Linear Algebra and its Applications 432 (2010) 1267–1276

Contents lists available at ScienceDirect

Linear Algebra and its Applications

journal homepage: www.elsevier.com/locate/laa

Generalized bicircular projections on JB*-triples☆

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ARTICLE INFO

Article history:
Received 20 August 2009
Accepted 25 October 2009
Available online 25 November 2009

Submitted by C.K. Li

AMS classification:
Primary: 47A65
Secondary: 17C65, 46L99, 47A62

Keywords:
JB*-triple
C*-algebra
Generalized bicircular projection
Isometry
Bicontractive projection
Hermitian projection

ABSTRACT

Let A be a JB*-triple and let $P : A \to A$ be a linear projection. It is proved that $P + \lambda(Id - P)$ is an isometry for some modulus one complex number $\lambda \neq 1$ if and only if either $\lambda = -1$, or P is hermitian. It is also proved that every rank one bicontractive projection on A is hermitian. The particular case when A is a C*-algebra is discussed through several examples.

1. Introduction

Bicircular projections and their generalizations have received a lot of attention recently. A basic problem is to determine the structure of these mappings on a given Banach space.

Let $\mathcal{X}$ be a complex Banach space and let $P : \mathcal{X} \to \mathcal{X}$ be a linear projection, that is a linear mapping with the property $P^2 = P$. By $\overline{P}$ we denote the projection $Id - P$, where Id is the identity operator on $\mathcal{X}$. A projection P is called bicircular if the mapping $P + \lambda\overline{P}$ is an isometry for *every* modulus one complex number λ. The study of bicircular projections is motivated by complex analysis and it is initiated in a series of papers by Stachó and Zalar [24–26].

☆ The author was supported by the Ministry of Science, Education and Sports of the Republic of Croatia (Project No. 037-0372784-2757).

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0024-3795/$ - see front matter
doi:10.1016/j.laa.2009.10.037

A bounded linear operator $T : \mathcal{X} \to \mathcal{X}$ is said to be hermitian if $e^{i\theta T}$ is an isometry for every $\theta \in \mathbb{R}$. A projection on $\mathcal{X}$ is a bicircular projection if and only if it is a hermitian projection, as shown by Jamison [20, Lemma 2.1]. Since many results concerning hermitian operators are known, this observation enables characterization of bicircular projections in numerous complex Banach spaces.

The notion of bicircular projections was generalized by Fošner, Li and the author in [16], by requiring that, for a linear projection P, the mapping $P + \lambda\overline{P}$ be an isometry for *some* modulus one complex number $\lambda \neq 1$. These projections are now known as generalized bicircular projections. They have been recently studied in a series of papers by Botelho and Jamison (e.g. [4–10]).

Let us emphasize that, for a linear projection P and a modulus one complex number λ, the mapping $P + \lambda\overline{P}$ is linear and surjective. Thus the study of generalized bicircular projections on a given complex Banach space depends on the knowledge of the structure of surjective linear isometries on that space.

Obviously, if P is a bicircular projection (respectively, generalized bicircular projection), then $\overline{P}$ is also a bicircular projection (respectively, generalized bicircular projection).

Every generalized bicircular projection is contractive [22, Corollary 2], that is $\|P\| \leqslant 1$. In fact, since P is generalized bicircular implies that $\overline{P}$ is generalized bicircular as well, every generalized bicircular projection is bicontractive, that is $\|P\| \leqslant 1$ and $\|\overline{P}\| \leqslant 1$. Considering that bicontractive projections have been studied in various complex Banach spaces, this result becomes another useful tool in investigations of generalized bicircular projections.

