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Banach manifolds of algebraic elements in the algebra $\mathcal{L}(H)$ of bounded linear operators*

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Abstract: Given a complex Hilbert space H, we study the manifold $\mathcal{A}$ of algebraic elements in $Z = \mathcal{L}(H)$. We represent $\mathcal{A}$ as a disjoint union of closed connected subsets M of Z each of which is an orbit under the action of G, the group of all C*-algebra automorphisms of Z. Those orbits M consisting of hermitian algebraic elements with a fixed finite rank r, $(0 < r < \infty)$ are real-analytic direct submanifolds of Z. Using the C*-algebra structure of Z, a Banach-manifold structure and a G-invariant torsionfree affine connection ∇ are defined on M, and the geodesics are computed. If M is the orbit of a finite rank projection, then a G-invariant Riemann structure is defined with respect to which ∇ is the Levi-Civita connection.

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1 Introduction

In this paper we are concerned with certain infinite-dimensional Grassmann manifolds in $Z\colon = \mathcal{L}(H)$, the space of bounded linear operators $z\colon H \to H$ in a complex Hilbert space H. Grassmann manifolds are a classical object in Differential Geometry and in recent years several authors have considered them in the Banach space setting. Besides the Grassmann structure, a Riemann and a Kähler structure has sometimes been defined even in the infinite-dimensional setting. Let us recall some aspects of the topic that are relevant for our purpose.

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The study of the manifold of minimal projections in a finite-dimensional simple formally real Jordan algebra was made by U. Hirzebruch in [4], who proved that such a manifold is a compact symmetric Riemann space of rank 1, and that every such a space arises in this way. Later on, Nomura in [13, 14] established similar results for the manifold of fixed finite rank projections in a topologically simple real Jordan-Hilbert algebra. In [7], the authors studied the Riemann structure of the manifold of finite rank projections in Z without the use of any global scalar product. As pointed out there, the Jordan-Banach structure of Z encodes information about the differential geometry of some manifolds naturally associated to it, one of which is the manifold of algebraic elements in Z. On the other hand, the Grassmann manifold of all projections in Z has been discussed by Kaup in [11]. See also [1, 8] for related results.

It is therefore reasonable to study the manifold of algebraic elements in Z. We restrict our considerations to the set $\mathcal{A}$ of all normal algebraic elements in Z that have finite rank. Normality allows us to use spectral theory which is an essential tool. In the case $H = \mathbb{C}^n$ all elements in Z are algebraic (as any square matrix is a root of its characteristic polynomial) and have finite rank, whereas for arbitrary H the set of *all* (finite and non finite rank) algebraic elements is norm total in Z, see [5] (Lemma 3.11). Under the above restrictions $\mathcal{A}$ is represented as a disjoint union of closed connected subsets M of Z, each of which is homogeneous and invariant under the natural action of G, the group of all C*-automorphisms of Z. Actually these sets are the orbits of G in $\mathcal{A}$. The family of these orbits is quite plentiful and different orbits may have quite different properties. If an orbit M contains a hermitian element then all elements in M are hermitian and M turns out to be a closed real-analytic direct submanifold of Z. Using algebraic tools, a real-analytic Banach-manifold structure and a G-invariant affine connection ∇ are defined on M in that case, and the ∇-geodesics are computed. For $a \in M$, the restriction to M of the *Peirce reflection* $S_{\mathbf{a}}$ on Z around the projection $\mathbf{a} := \operatorname{supp}(a)$ is a real-analytic involution of M for which a is a fixed point. The set $\mathsf{Fix}_M(S_{\mathbf{a}})$ of the fixed points of such involution is a direct real-analytic submanifold of Z. If a is a finite rank projection then M is a symmetric manifold.

For an orbit M and a point $a \in M$, the following conditions on T_aM are known to be equivalent: (1) T_aM is linearly homeomorphic to a Hilbert space, (2) T_aM is a reflexive Banach space, (3) the rank of a is finite. If these conditions hold for some $a \in M$, then this occurs for all $a \in M$. If in addition a is a finite rank projection, then a G-invariant Riemann structure can be defined on M. We take a JB*-triple system approach instead of the Jordan-algebra approach of [13, 14]. As noted in [1] and [6], within this context the algebraic structure of JB*-triple acts as a substitute for the Jordan algebra structure. Since M consists of elements with a fixed finite rank r, $(0 < r < \infty)$, the JB*-triple structure provides a *local scalar product* known as the *algebraic metric* of Harris ([2], prop. 9.12). Although Z is not a Hilbert space, the use of the algebraic scalar product allows us to define a G-invariant Riemann structure on M for which ∇ is the Levi-Civita connection.

2 Algebraic preliminaries.

For a complex Banach space X denote by $X_{\mathbb{R}}$ the underlying real Banach space, and let $\mathcal{L}(X)$ and $\mathcal{L}_{\mathbb{R}}(X)$ respectively be the Banach algebra of all bounded complex-linear operators on X and the Banach algebra of all bounded real-linear operators on $X_{\mathbb{R}}$. A complex Banach space Z with a continuous mapping $(a,b,c) \mapsto \{abc\}$ from $Z \times Z \times Z$ to Z is called a *JB*-triple* if the following conditions are satisfied for all $a,b,c,d \in Z$, where the operator $a \square b \in \mathcal{L}(Z)$ is defined by $z \mapsto \{abz\}$ and $[\,,\,]$ is the commutator product:

(1) $\{abc\}$ is symmetric complex linear in a, c and conjugate linear in b.
(2) $[a\square b,\, c\square d] = \{abc\}\square d - c\square\{dab\}$.
(3) $a\square a$ is hermitian and has spectrum ≥ 0.
(4) $\|\{aaa\}\| = \|a\|^3$.

If a complex vector space Z admits a JB*-triple structure, then the norm and the triple product determine each other. For $x,y,z \in Z$ we write $L(x,y)(z) = (x\square y)(z)$ and $Q(x,y)(z) := \{xzy\}$. Note that $L(x,y) \in \mathcal{L}(Z)$ whereas $Q(x,y) \in \mathcal{L}_{\mathbb{R}}(Z)$, and that the operators $L_a = L(a,a)$ and $Q_a = Q(a,a)$ commute. A *derivation* of a JB*-triple Z is an element $\delta \in \mathcal{L}(Z)$ such that $\delta\{zzz\} = \{(\delta z)zz\} + \{z(\delta z)z\} + \{zz(\delta z)\}$ and an *automorphism* is a bijection $\phi \in \mathcal{L}(Z)$ such that $\phi\{zzz\} = \{(\phi z)(\phi z)(\phi z)\}$ for $z \in Z$. The latter occurs if and only if ϕ is a surjective linear isometry of Z. The group $\mathsf{Aut}(Z)$ of automorphisms of Z is a real Banach-Lie group whose Banach-Lie algebra is the set $\mathsf{Der}(Z)$ of all derivations of Z. The connected component of the identity in $\mathsf{Aut}(Z)$ is denoted by $\mathsf{Aut}^\circ(Z)$. Two elements $x, y \in Z$ are *orthogonal* if $x\square y = 0$ and $e \in Z$ is called a *tripotent* if $\{eee\} = e$, the set of which is denoted by $\mathsf{Tri}(Z)$. For $e \in \mathsf{Tri}(Z)$, the set of eigenvalues of $e\square e \in \mathcal{L}(Z)$ is contained in $\{0,\, \frac{1}{2},\, 1\}$ and the topological direct sum decomposition, called the *Peirce decomposition* of Z,

