + \|\big(L_n(T_i)\big)h_j - \big(L(T_i)\big)h_j\| \leq 3\varepsilon$$

which completes the proof. □

It is therefore reasonable to study conditions for an isometry L of $\mathfrak{A}$ to preserve the family $\mathfrak{F}$. We first analyze the case of a C*-algebra. Let $\mathfrak{A} := \mathcal{L}(H)$; then the adjunction $T \mapsto T^*$ is a surjective linear isometry of $\mathfrak{A}$ which does not preserve collective compactness ([1] page 422 and example 2.6). This suggests that we should restrict our considerations to $\operatorname{Isom}(\mathfrak{A})_0$.

Proposition 3.4. *Let $\mathfrak{A} \subset \mathcal{L}(H)$ be a unital C*-algebra of operators acting on a Hilbert space H. Then every isometry $L \in \operatorname{Isom}(\mathfrak{A})_0$ preserves collective compactness.*

Proof. By a well known result of Kadison, every $L \in \operatorname{Isom}(\mathfrak{A})$ is of the form

$$L(A) = U\rho(A), \qquad A \in \mathfrak{A} \tag{3}$$

where $U = L(I)$ is a unitary element in $\mathfrak{A}$ and $\rho\colon \mathfrak{A} \to \mathfrak{A}$ is an isometry that satisfies

$$\rho(I) = I \quad \rho(A^2) = \rho(A)^2 \quad \rho(A^*) = \rho(A)^*,$$

that is, ρ is an element in the group $\operatorname{Aut}(\mathfrak{A}^+)$ of all *-automorphisms of the Jordan-C*-algebra $\mathfrak{A}^+$ associated to $\mathfrak{A}$. Let $\mathcal{U} \subset \mathfrak{A}$ be the connected component of I in the set of unitary elements of $\mathfrak{A}$. From (3), it is clear that $\operatorname{Isom}(\mathfrak{A})_0$ consists of the transformations

$$A \mapsto L(A) = U\rho(A) \qquad U \in \mathcal{U},\ \ \rho \in \operatorname{Aut}(\mathfrak{A}^+)_0.$$

By (2.3), in order to prove the statement it suffices to consider the elements $\rho \in \operatorname{Aut}(\mathfrak{A}^+)_0$. Assume that ρ lies in that group and that $\|\rho - \mathrm{Id}\| \le 2/3$, where the norm is the operator norm in $\mathcal{L}(\mathfrak{A})$. Then by ([2] lemma 1 page 26) we have

$$\rho(A) = UAU^* \qquad A \in \mathfrak{A},$$

where U is a unitary operator in the weak-operator closure of $\mathfrak{A}$ in $\mathcal{L}(H)$. Hence ρ preserves the family $\mathfrak{F}$ by (2.3). Finally, any element in $\operatorname{Aut}(\mathfrak{A}^+)_0$ preserves collective compactness since $\operatorname{Aut}(\mathfrak{A}^+)_0$ is a connected Lie group and therefore it is generated by any neighbourhood of its identity element. □

At this point, it is reasonable to ask what happens in the case of an arbitrary J*-algebra.

Proposition 3.5. *Let $\mathfrak{A}$ be a J*-algebra. Then, for every derivation $\delta \in \operatorname{Der}(\mathfrak{A})$, the following conditions are equivalent*

i) δ preserves the family $\mathfrak{F}$ of all collectively compact subsets in $\mathfrak{A}$,

ii) For each $t \in \mathbb{R}$, the isometry of $\mathfrak{A}$ given by $g(t) := \exp t\delta$ preserves the family $\mathfrak{F}$.

Proof. i)⇒ii). Assume that $\delta \in \operatorname{Der}(\mathfrak{A})$ preserves the family $\mathfrak{F}$. We have to show that $\exp t\delta \in G$ preserves $\mathfrak{F}$. Obviously we may assume $t = 1$. Let $\mathcal{W} \in \mathfrak{F}$. We have to prove that $\big((\exp\delta)\mathcal{W}\big)B_H$ is totally bounded in K. Let $\varepsilon > 0$ be given. By the properties of the exponential, there is an index $N \in \mathbb{N}$ such that $\sum_{N+1}^{\infty} \frac{1}{n!}\|\delta\|^n \leq \varepsilon$. Since $\mathcal{W} \subset B_{\mathfrak{A}}$, we have $\|T\| \leq 1$ for all $T \in \mathcal{W}$ and hence

$$\|\sum_{N+1}^{\infty} \frac{1}{n!}(\delta^n T)h\| \leq \sum_{N+1}^{\infty} \frac{1}{n!}\|\delta\|^n \|T\|\|h\| \leq \sum_{N+1}^{\infty} \|\delta\|^n \leq \varepsilon$$

holds for all $T \in \mathcal{W}$ and all $h \in B_H$. To shorten the notation, set $p(\delta) := I + \frac{1}{1!}\delta + \cdots + \frac{1}{n!}\delta^n$. By assumption δ preserves $\mathfrak{F}$, hence each of the sets $\mathcal{W}, \delta\mathcal{W} \cdots, \delta^n\mathcal{W}$ is collectively compact and so is $p(\delta)\mathcal{W}$ due to (2.2) and the inclusion

$$p(\delta)\mathcal{W} \subset \mathcal{W} + \delta\mathcal{W} + \cdots + \frac{1}{n!}\delta^n\mathcal{W}.$$