A linear isometry $T : \mathcal{X} \to \mathcal{X}$ satisfying $T^2 = Id$ is called an isometric reflection. It is easy to verify that the average of the identity with an isometric reflection is a generalized bicircular projection. These mappings are the only generalized bicircular projections in various settings (see e.g. [16,5,9,14,19]). More precisely, in many cases it turns out that, if P is a linear projection such that $P + \lambda\overline{P}$ is an isometry for some modulus one complex number $\lambda \neq 1$, then either $\lambda = -1$, or P is hermitian (in particular, $2P - Id$ is an isometry). The aim of this paper is to prove that this is true for a large class of complex Banach spaces, known as JB*-triples, as well as to prove that the additional assumption that P is rank one yields the conclusion that P must be hermitian.

A JB*-triple is a complex Banach space A together with a continuous triple product $\{\cdot\cdot\cdot\} : A \times A \times A \to A$ such that

(i) $\{xyz\}$ is linear in x and z and conjugate linear in y;
(ii) $\{xyz\}$ is symmetric in the outer variables, i.e., $\{xyz\} = \{zyx\}$;
(iii) for any $x \in A$, the operator $\delta(x) : A \to A$ defined by $\delta(x)y = \{xxy\}$ is hermitian with nonnegative spectrum;
(iv) the following "main identity" holds:

$$\delta(x)\{abc\} = \{\delta(x)a, b, c\} - \{a, \delta(x)b, c\} + \{a, b, \delta(x)c\};$$

(v) for every $x \in A$, $\|\{xxx\}\| = \|x\|^3$.

The notion of a JB*-triple can be regarded as a simultaneous generalization of complex Hilbert spaces and C*-algebras: a complex Hilbert space is a JB*-triple with respect to the triple product defined by

$$\{xyz\} = \frac{1}{2}(\langle x, y\rangle z + \langle z, y\rangle x)$$

and a C*-algebra is a JB*-triple with respect to the triple product defined by

$$\{xyz\} = \frac{1}{2}(xy^*z + zy^*x).$$

Other examples of JB*-triples are JB*-algebras as well as some Lie algebras.

An element a in a JB*-triple A is called a tripotent if $\{aaa\} = a$, and it is called a minimal tripotent, or an atom, if it is a nonzero tripotent with the property $\{aAa\} = \mathbb{C}a$. Every nonzero tripotent is norm one.

A linear subspace J of a JB*-triple A is called an ideal of A if $\{AAJ\} + \{AJA\} \subseteq J$. A JB*-triple is said to be a prime JB*-triple if the intersection of two nonzero norm closed ideals of A is always nonzero. If J and K are norm closed ideals of A, then

$$J \cap K = \{JKA\} = \{JAK\} = \{AKJ\} = \{KAJ\} = \{KJA\} = \{AJK\}. \tag{1}$$

2. Main results

Theorem 2.1 is a general result characterizing generalized bicircular projections acting on a JB*-triple. Its proof relies on deep results concerning the structure of surjective linear isometries and the structure of bicontractive linear projections on JB*-triples (for the historical survey of these results see e.g. [23], where a comprehensive list of references on JB*-triples can be found).

The class of all bicontractive projections on a JB*-triple A coincides with the class of all generalized bicircular projections on A, and it contains the class of all hermitian projections on A.

Theorem 2.1. *Let A be a JB*-triple and let $P : A \to A$ be a linear projection. Then $P + \lambda\overline{P}$ is an isometry for some modulus one complex number $\lambda \neq 1$ if and only if one of the following holds:*

(i) $\lambda = -1$ *and* $P = \frac{1}{2}(Id + T)$ *for some isometric reflection* $T : A \to A$,
(ii) P *is hermitian* ($\equiv$ *bicircular*).

Proof. Let $P : A \to A$ be a linear projection such that $P + \lambda\overline{P}$ is an isometry for some modulus one complex number $\lambda \neq 1$. For every modulus one complex number μ, let us define the mapping $T_\mu : A \to A$ by $T_\mu = P + \mu\overline{P}$. Recall that all T_μ are linear and surjective. Since T_λ is an isometry, as stated in Section 1, P is a bicontractive projection. According to [17, Theorem 4], this implies that T_{-1} is an isometry such that $P = \frac{1}{2}(Id + T_{-1})$ and ${T_{-1}}^2 = Id$.