$$Z = Z_1(e) \oplus Z_{1/2}(e) \oplus Z_0(e) \tag{1}$$

holds. Here $Z_k(e)$ is the k- eigenspace of $e\square e$ and the *Peirce projections* are

$$P_1(e) = Q^2(e), \qquad P_{1/2}(e) = 2(e\square e - Q^2(e)), \qquad P_0(e) = \mathrm{Id} - 2e\square e + Q^2(e).$$

We will use the *Peirce rules* $\{Z_i(e)\, Z_j(e)\, Z_k(e)\} \subset Z_{i-j+k}(e)$ where $Z_l(e) = \{0\}$ for $l \neq 0, 1/2, 1$. In particular, every Peirce space is a JB*-subtriple of Z and $Z_1(e)\square Z_0(e) = \{0\} = Z_0(e)\square Z_1(e)$.

A JB*-triple Z may have no non-zero tripotents however the set of them is plentiful if Z is a dual Banach space.

Let $\mathbf{e} = (e_1, \cdots, e_n)$ be a finite sequence of non-zero mutually orthogonal tripotents $e_j \in Z$, and define for all integers $0 \le j, k \le n$ the linear subspaces

$$\begin{aligned}
Z_{jj}(\mathbf{e}) &= Z_1(e_j) & 1 \le j \le n,\\
Z_{jk}(\mathbf{e}) = Z_{kj}(\mathbf{e}) &= Z_{1/2}(e_j) \cap Z_{1/2}(e_k) & 1 \le j, k \le n, \quad j \ne k,\\
Z_{0j}(\mathbf{e}) = Z_{j0}(\mathbf{e}) &= Z_{1/2}(e_j) \cap \bigcap_{k \ne j} Z_0(e_k) & 1 \le j \le n,\\
Z_{00}(\mathbf{e}) &= \bigcap_{1 \le j \le n} Z_0(e_j).
\end{aligned} \tag{2}$$

Then the following topologically direct sum decomposition, called the *joint Peirce decomposition* relative to the family $\mathbf{e}$, holds

$$Z = \Big(\bigoplus_{0 \le k \le n} Z_{k0}(\mathbf{e})\Big) \oplus \Big(\bigoplus_{1 \le k < j \le n} Z_{kj}(\mathbf{e})\Big) \oplus \Big(\bigoplus_{1 \le k \le n} Z_{kk}(\mathbf{e})\Big). \tag{3}$$

The Peirce spaces multiply according to the rules $\{Z_{jm} Z_{mn} Z_{nk}\} \subset Z_{jk}$, and all products that cannot be brought to this form (after reflecting pairs of indices if necessary) vanish. The projectors $P_{kj}(\mathbf{e})\colon Z \to Z_{kj}(\mathbf{e})$, called *joint Peirce projectors*, are invariant under the group $\mathsf{Aut}(Z)$, that is, they satisfy

$$P_{kj}(h(\mathbf{e})) = h P_{kj}(\mathbf{e}) h^{-1}, \qquad h \in \mathsf{Aut}(Z),$$

where $h(\mathbf{e}) := (h(e_1), \cdots h(e_n))$, and the explicit formula for the $P_{kj}(\mathbf{e})$ can be found in [5] (Lemma 3.15). If W is a complex Banach space with an involution *, then its selfadjoint part $W_s := \{w \in W \colon w^* = w\}$ is a purely real Banach space. In the joint Peirce decomposition of Z relative to the orthogonal family $\mathbf{e} := (e_1, \cdots, e_n)$ every Peirce space $Z_{jk}(\mathbf{e})$, $(0 \le j \le k \le n)$, is invariant under the natural involution * of Z, hence they are complex Banach spaces with involution too.

Recall that every C*-algebra Z is a JB*-triple with respect to the triple product $2\{abc\} := (ab^*c + cb^*a)$. In that case, every projection in Z is a tripotent and more generally the tripotents are precisely the partial isometries in Z. C*-algebra derivations and C*-automorphisms are derivations and automorphisms of Z as a JB*-triple though the converse is not true. More precisely, for $Z = \mathcal{L}(H)$, the group of C*-algebra automorphisms consists of those elements in $\mathsf{Aut}(Z)$ that fix the unit of Z, i.e., $\mathsf{G} = \{g \in \mathsf{Aut}(Z) \colon g(1) = 1\}$.

We refer to [9], [11], [15] and the references therein for the background of JB*-triple theory, and to [12] for the finite dimensional case.

3 Banach manifolds of algebraic elements in $\mathcal{L}(H)$.

From now on, Z will denote the C*-algebra $\mathcal{L}(H)$. An element $a \in Z$ is said to be *algebraic* if it satisfies the equation $p(a) = 0$ for some non identically null polynomial $p \in \mathbb{C}[X]$. By elementary spectral theory $\sigma(a)$, the spectrum of a in Z, is a finite set whose elements are roots of the algebraic equation $p(\lambda) = 0$. In case a is *normal* we have

$$a = \sum_{\lambda \in \sigma(a)} \lambda\, e_\lambda$$

where λ and e_λ are, respectively, the spectral values and the corresponding spectral projections of a. If $0 \in \sigma(a)$ then e_0, the projection onto $\ker(a)$, satisfies $e_0 \neq 0$ but in the above representation the summand $0\, e_0$ is null and will be omitted. Thus for normal algebraic elements $a \in Z$ we have

$$a = \sum_{\lambda \in \sigma(a) \setminus \{0\}} \lambda\, e_\lambda \tag{4}$$

In particular, in (4) the numbers λ are non-zero pairwise distinct complex numbers and the e_λ are pairwise orthogonal non-zero projections. We say that a has *finite rank* if $\dim a(H) < \infty$, which always occurs if $\dim(H) < \infty$. Set $r_\lambda := \operatorname{rank}(e_\lambda)$. Then a has finite rank if and only if $r_\lambda < \infty$ for all $\lambda \in \sigma(a) \setminus \{0\}$ (the case $0 \in \sigma(a)$ and $\dim \ker a = \infty$ may occur and still a has finite rank).