Therefore there are finite sets

$$\{T_1, \cdots, T_r\} \subset \mathcal{W} \qquad \{h_1, \cdots, h_s\} \subset B_H$$

with the following property: for each $T \in \mathcal{W}$ and each $h \in B_H$ there is a pair (T_i, h_j) such that

$$\|\big(p(\delta)T\big)h - \big(p(\delta)T_i\big)h_j\| \leq \varepsilon.$$

Now

$$\|\big((\exp\delta)T\big)h - \big(p(\delta)T_i\big)h_j\| = \|\Big([p(\delta) + \sum_{N+1}^{\infty} \frac{1}{n!}\delta^n]T\Big)h - \big(p(\delta)T_i\big)h_j\| \leq$$
$$\|\big(p(\delta)T\big)h - \big(p(\delta)T_i\big)h_j\| + \|\Big(\sum_{N+1}^{\infty} \frac{1}{n!}\delta^n T\Big)h\| \leq 2\varepsilon$$

which proves the claim.

Assume that $t \in \mathbb{R} \mapsto g(t) \in F$ is a one-parameter group of surjective linear isometries of $\mathfrak{A}$ such that each $g(t)$ preserves the family $\mathfrak{F}$. Due to the inclusion $F \subset G$, we have a one-parameter group in G and therefore there exists a derivation $\delta \in \mathrm{Der}(\mathfrak{A})$ such that $g(t) = \exp t\delta$ for $t \in \mathbb{R}$. We claim that δ preserves the family $\mathfrak{F}$. In order to prove it, we have to check that whenever $\mathcal{W} \subset B_{\mathfrak{A}}$ is a collectively compact subset of operators in $B_{\mathfrak{A}}$, the set $\delta(\mathcal{W}) \subset \mathfrak{A}$ is collectively compact, that is, $\delta(\mathcal{W})B_H$ is totally bounded in K. Let $\mathcal{W}$ be as mentioned and let $\varepsilon > 0$ be given. From the definition of the exponential we have

$$\|\frac{1}{t}\big(\exp t\delta - I) - \delta\big)\| \leq \varepsilon$$

for sufficiently small values of $|t| \leq \tau$, $t \neq 0$, the norm being the operator norm in $\mathcal{L}(\mathfrak{A})$. Fix any $t = t_0$ in the above conditions, and set

$$f(t_0) := \frac{1}{t}\big(\exp t_0\delta - I) - \delta$$

to shorten the writing. Since $M := \sup_{T\in\mathcal{W}} \|T\| < \infty$, we have for all $T \in \mathcal{W}$ and all $h \in H$

$$\|\big(f(t_0)T\big)h - \delta(T)h\| \le \|f(t_0) - \delta\|\|T\|\|h\| \le M\varepsilon.$$

By assumption $g(t_0) = \exp t_0\delta$ preserves the family $\mathfrak{F}$, therefore the sets $(\exp t_0\delta)\mathcal{W} \subset \mathfrak{A}$ and $\mathcal{W}$ are collectively compact, hence by (2.2) $f(t_0)(\mathcal{W}) = \frac{1}{t_0}(\exp t_0\delta - I)(\mathcal{W}) \subset \mathfrak{A}$ is collectively compact, too. Hence $\big(f(t_0)\mathcal{W}\big)B_H \subset K$, is totally bounded. Thus there are finite sets

$$\{T_1, \cdots, T_r\} \subset \mathcal{W} \qquad \{h_1, \cdots, h_s\} \subset H$$

with the following property: For each $T \in \mathcal{W}$ and $h \in B_H$ there is a pair (T_i, h_j) such that

$$\|\big(f(t_0)T\big)h - \big(f(t_0)T_i\big)h_j\| \le \varepsilon.$$

Clearly $\{\delta(T_i)h_j : 1 \le i \le r,\ 1 \le j \le s\}$ is a finite subset of $\delta(\mathcal{W})B_H$ and by construction

$$\|\delta(T)h - \delta(T_i)h_j\| \le \|\delta(T)h - \big(f(t_0)T\big)h\| + \|\big(f(t_0)T\big)h - \big(f(t_0)T_i\big)h_j\| + \\ \|\big(f(t_0)T_i\big)h_j - \delta(T_i)h_j\| \le \varepsilon\|T\| + \varepsilon + \varepsilon\|T_i\| \le (2M+1)\varepsilon$$

for all $T \in \mathcal{W}$ and all $h \in B_H$, which shows that $\delta(\mathcal{W})B_H$ is totally bounded. □