Assume that $\lambda \neq -1$.

According to [21] (see also [12, Theorem D]), every surjective linear isometry $T_\mu : A \to A$ satisfies

$$T_\mu(\{xyz\}) = \{T_\mu(x)T_\mu(y)T_\mu(z)\} \quad (x, y, z \in A). \tag{2}$$

If we define (cf. the proof of [25, Proposition 3.4]), for fixed $x, y, z \in A$,

$$\begin{aligned}
a &= \{P(x)\overline{P}(y)P(z)\},\\
b &= \{P(x)P(y)P(z)\} + \{P(x)\overline{P}(y)\overline{P}(z)\} + \{\overline{P}(x)\overline{P}(y)P(z)\} - P\{xyz\},\\
c &= \{\overline{P}(x)\overline{P}(y)\overline{P}(z)\} + \{\overline{P}(x)P(y)P(z)\} + \{P(x)P(y)\overline{P}(z)\} - \overline{P}\{xyz\},\\
d &= \{\overline{P}(x)P(y)\overline{P}(z)\},
\end{aligned}$$

then (2) is equivalent to

$$a + \mu b + \mu^2 c + \mu^3 d = 0. \tag{3}$$

Since T_1, T_{-1}, T_λ and $T_{-\lambda} = T_{-1}T_\lambda$ are isometries, solving the system obtained inserting $\mu = \pm 1, \pm\lambda$ in (3), we conclude $a = b = c = d = 0$. Hence, (3) holds for every modulus one complex number μ. Then (2) also holds for every modulus one complex number μ. In particular,

$$T_\mu(\{xxx\}) = \{T_\mu(x)T_\mu(x)T_\mu(x)\} \quad (x \in A).$$

This yields

$$\begin{aligned}
\|T_\mu(x)\|^3 = \; &\|\{T_\mu(x)T_\mu(x)T_\mu(x)\}\| = \|T_\mu(\{xxx\})\|\\
&\leqslant \|T_\mu\| \cdot \|\{xxx\}\| = \|T_\mu\| \cdot \|x\|^3 \quad (x \in A),
\end{aligned}$$

so $\|T_\mu\|^3 \leqslant \|T_\mu\|$ and finally $\|T_\mu\| \leqslant 1$ for every modulus one complex number μ. Thus, for every $x \in A$,

$$\|x\| = \|T_{\overline{\mu}}T_\mu(x)\| \leqslant \|T_\mu(x)\| \leqslant \|x\|$$

which implies $\|T_\mu(x)\| = \|x\|$. Hence, T_μ is an isometry for every modulus one complex number μ, so P is bicircular ($\equiv$ hermitian).

Conversely, if (i) or (ii) holds, then $P - \overline{P}$ is an isometry. $\square$

In the setting of general complex Banach spaces, if P is not hermitian, then $\lambda^n = 1$ for some $n \in \mathbb{N} \setminus \{1\}$ [22, Theorem 1]. According to Theorem 2.1, in the particular case of JB*-triples, $n = 2$. Let us also mention that, for every complex number λ such that $\lambda^n = 1$ for some $n \in \mathbb{N} \setminus \{1\}$, there exist a complex Banach space $\mathcal{X}$ and a nonhermitian projection P on $\mathcal{X}$ such that $P + \lambda\overline{P}$ is an isometry [22, Theorem 3].

Let us notice that, regardless of which one of the conclusions (i) and (ii) from Theorem 2.1 holds, the mapping $P - \overline{P}$ is an isometry, thus $P = \frac{1}{2}(Id + T)$ for some isometric reflection $T : A \to A$. Theorem 2.1 points out the fact that the assumption of the existence of a modulus one complex number $\lambda \notin \{-1, 1\}$ such that $P + \lambda\overline{P}$ is an isometry, yields the conclusion that P must be hermitian.