Hence every finite rank normal algebraic element $a \in Z$ gives rise to: (i) a positive integer n which is the cardinal of $\sigma(a) \setminus \{0\}$, (ii) an ordered n-tuple $(\lambda_1, \cdots, \lambda_n)$ of numbers in $\mathbb{C} \setminus \{0\}$, which is the set of the pairwise distinct non-zero spectral values of a, (iii) an ordered n-tuple $(e_1, \cdots, e_n)$ of non-zero pairwise orthogonal projections, and (iii) an ordered n-tuple $(r_1, \cdots, r_n)$ where $r_k \in \mathbb{N} \setminus \{0\}$ is the rank of the spectral projection e_k.

The spectral resolution of a is unique except for the order of the summands in (4), therefore these three n-tuples are uniquely determined up to a permutation of the indices $(1, \cdots, n)$. The operator a can be recovered from the set of the first two ordered n-tuples, a being given by (4).

Given the n-tuples $\Lambda := (\lambda_1, \cdots, \lambda_n)$ and $R := (r_1, \cdots, r_n)$ in the above conditions, we let

$$M(n,\, \Lambda,\, R) := \{ \sum_k \lambda_k e_k : \quad e_j e_k = 0 \ \text{ for } j \neq k,\ \operatorname{rank}(e_k) = r_k,\ 1 \leq j,\, k \leq n \} \tag{5}$$

be the set of the elements (4) where the coefficients λ_k and ranks r_k are given and the e_k range over non-zero, pairwise orthogonal projections of rank r_k. For instance, for $n = 1$, $\Lambda = \{1\}$ and $R = \{r\}$ we obtain the manifold of projections with a given finite rank r, that was studied in [7].

The involution $z \mapsto z^*$ on Z is a C*-algebra antiautomorphism that fixes every projection, preserves normality, orthogonality and ranks, hence it maps the set $\mathcal{A}$ onto itself. For the n-tuple $\Lambda = (\lambda_1, \cdots, \lambda_n)$ we set $\Lambda^* := (\bar{\lambda}_1, \cdots, \bar{\lambda}_n)$. Then $z \mapsto z^*$ induces a map $M(n, \Lambda, R) \to M(n, \Lambda, R)^*$ where $M(n, \Lambda, R)^* = \{z^* \colon z \in M\} = M(n, \Lambda^*, R)$, and $\Lambda \subset \mathbb{R}^n$ if and only if $M(n, \Lambda, R)$ consists of hermitian elements.

To a normal algebraic element $a = \sum_{\lambda \in \sigma(a) \setminus \{0\}} \lambda e_\lambda$ we associate $\mathbf{a}$, called the *support* of a, and $\mathbf{e}$ where

$$\mathbf{a} = \operatorname{supp}(a) := \sum_{\lambda \in \sigma(a) \setminus \{0\}} e_\lambda = e_1 + \cdots + e_n, \qquad \mathbf{e} := \mathbf{e}(a) := (e_1, \cdots, e_n).$$

Proposition 3.1. Let $\mathcal{A}$ and $\mathcal{H}$ be the set of all normal (respectively, hermitian) algebraic elements of finite rank in Z, and let $M(n,\Lambda,R)$ be defined as in (5). Then

$$\mathcal{A} = \bigcup_{n,\,\Lambda,\,R} M(n,\,\Lambda,\,R), \qquad \mathcal{H} = \bigcup_{n,\,\Lambda=\Lambda^*,\,R} M(n,\,\Lambda,\,R) \tag{6}$$

is a disjoint union of G-invariant closed connected subsets of Z on each of which the group G acts transitively. The sets $M = M(n,\,\Lambda,\,R)$ are the orbits of G in $\mathcal{A}$ (respectively, in $\mathcal{H}$).

Proof. It suffices to prove the statements concerning $\mathcal{A}$. We have seen that $\mathcal{A} \subset \bigcup_{n,\,\Lambda,\,R} M(n,\,\Lambda,\,R)$. Conversely, let a belong to some $M(n,\,\Lambda,\,R)$ hence we have $a = \sum_k \lambda_k e_k$ for some orthogonal projections e_k. Then $\mathsf{Id} = (e_1 + \cdots + e_n) + f$ where f is the projection onto $\ker(a)$ if $0 \in \sigma(a)$ and $f = 0$ otherwise. The above properties of the e_k, f yield easily $ap(a) = 0$ or $p(a) = 0$ according to the cases, where $p \in \mathbb{C}[X]$ is the polynomial $p(z) = (z - \lambda_1). \cdots .(z - \lambda_n)$. Hence $a \in \mathcal{A}$. Clearly (6) is union of disjoint subsets.

Fix one of the sets $M := M(n,\,\Lambda,\,R)$ and take any pair $a, b \in M$. Then

$$a = \lambda_1 p_1 + \cdots + \lambda_n p_n, \qquad b = \lambda_1 q_1 + \cdots + \lambda_n q_n.$$

In case $0 \in \sigma(a)$, set $p_0 := \mathsf{Id} - \sum_k p_k$ and $q_0 := \mathsf{Id} - \sum_k q_k$. Since rank p_k=rank $q_k < \infty$, the projections p_k and q_k are unitarily equivalent and so are p_0 and q_0. Let us choose orthonormal basis $\mathcal{B}_k^p$ and $\mathcal{B}_k^q$ in the ranges $p_k(H)$ and $q_k(H)$ for $k = 0,\,1,\cdots,n$. Then $\bigcup_k \mathcal{B}_k^p$ and $\bigcup_k \mathcal{B}_k^q$ are two orthonormal basis in H. The unitary operator $U \in Z$ that exchanges these basis satisfies $Ua = b$. In particular, M is the orbit of any of its points under the action of the unitary group of H. Since this group is connected and its action on Z is continuous, M is connected.

By the orthogonality properties of the e_k, the successive powers of a have the expression

$$a^l = \lambda_1^l e_1 + \cdots + \lambda_n^l e_n, \qquad 1 \le l \le n,$$

where the determinant $\det(\lambda_k^l) \neq 0$ does not vanish since it is a Vandermonde determinant and the λ_k are pairwise distinct. Thus the e_k are polynomials in a whose coefficients are rational functions of the λ_k.