For $U, V \in \mathfrak{A}$, the operator $U\square V \in \mathcal{L}(\mathfrak{A})$ is given by $(U\square V)X := \frac{1}{2}(UV^*X + XV^*U)$, $(X \in \mathfrak{A})$ and $U\square V - V\square U$ is a derivation of $\mathfrak{A}$. We let $\mathrm{Inder}(\mathfrak{A})$ be the closure (in the operator norm of $\mathrm{Der}(\mathfrak{A})$) of the *real* linear span of the set of derivations that have the above form

$$\mathrm{Inder}(\mathfrak{A}) := \overline{\mathrm{span}}\{U\square V - V\square U \colon U, V \in \mathfrak{A}\}$$

It is clear from (2.3) that $U\square V - V\square U$ preserves collective compactness in $\mathfrak{A}$ and so do the elements of $\mathrm{Inder}(\mathfrak{A})$ as one can see by a routine argument on total boundedness similar to that made in (3.3). More generally let $\mathrm{Inder}_{\mathfrak{F}}(\mathfrak{A})$ denote the closure of $\mathrm{Inder}(\mathfrak{A})$ in $\mathrm{Der}(\mathfrak{A})$ relative to the topology τ of uniform convergence on the sets $\mathcal{W}$ in $\mathfrak{F}$. Then

Lemma 3.6. *If $\mathfrak{A}$ is a J*-algebra then $\mathrm{Inder}_{\mathfrak{F}}(\mathfrak{A})$ is a Lie subalgebra of $\mathrm{Der}(\mathfrak{A})$ and each element in it preserves collective compactness in $\mathfrak{A}$.*

In general $\mathrm{Inder}_{\mathfrak{F}}(\mathfrak{A})$ is a proper subset of $\mathrm{Der}(\mathfrak{A})$. For a detailed study of derivations of a JB*-triple see [5].

From now on, we assume that the J*-algebra $\mathfrak{A}$ is a special Cartan factor. This will provide us with detailed information on the group $\mathrm{Isom}(\mathfrak{A})$. Indeed, for rectangular Cartan factors, which are the spaces $\mathfrak{A} := \mathcal{L}(H, K)$ with Hilbert spaces H, K, $\dim H \le \dim K$, every surjective isometry has one of the forms (see [8] Satz 4)

$$L_{U,V}(T) = VTU \qquad L_{U,V}(T) = VT'U \qquad T \in \mathfrak{A}$$

where T' stands for the transpose of T, U and V are unitary operators on H and K, respectively, and the second form does not occur unless $\dim H = \dim K$.

Let H be a Hilbert space with a conjugation $x \mapsto \overline{x}$, $(x \in H)$, and let $T \mapsto T'$, where $T'x := \overline{T^*(\overline{x})}$, be the associated transposition on $\mathcal{L}(H)$. The symmetric and anti-symmetric Cartan factors are the spaces $\mathcal{L}(H)^\epsilon := \{T \in \mathcal{L}(H) : T = \epsilon T'\}$, where $\epsilon = +1$ and $\varepsilon = -1$, respectively. The surjective linear isometries of these factors are the mappings

$$L_U(T) = UTU' \qquad T \in \mathcal{L}(H)^\epsilon,$$

where U is a unitary operator on H. Thus, for Cartan factors of type I with $\dim H \neq \dim K$ and for Cartan factors of type II or III all surjective linear isometries are spatial, that is, they are induced by isometries of the underlying Hilbert spaces H, K, and hence they preserve collective compactness. Cartan factors of type I with $\dim H = \dim K$ also have the isometry $T \mapsto T'$ which is not induced by any isometry of H and does not preserve collective compactness in $\mathcal{L}(H)$.

By ([12] prop 9), the only compact operator contained in an infinite-dimensional spin factor (Cartan factor of type IV) is the null operator. Hence, in this case the family $\mathfrak{F}$ only contains the set $\mathcal{W} = \{0\}$ and therefore any $L \in \mathrm{Isom}\mathfrak{A}$ preserves $\mathfrak{F}$.