Remark 2.1. Let H be a complex Hilbert space and let $P : H \to H$ be a bicontractive projection. Since P is bounded, $P(H)$ is a closed subspace of H. Thus $P^* = P$. Let $T : H \to H$ be defined by $T = 2P - Id$. Then $T^* = T$ and $T^2 = Id$, so T is an isometry. For every modulus one complex number λ and every $x \in H$,

$$\begin{aligned}\|(P + \lambda\overline{P})(x)\|^2 &= \tfrac{1}{4}\|(1-\lambda)T(x) + (1+\lambda)x\|^2 \\ &= \tfrac{1}{4}\Big(|1-\lambda|^2\|T(x)\|^2 + |1+\lambda|^2\|x\|^2 \\ &\quad + (\overline{\lambda} - \lambda)\left(\langle T(x), x\rangle - \langle x, T(x)\rangle\right)\Big) = \|x\|^2.\end{aligned}$$

Hence, every bicontractive ($\equiv$ generalized bicircular) projection acting on a complex Hilbert space is a hermitian ($\equiv$ bicircular) projection.

In general, the class of all bicontractive ($\equiv$ generalized bicircular) projections does not coincide with the class of all hermitian ($\equiv$ bicircular) projections, see [26, Example 2]. However, it will be proved in Theorem 2.2 that every bicontractive projection of rank one, acting on a JB*-triple, is hermitian. Theorem 2.2 is an extension of [26, Theorem 1].

Theorem 2.2. *Let A be a JB*-triple and let $P : A \to A$ be a rank one linear projection. Then the following conditions are mutually equivalent:*

(i) *P is bicontractive,*
(ii) *P is hermitian,*
(iii) *there exist an atom $a \in A$ such that $P(A) = \mathbb{C}a$ and $\{a\overline{P}(A)a\} = 0$, and two norm closed ideals $H \supseteq P(A)$ and $J \subseteq Ker(P)$ of A such that $A = H \oplus J$, where H is isometrically isomorphic to a Hilbert space.*

Proof. (ii) $\Longrightarrow$ (i): Trivial since $2P - Id$ is an isometry.

(i) $\Longrightarrow$ (iii): Since P is a bicontractive projection, [17, Theorem 4] implies that $T = 2P - Id$ is a surjective linear isometry; $T^2 = Id$. According to [21],

$$T(\{xyz\}) = \{T(x)T(y)T(z)\} \quad (x, y, z \in A).$$

Let $a \in A$ be norm one such that $P(x) = f(x)a$ for some bounded linear functional f on A.

From $P^2 = P$ we get $f(a) = 1$. Then $P(a) = a$, so $T(a) = a$. Then

$$T(\{aaa\}) = \{T(a)T(a)T(a)\} = \{aaa\},$$

which implies

$$P(\{aaa\}) = \{aaa\}.$$

Then

$$|f(\{aaa\})| = \|f(\{aaa\})a\| = \|P(\{aaa\})\| = \|\{aaa\}\| = \|a\|^3 = 1.$$

Thus $\{aaa\} = \mu a$ for some modulus one $\mu \in \mathbb{C}$. Let $\delta(a) : A \to A$ be defined by $\delta(a)x = \{aax\}$. Then $\delta(a)a = \mu a$, so μ is an element in the spectrum of $\delta(a)$. By the definition of a JB*-triple, the spectrum of $\delta(a)$ is nonnegative. This yields $\mu = 1$ and $\{aaa\} = a$. Hence, a is a tripotent.

To every tripotent $a \in A$, there corresponds a Peirce decomposition (see [18, Section 1], and also [23, Section 2.2])

$$A = A_1 \oplus A_{1/2} \oplus A_0,$$

with $\{A_1A_0A\} = \{A_0A_1A\} = \{AA_0A_1\} = \{AA_1A_0\} = 0$, where $A_\alpha = \{x \in A : \{aax\} = \alpha x\}$ for every $\alpha \in \{1, 1/2, 0\}$. Let $P_\alpha : A \to A_\alpha$ be the corresponding Peirce projections.