Now we show that M is a closed subset of Z. Let $w \in \overline{M}$ and let $(z_\mu)_{\mu\in\mathbb{N}}$ be a sequence in M such that $\lim_{\mu\to\infty} z_\mu = w$. We have to show that $w \in M$. Each point z_μ has a spectral resolution of the form

$$z_\mu = \lambda_1 e_{1\mu} + \cdots + \lambda_n e_{n\mu}, \qquad \mu \in \mathbb{N}, \tag{7}$$

where the spectral values $\Lambda = (\lambda_1, \cdots, \lambda_n)$ are fixed. By the above, each projection $e_{k\mu}$, $(1 \le k \le n)$, is a polynomial in z_μ, say

$$e_{k\mu} = f_{1k}(\Lambda) z_\mu + f_{2k}(\Lambda) z_\mu^2 + \cdots + f_{nk}(\Lambda) z_\mu^n, \qquad 1 \le k \le n, \qquad \mu \in \mathbb{N}, \tag{8}$$

where the coefficients $f_{kj}(\Lambda)$ are rational functions of the spectral values $\Lambda = (\lambda_1, \cdots, \lambda_n)$ and do not depend on the index $\mu \in \mathbb{N}$. Since $\lim_{\mu\to\infty} z_\mu = w$ and the power operation in Z is continuous, the expression (8) yields the existence of the limit

$$e_k := \lim_{\mu\to\infty} e_{k\mu} = f_{1k}(\Lambda)w + f_{2k}(\Lambda)w^2 + \cdots + f_{nk}(\Lambda)w^n, \qquad 1 \le k \le n.$$

In particular, each of the sequences $(e_{k\mu})_{\mu\in\mathbb{N}}$, $(1 \le k \le n)$, is a Cauchy sequence in Z and more precisely in the subset of Z that consists of the projections that have a fixed given finite rank r_k. Since the latter set is closed, we have $\operatorname{rank}(e_k) = r_k$. Taking the limit for $\mu \to \infty$ in (7) we get $w = \lambda_1 e_1 + \cdots + \lambda_n e_n$ which shows $w \in M$. This completes the proof. □

To establish our main result [Theorem (3.4) below] we need some notation and technical results. To a normal algebraic element $a \in Z$ with spectral resolution $a = \sum_k \lambda_k e_k$, we associate the Peirce space

$$\Xi(\mathbf{e}) := Z_{1/2}(e_1) + \cdots + Z_{1/2}(e_n) \subset Z. \tag{9}$$

Remark that $\Xi(\mathbf{e})$ is linearly homeomorphic to a closed subspace of the product $Z_{1/2}(e_1)\times \cdots \times Z_{1/2}(e_n)$. Indeed, the spaces $Z_{1/2}(e_k)$, $(1 \le k \le n)$, are not direct summands in $\Xi(\mathbf{e})$, however by ([12], th. 3.14 (3)) and ([5], lemma 3.15), $\Xi(\mathbf{e})$ is a topologically complemented subspace of Z and we have

$$\Xi(\mathbf{e}) = \Big(\bigoplus_{1\le r<s\le n} Z_{rs}\Big) \oplus \Big(\bigoplus_{1\le k\le n} Z_{k0}\Big). \tag{10}$$

Hence each $u \in \Xi(\mathbf{e})$ determines in a unique way the projections $u_{r,s}$ and $u_{k,0}$ of u onto the subspaces $Z_{r,s}(\mathbf{e})$ and $Z_{k,0}(\mathbf{e})$, which in turn give in a unique way vectors $u_k := u_{k,0} + \sum_{r\ne k} u_{r,k}$ satisfying $u_k \in Z_{1/2}(e_k)$ and $u = \sum_{1\le k\le n} u_k$. The map $\phi\colon u \mapsto (u_1, \cdots, u_n)$, $(u \in \Xi(\mathbf{e}))$, where the u_k have been just defined, is injective since (10) is a direct sum, hence it is an isomorphism onto the image $\phi\big(\Xi(\mathbf{e})\big) \subset \prod_{k=1}^n Z_{1/2}(e_k)$. When this product space is endowed with the norm of the supremum, ϕ is continuous by the continuity of the Peirce projectors and the inverse $\phi^{-1}\colon (u_1, \cdots, u_n) \mapsto u = \sum_k u_k$ is also continuous. In particular $\phi\big(\Xi(\mathbf{e})\big)$ is closed in $\prod_{k=1}^n Z_{1/2}(e_k)$ and we shall always identify $\Xi(\mathbf{e})$ with its image $\phi\big(\Xi(\mathbf{e})\big) \subset \prod_{k=1}^n Z_{1/2}(e_k)$.

We define JB*-triple inner derivation valued map $\Phi_{\mathrm{a}}\colon \Xi(\mathbf{e}) \to \mathsf{Der}(Z)$ by

$$\Phi_{\mathrm{a}}(u) := \sum_{1\le k\le n} (e_k \Box u_k - u_k \Box e_k) \qquad u = (u_1, \cdots, u_n) \in \Xi(\mathbf{e}). \tag{11}$$

Remark that all Peirce spaces $Z_{k,j}(\mathbf{e})$ as well as $\Xi(\mathbf{e})$ are invariant under the canonical adjoint operation of $Z = \mathcal{L}(Z)$. By ([5], lemma 3.15) for $1 \le k \ne j \le n$ the Peirce projector onto the space $Z_{kj}(\mathbf{e}) = Z_{1/2}(e_k) \cap Z_{1/2}(e_j)$ is the operator $P_{kj}(\mathbf{e}) = 4Q(e_k, e_j)^2$. Therefore the map $Z_{kj}(\mathbf{e}) \to Z_{kj}(\mathbf{e})$ defined by

$$w \mapsto w^\# := 2Q(e_k, e_j)w \qquad (w \in Z_{kj}(\mathbf{e}))$$

is a conjugate-linear involution on $Z_{kj}(\mathbf{e})$ which induces a decomposition of this space into the direct sum of the $\pm$-eigensubspaces of $Q(e_k, e_j)$. Finally by ([1] th. 3.1), for $u = (u_1, \cdots, u_n)$ in the selfadjoint part $\Xi(\mathbf{e})_s$ of $\Xi(\mathbf{e})$ the triple derivation $\Phi_{\mathrm{a}}(u)$ is actually a C*-algebra derivation and we define the space

$$\mathsf{Der}_{\mathrm{a}}(Z) := \{\Phi_{\mathrm{a}}(u) \,:\, u = u^* \in \Xi(\mathbf{e})_s\}.$$

Lemma 3.2. Let $a = \sum_k \lambda_k e_k$ be the spectral resolution of a normal algebraic element in Z. Let $u = \sum_k u_k$ where $u_k \in Z_{1/2}(e_k)$ and $u_k = u_k^*$ are selfadjoint elements for $k = 1, \cdots, n$. Let $u_k = u_{k0} + \sum_{r \neq k} u_{rk}$ be the joint Peirce decomposition of u_k relative to $\mathbf{e}$. Then

$$[\Phi_a(u)]a = \frac{1}{2} \sum_{\substack{1 \leq j,\, k \leq n \\ j \neq k}} (\lambda_j - \lambda_k) u_{kj} - \frac{1}{2} \sum_{1 \leq k \leq n} \lambda_k u_{k0}. \tag{12}$$