We consider the open ball $B_\mathfrak{A}$ as a Banach manifold in the canonical atlas (consists of the identity map $\mathrm{Id}\colon B_\mathfrak{A} \to B_\mathfrak{A}$ as a local chart). In this local coordinate, each holomorphic vector field can be represented in the form $\mathcal{X} = h\frac{\partial}{\partial X}$ where $h\colon B_\mathfrak{A} \to \mathfrak{A}$ is a holomorphic map, and we shall identify $\mathcal{X}$ with the function h with no danger of confusion. The set $\mathfrak{g} := \mathrm{aut}(B_\mathfrak{A})$ of complete holomorphic vector fields on $B_\mathfrak{A}$ is a Lie algebra in the usual vector space operations and the Jacobi bracket. We have the vector space direct sum decomposition

$$\mathfrak{g} = \mathfrak{L} \oplus \mathfrak{P} \tag{4}$$

where $\mathfrak{L} = \mathrm{Der}(\mathfrak{A})$ is the Lie algebra of all triple derivations of $\mathfrak{A}$ and $\mathfrak{P}$ consists of the quadratic vector fields $\mathcal{X} = Q_A\frac{\partial}{\partial X}$ where $Q_A(X) = A - XA^*X$ for $A \in \mathfrak{A}$. Moreover

$$[\mathfrak{L}, \mathfrak{L}] \subset \mathfrak{L}, \quad [\mathfrak{L}, \mathfrak{P}] \subset \mathfrak{P}, \quad [\mathfrak{P}, \mathfrak{P}] \subset \mathfrak{L}$$

and more precisely

$$[\delta_1, \delta_2] = \delta_1\delta_2 - \delta_2\delta_1, \quad [\delta, Q_A] = Q_{\delta(A)}, \quad [Q_A, Q_B] = 2(A\Box B - B\Box A). \tag{5}$$

Finally, for $\mathcal{X} = h\frac{\partial}{\partial X} \in \mathfrak{g}$ the expression $\|\mathcal{X}\| := \sup\limits_{\|X\| \leq 1} \|h(X)\|$ defines a norm with respect to which $\mathfrak{g}$ is a Banach-Lie algebra and the decomposition (4) is topological.

We say that a vector field $X = h\frac{\partial}{\partial X}$ preserves the family $\mathfrak{F}$ if so does the function h, that is, $h(\mathfrak{F}) \subset \mathfrak{F}$. Define

$$\mathfrak{h} := \{\mathcal{X} = h\frac{\partial}{\partial X} \in \mathfrak{g} : h(\mathfrak{F}) \subset \mathfrak{F}\}$$

Proposition 3.7. *$\mathfrak{h}$ is a closed Lie subalgebra of $\mathfrak{g}$ and we have the topological vector space direct sum decomposition*

$$\mathfrak{h} = \mathfrak{L}_\mathfrak{F} \oplus \mathfrak{P}_\mathfrak{F}$$

where $\mathfrak{L}_{\mathfrak{F}}$ *consists of those derivations* $\delta \in \mathrm{Der}(\mathfrak{A})$ *that preserve* $\mathfrak{F}$ *and* $\mathfrak{P}_{\mathfrak{F}} := \{Q_A : A \in \mathcal{K}(H, H)\}$. *Moreover,* F *is a closed Banach-Lie subgroup of* G *whose Lie algebra is* $\mathfrak{h}$.

Proof. From (5) and (2.2) it is clear that $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$. To show that $\mathfrak{h}$ is closed in $\mathfrak{g}$, let $\mathcal{X}_n = \delta_n + Q_{A_n}$ be a sequence in $\mathfrak{h}$ with $\mathcal{X}_n \to \mathcal{X}$ where $\mathcal{X} = \delta + Q_A \in \mathfrak{g}$. Since the decomposition (4) is topological, the canonical projections from $\mathfrak{h}$ onto $\mathfrak{L}$ and $\mathfrak{P}$ are continuous, and hence we have $\delta_n \to \delta$ and $Q_{A_n} \to Q_A$. By assumption $A_n \in \mathcal{K}(H, K)$ are compact operators and the relation $A_n = Q_{A_n}(0) \to Q_A(0) = A$ gives that A is compact and so $Q_A \in \mathfrak{P}_{\mathfrak{F}}$. The proof that $\mathfrak{L}_{\mathfrak{F}}$ is closed is a repetition of the arguments made in (3.3). The fact that F is a Lie subgroup of G follows by standard arguments from (3.1) and (3.5). □

Corollary 3.8. *Let* $g = M_A L$ *be a holomorphic automorphism of the unit ball* $B_{\mathfrak{A}}$ *of a special Cartan factor* $\mathfrak{A}$. *If* $\mathfrak{A}$ *is of type I with* $\dim H \neq \dim K$ *or of type II or III then* g *preserves collective compactness if and only if* $A = g(0)$ *is a compact operator. If* $\mathfrak{A}$ *is of type I with* $\dim H = \dim K$ *then the same is true for the automorphisms* g *in the connected component of the identity of* G. *If* $\mathfrak{A}$ *is of type IV and* $\dim \mathfrak{A} = \infty$ *then* g *preserves collective compactness if and only if* g *is a surjective linear isometry of* $\mathfrak{A}$.

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