Proof. First we check that

$$\{e_k u_k e_j\} = Q(e_k, e_j)(u_k) = \frac{1}{2} u_{kj} \quad \text{for } k \neq j \quad \text{and } \{e_k u_k e_k\} = 0 \quad (1 \leq j, k \leq n). \tag{13}$$

Clearly $\{e_k u_k e_k\} = Q(e_k) u_k \in Q(e_k) Z_{1/2}(e_k) = 0$ by the Peirce rules. For $k \neq j$ we have $u_{k,0} \in Z_{1/2}(e_k) \cap \bigcap_{r \neq k} Z_0(e_r) \subset Z_0(e_j)$ hence $\{e_k u_{k0} e_j\} = 0$. By ([5], lemma 3.15) for $1 \leq k \neq j \leq n$ we have $P_{kj}(\mathbf{e}) = 4Q(e_k, e_j)^2$. Since the u_k are $*$-selfadjoint (hence also $\#$-selfadjoint), we have by the Peirce rules

$$\{e_k \sum_{\substack{1 \leq r \leq n \\ r \neq k}} u_{kr}\, e_j\} = \sum_{\substack{1 \leq r \leq n \\ r \neq k}} \{e_k\, u_{kr}\, e_j\} = \{e_k\, u_{kj}\, e_j\} = Q(e_k, e_j) u_{kj} = \frac{1}{2} u_{kj}^{\#} = \frac{1}{2} u_{kj}.$$

As a consequence

$$\Big(\sum_{1 \leq k \leq n} e_k \Box u_k \Big) a = \sum_{1 \leq k \leq n} \sum_{1 \leq j \leq n} \lambda_j \{e_k\, u_{kj}\, e_j\} = \frac{1}{2} \sum_{\substack{1 \leq k, j \leq n \\ j \neq k}} \lambda_j u_{kj}. \tag{14}$$

Next we use $u_k \in Z_{1/2}(e_k)$ to compute

$$\Big(\sum_{1 \leq k \leq n} u_k \Box e_k \Big) a = \sum_{1 \leq k \leq n} \sum_{1 \leq j \leq n} \lambda_j \{u_k\, e_k\, e_j\} = \sum_{1 \leq j \leq n} \{u_j,\, e_j,\, e_j\} = \sum_{1 \leq j \leq n} \lambda_j u_{j0} + \frac{1}{2} \sum_{\substack{1 \leq j, k \leq n \\ j \neq k}} \lambda_j u_{jk}. \tag{15}$$

Collecting the results in (14) and (15) and using $u_{kj} = u_{jk}$ one gets (12). □

Corollary 3.3. Assume that in lemma (3.2) the algebraic element a is hermitian. Then the map $\Phi_{\mathrm{a}} \colon u \mapsto [\Phi_a(u)](\cdot)$ is a real-linear isomorphism of the Banach space $\Xi(\mathbf{e})_s$ onto $\mathsf{Der}_{\mathrm{a}}(Z)$.

Proof. If a is hermitian then the λ_k are real numbers, hence $[\Phi_a(u)](\cdot) \in \mathsf{Der}_{\mathrm{a}}(Z)$. Clearly $u \mapsto \Phi_{\mathrm{a}}(u)$ is a real-linear map. By (12) the relation $\Phi_{\mathrm{a}}(u) = 0$ implies $u_{k,j} = 0 = u_{k,0}$

since $\lambda_k \neq \lambda_j$ and $\lambda_j \neq 0$, therefore $u = 0$. Moreover Φ_a is surjective. Indeed, let $\delta \in \mathsf{Der}_a(Z)$ be arbitrarily given. Then $\delta = \sum_k (v_k \Box e_k - e_k \Box v_k)$ for some $v = (v_1, \cdots, v_n)$ in $\Xi(\mathbf{e})_s$, and by (12) we can recover v from the value $\delta(a)$ that the derivation δ takes at the point $a \in Z$. If we let $\pi_{k,j}(\mathbf{e})\colon Z \to Z_{k,j}(\mathbf{e})$ denote the Peirce joint projection relative to the family $\mathbf{e}$, then (12) reads

$$v_{k,j} = \frac{2}{\lambda_j - \lambda_k} \pi_{k,j}(\mathbf{e})\big(\delta(a)\big), \qquad v_{k,0} = \frac{-2}{\lambda_k} \pi_{k,0}(\mathbf{e})\big(\delta(a)\big).$$

Since the evaluation at a and the Peirce projections are continuous, so is Φ_a^{-1}. □

Recall that a subset $M \subset Z$ is called a *real analytic* submanifold if to every $a \in M$ there are open subsets $P, Q \subset Z$ and a closed real-linear subspace $X \subset Z$ with $a \in P$ and $\phi(P \cap M) = Q \cap X$ for some bianalytic map $\phi\colon P \to Q$. If to every $a \in M$ the linear subspace $X = T_a M$, called the *tangent space* to M at a, can be chosen to be topologically complemented in Z then M is called a *direct submanifold* of Z.

Theorem 3.4. The selfadjoint orbits $M = M(n, \Lambda, R)$ defined in (5) are closed real analytic direct submanifolds of Z, the tangent space at the point $a \in M$ is the selfadjoint part of space $\Xi(\mathbf{e})$ defined in (9) and a local chart at a is given by

$$u = \sum_k u_k \mapsto [\exp \Phi_a(u)](a), \qquad u \in \Xi(\mathbf{e})_s, \tag{16}$$

with $\Phi_a(u) = \sum_k (e_k \Box u_k - u_k \Box e_k)$.

Proof. Fix one of the sets $M = M(n, \Lambda, R)$ with $M = M^*$ and a point $a \in M$ with spectral resolution $a = \sum_k \lambda_k e_k$. We know by (3.1) that M is closed in Z. By the orthogonality properties of the e_k, the successive powers of a have the expression

$$a^l = \lambda_1^l e_1 + \cdots + \lambda_n^l e_n, \qquad 1 \leq l \leq n,$$

where the determinant $\det(\lambda_k^l) \neq 0$ does not vanish since it is a Vandermonde determinant and the λ_k are pairwise distinct. Thus the e_k are polynomials in a whose coefficients are rational functions of the λ_k.

Next we show that the tangent space $T_a M$ to M at a can be identified with a real vector subspace of $\Xi(\mathbf{e})_s$. Consider a smooth curve $t \mapsto a(t)$, $t \in I$, through $a \in M$ where I is a neighbourhood of $0 \in \mathbb{R}$ and $a(0) = a$. Each $a(t)$ has a spectral resolution

$$a(t) = \lambda_1 e_1(t) + \cdots + \lambda_n e_n(t),$$

therefore the maps $t \mapsto e_k(t)$, $(1 \leq k \leq n)$, are smooth curves in the manifolds M_k of the projections in Z that have fixed finite rank $r_k = \operatorname{rank}(e_k)$, whose tangent spaces at $e_k = e_k(0)$ are the real spaces $Z_{1/2}(e_k)_s$ (see [1] or [7]). Therefore

$$u_k := \frac{d}{dt}|_{t=0} e_k(t) \in Z_{1/2}(e_k)_s, \qquad 1 \leq k \leq n.$$

By assumption a is hermitian, hence $\sigma(a) \subset \mathbb{R}$ and the tangent vector to $t \mapsto a(t)$ at $t = 0$ then satisfies $u = \frac{d}{dt}_{|t=0} a(t) = \sum_k \lambda_k u_k \in \Xi(\mathbf{e})_s$, thus $T_a M$ can be identified with a vector subspace of $\Xi(\mathbf{e})_s$. In fact $T_a N$ coincides with that space as it easily follows from the following result that should be compared with ([1] th. 3.3)

Indeed, as shown above we have $Z = \Xi(\mathbf{e})_s \oplus Y$ for a certain direct subspace Y. The mapping $\Xi(\mathbf{e})_s \oplus Y \to Z$ defined by $(x, y) \mapsto F(x, y) := (\exp \Phi_a(x))y$ is a real-analytic and its Fréchet derivative at $(0, a)$ is

$$\frac{\partial F}{\partial x}|_{(0,a)}(u, v) = [\Phi_a(u)]a,$$
$$\frac{\partial F}{\partial y}|_{(0,a)}(u, v) = \big(\exp \Phi_a(0)\big)v = v,$$

which is invertible according to (3.3). By the implicit function theorem there are open sets U, V with $0 \in U \subset X$ and $a \in V \subset Y$ such that $W := F(U \times V)$ is open in Z and $F\colon U \times V \to W$ is bianalytic and the image $F(U)$ is a direct real analytic submanifold of Z. So it remains to show that $F(U) = W \cap M$.

The operator $\Phi_a(u) = \sum_k (u_k \Box e_k - e_k \Box u_k)$, $u \in \Xi_s(a)$, is an inner C*-algebra derivation of Z, hence $h := \exp \Phi_a(z)$ is a C*-algebra automorphism of Z. Actually h lies in $\mathsf{Aut}^\circ(Z)$, the identity connected component. In particular h preserves the algebraic character and the spectral decomposition, hence it preserves M and so

$$F(U) = \{(\exp \Phi_a(u))a : z \in U\} \subset M.$$

To complete the proof, let $x \in \Xi_s(\mathbf{e})$ be given. By (3.3) the operator $\Phi_a(\cdot)$ is a surjective real linear homeomorphism of $\Xi_s(\mathbf{e})$ hence $u := \Phi_a^{-1}(x) \in \Xi_s(\mathbf{e})$, and by the above paragraph $t \mapsto (\exp \Phi_a(tw))a$, $|t| < \delta$ for some $\delta > 0$, is a curve in M whose tangent vector at a is $\Phi_a(u) = x$. Thus $\Xi_s(\mathbf{e}) \subset T_a M$. □

The proof of (3.4) has the following corollaries

Corollary 3.5. The action of the Banach Lie group $\mathsf{G} = \mathsf{Aut}(Z)$ on M admits local real-analytic cross sections, more precisely: To every $a \in M$, there is an open neighbourhood N_a of a in M and a real-analytic function $\chi : N_a \to \mathsf{G}$ such that $[\chi(b)](a) = b$ for all $b \in N_a$.

Proof. According to the proof of theorem (3.4), for each element b in a neighbourhood N_a of a there is a unique $u \in \Xi(\mathbf{e})$, say $u = u(b)$, such that $[\exp \Phi_a\big(u(b)\big)](a) = b$. Set $\chi(b) := \exp \Phi_a\big(u(b)\big) \in \mathsf{G}$. Then $b \mapsto \chi(b)$ satisfies the requirements. □

Corollary 3.6. If $\dim Z < \infty$ then the selfadjoint sets $M = M(n, \Lambda, R)$ are compact real analytic direct submanifolds of Z.

Let M be a real analytic manifold and TM its corresponding tangent bundle. Recall that a *norm on* TM is a lower semicontinuous function $\alpha\colon TM \to \mathbb{R}$ such that the restriction of α to to every tangent space T_xM, $x \in M$, is a norm on T_xM with the following property: there is a neighbourhood N of x in M which can be realized as a domain in a real Banach space E such that

$$c\|a\| \leq \alpha(u, a) \leq C\|a\|$$

for all $(u, a) \in TN \approx N \times E$ and suitable constants $0 < c \leq C$. The manifold M together with a fixed norm α on TM is called a *real Banach manifold*. If $(\widetilde{M}, \tilde{\alpha})$ is another real Banach manifold, then we say that a real analytic mapping $\phi\colon M \to \widetilde{M}$ is a *contraction* if $\tilde{\alpha} \circ T_\phi \leq \alpha$ and we say that ϕ is an *isometry* if $\tilde{\alpha} \circ T_\phi = \alpha$.

Let M be a connected real analytic Banach manifold with a norm α and denote by L the group of all real analytic surjective isometries of $g\colon M \to M$. An element s in L is called an *involution* of M if $s^2 = \mathsf{Id}_M$ and an involution s is called a *symmetry* at the point $x \in M$ if x is an isolated fixed point of s. Such an involution is unique if it exists. A connected real analytic Banach manifold M is said to be *symmetric* if there exists a symmetry at every point $x \in M$. A mapping $h\colon M \to \widetilde{M}$ is said to be a *morphism* of the symmetric manifolds M and $\widetilde{M}$ if h is real analytic and $h \circ s_x = s_{h(x)} \circ h$ holds for all $x \in M$.

Theorem 3.7. Let $\mathcal{H} = \bigcup_{n,\Lambda,R} M(n, \Lambda, R)$ be the set of all hermitian algebraic elements of finite rank in $Z = \mathcal{L}(H)$. Then each component $M = M(n, \Lambda, R)$ is a closed real-analytic direct Banach submanifold of Z. For each $a \in M$, the Peirce reflection $S_{\mathbf{a}}$ in Z around the support $\mathbf{a} = \mathrm{supp}(a)$ of a is real-analytic involution M for which a is a fixed point. The set $\mathsf{Fix}_M(S_{\mathbf{a}})$ of fixed points of $S_{\mathbf{a}}$ in M is real-analytic direct submanifold of Z. If M is the orbit of a finite rank projection them M is a symmetric manifold.

Proof. Fix any orbit $M(n, \Lambda, R)$ and any point $a \in M$. Set $\mathbf{e} = (e_1, \cdots, e_n)$ where $a = \sum_k \lambda_k e_k$ is the spectral resolution of a. Let N and $E := T_aM \approx \Xi(\mathbf{e})_s$ denote the neighbourhood of a in M and the Banach space for which the tangent bundle satisfies $TN \approx N \times E$. Define a function $\alpha\colon N \times E \to \mathbb{R}$ by

$$\alpha(b, u) := \|u\|, \qquad b \in N \qquad u \in \Xi(\mathbf{e})_s,$$

where $\|\cdot\|$ is the operator norm on Z. Since M is an orbit under the group G, we can extend α in a unique way to a G-invariant norm on M in a natural way. Thus (M, α) is a Banach manifold for which G (and in fact $\mathsf{Aut}(Z)$) acts as a group of isometries.

For a tripotent $e \in \mathsf{Tri}(Z)$, the *Peirce reflection* around e is the linear map $S_e\colon = \mathsf{Id} - P_{1/2}(e)$ or in detail $z = z_1 + z_{1/2} + z_0 \mapsto S_e(z) = z_1 - z_{1/2} + z_0$ where z_k are the Peirce e-projections of z, $(k = 1, 1/2, 0)$. Recall that S_e is an involutory triple automorphism of Z with $S_e(e) = e$, and clearly the set $\mathsf{Fix}_Z(S_e)$ of the fixed points of S_e in Z is $\mathsf{Fix}_Z(S_e) = \{z \in Z\ :\ P_{1/2}(e)z = 0\}$. If e is a projection in $Z = \mathcal{L}(H)$ (taken as a tripotent) then S_e is a C*-algebra automorphism of Z, hence S_e preserves the set of projections, the orthogonality relations and ranks as well as the hermitian character of the elements in Z. In particular, S_e transforms each orbit M onto another orbit $\widetilde{M} := S_eM$ of the set $\mathcal{A}$ of algebraic elements. Given $a \in M$, the preceding considerations apply to the projection $\mathbf{a} = \operatorname{supp}(a)$. By the Peirce rules we have $Q(\mathbf{a})a = \{\sum_j e_j \sum_k \lambda_k e_k \sum_l e_l\} = \sum_k \lambda_k e_e = a$, hence $P_1(\mathbf{a})a = a$ and $S_\mathbf{a}(a) = a$, therefore $S_\mathbf{a} M = M$ and $S_\mathbf{a}|M$ is a real-analytic involution of M for which a is a fixed point. For $n = 1$ it is known that M is symmetric ([1], [14] prop. 4.3). Thus we analyze the the set $\mathsf{Fix}_M(S_\mathbf{a})$ of the fixed points of $S_\mathbf{a}$ in M for $n > 1$. By the previous discussion

$$\mathsf{Fix}_M(S_\mathbf{a}) = M \cap \mathsf{Fix}_Z(S_\mathbf{a}) = M \cap \{z \in Z \,:\, P_{1/2}(\mathbf{a})z = 0\} = M \cap \mathsf{ker} P_{1/2}(\mathbf{a}), \tag{17}$$

which is a real analytic submanifold of M. The points of M in a neighbourhood U of a in M have the form $z = [\exp \Phi_\mathrm{a}(u)]a$. Hence any smooth curve $t \mapsto z(t)$ in $\mathsf{Fix}_U(S_\mathbf{a})$ passing through a with tangent vector $u \in \Xi(\mathbf{e})_s$ has the form $z(t) = [\exp \Phi_\mathrm{a}(tu)]a$ and will therefore satisfy $P_{1/2}(\mathbf{a})[\exp \Phi_\mathrm{a}(tu)]a = 0$ for all t in some interval around $t = 0$. By taking the derivative at $t = 0$ we get

$$P_{1/2}(\mathbf{a})[\Phi_\mathrm{a}(u)]a = 0,$$

the tangent space to $\mathsf{Fix}_M(S_\mathbf{a})$ at a being the set of solutions $u \in \Xi(\mathbf{e})$ of the above equation. By (10) it suffices to find the solutions in the subspaces $Z_{k,j}(\mathbf{e})$ and $Z_{k0}(\mathbf{e})$. Using the Peirce rules together with (12) and the expression $P_{1/2}(\mathbf{a}) = 2\big(\mathbf{a}\Box\mathbf{a} - Q^2(\mathbf{a})\big)$ it is a routine exercise to show that

$$\{u \in \Xi(\mathbf{e})_s \,:\, P_{1/2}(\mathbf{a})[\Phi_\mathrm{a}(u)]a = 0\} = \bigoplus_{1 \le k \le n} Z_{k0}(\mathbf{e})_s.$$

Now for $n \ge 2$ (and $\dim H \le 3$) it is immediate to see that we have $Z_{k0}(\mathbf{e}) \neq \{0\}$ for some $1 \le k \le n$, hence[‡] $\mathsf{Fix}_M(S_\mathbf{a})$ does not reduce to an isolated point and $S(\mathbf{a})$ is not a symmetry of M. Note that if M is symmetric then the symmetry of M around a must be $S(\mathbf{a})$. □

4 The Jordan connection on $M(n, \Lambda, R)$

By (3.4) the tangent space TM_a to M at the point a can be identified with the real space $\Xi(\mathbf{e})_s$, a direct summand in Z, the projector onto which is denoted by $P_\Xi(\mathbf{e})$. As any Peirce projector, $P_\Xi(\mathbf{e})$ is $\mathsf{Aut}(Z)$-invariant, that is,

$$P_\Xi(h(\mathbf{e})) = h\, P_\Xi(\mathbf{e})\, h^{-1}, \qquad h \in \mathsf{Aut}(Z). \tag{18}$$

Recall that a smooth vector field X on M is a smooth function $X\colon M \to TM$ such that $\pi \circ X = \mathsf{Id}_M$, where $\pi\colon TM \to M$ is the canonical projection. Thus $X(x)$, the value of X at $x \in M$, is a pair $X(x) = (x, X_x)$ where $X_x \in T_x M$. For all points x in a neighborhood of a, the tangent spaces $T_x M$ are unambiguously identified with the Banach space $E \approx \Xi(\mathbf{e})_s \hookrightarrow Z$, hence smooth vector fields on M will be locally identified with smooth Z-valued functions $X\colon M \to Z$ such that $X(x) \in \Xi(\mathbf{e})_s$ for all $x \in M$.

[‡] When $n = 1$ the all summands $Z_{k0}(\mathbf{e})$ reduce to 0.

We let $\mathfrak{D}(M)$ be the Lie algebra of smooth vector fields on M. For $Y \in \mathfrak{D}(M)$, we let Y'_a be the Fréchet derivative of Y at a. Thus Y'_a is a bounded linear operator $Z \to Z$, hence $Y'_a X_a \in Z$ and it makes sense to take the projection $P_\Xi(\mathbf{e}) Y'_a X_a \in \Xi(\mathbf{e})_s \approx T_a M$.

Definition 4.1. We define a connection ∇ on M by

$$(\nabla_X Y)_a := P_\Xi(\mathbf{e})\, Y'_a X_a, \qquad X, Y \in \mathfrak{D}(M), \qquad a \in M.$$

Note that if a is a projection, then ∇ coincides with the affine connection defined in ([1] def 3.6) and [7]. It is a matter of routine to check that ∇ is an affine connection on M, that it is G- invariant and torsion-free, i. e.,

$$g\,(\nabla_X Y) = \nabla_{g(X)}\, g\,(Y), \qquad g \in \mathsf{G},$$

where $(g\,X)_a := g'_a\,(X_{g_a^{-1}})$ for all $X \in \mathfrak{D}(M)$, and

$$T(X, Y) := \nabla_X Y - \nabla_Y X - [XY] = 0, \qquad X, Y \in \mathfrak{D}(M).$$

Since ∇ has been defined in terms of the Jordan structure of Z we refer to it as the *Jordan connection* on M.

Theorem 4.2. Let the manifold M be defined as in (5). Then the ∇-geodesics of M through the point $a \in M$ are the curves $\gamma(t) := [\exp t\,\Phi_a(u)]a$, $(t \in \mathbb{R})$, where $a \in M$ and $u \in \Xi(\mathbf{e})_s$.

Proof. Recall that the geodesics of ∇ are the curves $t \mapsto \gamma(t) = \sum_k \lambda_k e_k(t) \in M$ that satisfy the second order ordinary differential equation

$$\big(\nabla_{\dot\gamma(t)}\,\dot\gamma(t)\big)_{\gamma(t)} = 0.$$

Let $u \in \Xi(\mathbf{e})_s$. Then $\Phi_a(u)$ is an inner C*-algebra derivation of Z and $h(t) := \exp t\,\Phi_a(u)$ is an inner C*-automorphism of Z. Thus $h(t)a \in M$ and $t \mapsto \gamma(t)$ is a curve in the manifold M. Clearly $\gamma(0) = a$ and taking the derivative with respect to t at $t = 0$ we get by the Peirce rules

$$\dot\gamma(t) = \Phi_a(u)\gamma(t) = h(t)[\Phi_a(u)]a, \qquad \dot\gamma(0) = [\Phi_a(u)]a \in \Xi(\mathbf{e})_s,$$
$$\ddot\gamma(t) = [\Phi_a(u)^2]\gamma(t) = h(t)[\Phi_a(u)^2]a, \qquad \ddot\gamma(0) = \Phi_a(u)\dot\gamma(0) \in [\Phi_a(u)]\Xi(\mathbf{e})_s.$$

In particular $P_\Xi(\mathbf{e})[\Phi_a(u)^2]a = 0$. The definition of ∇ and the relation (18) give

$$\big(\nabla_{\dot\gamma(t)}\,\dot\gamma(t)\big)_{\gamma(t)} = P_\Xi(\boldsymbol{\gamma}(t))\,\Big(\dot\gamma(t)'_{\gamma(t)}\,\dot\gamma(t)\Big) = P_\Xi(\boldsymbol{\gamma}(t))\,\ddot\gamma(t) =$$
$$P_\Xi\big(\mathbf{h}(t)a\big)\,h(t)[\Phi_a(u)]a = h(t)P_\Xi\big(\mathbf{e}\big)\,[\Phi_a(u)^2]a = 0$$

for all $t \in \mathbb{R}$. Recall that by (3.3) the mapping $u \mapsto [\Phi_a(u)]a$ is a linear homeomorphism of $\Xi(\mathbf{e})_s$. Since geodesics are uniquely determined by the initial point $\gamma(0) = a$ and the initial velocity $\dot{\gamma}(0) = [\Phi_a(u)]a$, the above shows that family of curves in (4.2) with $a \in M$ and $u \in T_a M \approx \Xi(\mathbf{e})_s$ are all geodesics of the connection ∇. □

Proposition 4.3. Let $\mathcal{H} = \bigcup_{n,\Lambda,R} M(n, \Lambda, R)$ be the set of all hermitian algebraic elements of finite rank in $Z = \mathcal{L}(H)$. Then each component M for which $n = 1$ admits a G-invariant Riemann structure for which ∇ is the Levi-Civita connection.

Proof. First we assume that $\Xi(\mathbf{e})$ is closed under the operation of taking triple product. Suppose that $\operatorname{rank}(a) = r < \infty$ for $a \in M$. Then $\operatorname{rank}(e_k) \leq r < \infty$, $(1 \leq k \leq n)$, hence the JB*-subtriple $Z_{1/2}(e_k)$ has finite rank and so $Z_{1/2}(e_k)$ is a reflexive Banach space (see [10] or [2] prop. 9.11). The closed subspace $Z_{k0}(\mathbf{e}) = Z_{1/2}(e_k) \cap \bigcap_{j \neq k} Z_0(e_j)$ is also reflexive and so is the *finite* ℓ_∞-direct sum $\Xi(\mathbf{e}) = \bigoplus_{1 \leq k \leq n} Z_{k0}(\mathbf{e})$. But $\Xi(\mathbf{e})$ is a JB*-triple by assumption and being reflexive is linearly homeomorphic to a Hilbert space. Thus the tangent space $T_a M \approx \Xi(\mathbf{e})_s$ is linearly homeomorphic to a real Hilbert space under a suitable scalar product. We may take for instance the *algebraic inner product* on $\Xi(\mathbf{e})_s$ (denoted by $\langle \cdot, \cdot \rangle$) ([2] page 161) and we can define a Riemann metric on M by

$$g_a(X, Y) := \langle X_{\mathbf{a}}, Y_{\mathbf{a}} \rangle, \qquad X, Y \in \mathfrak{D}(M), \qquad a \in M. \tag{19}$$

Remark that g has been defined in algebraic terms, hence it is G-invariant. Moreover, ∇ is compatible with the Riemann structure, i. e.

$$X\, g(Y, W) = g(\nabla_X Y, W) + g(Y, \nabla_X W), \qquad X, Y, W \in \mathfrak{D}(M).$$

Therefore, ∇ is the only Levi-Civita connection on M and each symmetry of M (as induced by a Peirce reflection) is an isometry.

Remark that for $n = 1$ the Peirce joint decomposition of Z relative to $\mathbf{e} = e$ reduces to (1) and so $\Xi(\mathbf{e}) = Z_{1/2}(e)$ is a subtriple of Z. Actually this is the only case in which $\Xi(\mathbf{e})$ is closed under triple product. □

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