Let $\alpha \in \{1/2, 0\}$ and let $x \in A_\alpha$. Then

$$\begin{aligned}(1-\alpha)P(x) = f(x)a - \alpha P(x) &= \{aaP(x)\} - \alpha P(x)\\ &= \frac{1}{2}(\{aaT(x)\} + \{aax\}) - \frac{1}{2}\alpha(T(x) + x)\\ &= \frac{1}{2}(\{aaT(x)\} - T(\alpha x)) = \frac{1}{2}(T(\{aax\}) - T(\{aax\})) = 0,\end{aligned}$$

which implies $P(x) = 0$. Hence, $A_{1/2} \oplus A_0 \subseteq Ker(P)$, so $P = P_1$. Then for every $x \in A_1$ we have $x = P_1(x) = P(x) = f(x)a \in \mathbb{C}a$. Thus $A_1 \subseteq \mathbb{C}a$. Since the converse also holds, $A_1 = \mathbb{C}a$.

Note that $x \in A_\alpha$ if and only if $\delta(a)x = \alpha x$. For every $y \in A_{1/2}$, $z \in A_0$, the main identity

$$\delta(a)\{ayz\} = \{\delta(a)a, y, z\} - \{a, \delta(a)y, z\} + \{a, y, \delta(a)z\}$$

yields

$$\delta(a)\{ayz\} = \frac{1}{2}\{ayz\}.$$

Hence, $\{ayz\} \in A_{1/2}$. On the other hand, $P(y) = P(z) = 0$, thus $T(y) = -y$, $T(z) = -z$. Then we have

$$T(\{ayz\}) = \{T(a)T(y)T(z)\} = \{ayz\},$$

which implies $P(\{ayz\}) = \{ayz\}$. This yields $\{ayz\} = P_1(\{ayz\}) \in A_1$. Finally, $\{ayz\} \in A_{1/2} \cap A_1 = 0$, that is $\{aA_{1/2}A_0\} = 0$. Since $\{aA_0A_0\} \subseteq \{A_1A_0A\} = 0$ and $\{aA_1A_0\} \subseteq \{A_0A_1A\} = 0$, we conclude $\{aAA_0\} = 0$.

By [11, Proposition 2.1], applied for $X = A_0$, $Y = \mathbb{C}a$, there exist norm closed ideals J and H of A such that

$$\begin{aligned}&J = \{x \in A : \{aAx\} = 0\} \supseteq A_0,\\ &H = \{x \in A : \{xAJ\} = 0\} \supseteq \mathbb{C}a,\\ &J \cap H = 0.\end{aligned}$$

Since $a \in H$, we have

$$A_{1/2} \subseteq \{aaA\} \subseteq \{HAA\} \subseteq H.$$

Hence,

$$\mathbb{C}a \oplus A_{1/2} \subseteq H, \quad A_0 \subseteq J.$$

Since $A = \mathbb{C}a \oplus A_{1/2} \oplus A_0$ and $J \cap H = 0$, we conclude

$$H = \mathbb{C}a \oplus A_{1/2}, \quad J = A_0.$$

Let $x \in A_\alpha$, $\alpha \in \{1/2, 0\}$. From

$$\delta(a)\{axa\} = \{\delta(a)a, x, a\} - \{a, \delta(a)x, a\} + \{a, x, \delta(a)a\}$$

we get

$$\delta(a)\{axa\} = (2-\alpha)\{axa\}. \tag{4}$$

Since

$$\{axa\} \subseteq \{aAA\} \subseteq \{HAA\} \subseteq H = A_1 \oplus A_{1/2},$$

there exist $y \in A_1, z \in A_{1/2}$ such that $\{axa\} = y + z$. Then (4) can be written as

$$\delta(a)y + \delta(a)z = (2-\alpha)y + (2-\alpha)z.$$

Since $\delta(a)y = y$ and $\delta(a)z = \frac{1}{2}z$, this implies

$$(\alpha - 1)y + \left(\alpha - \frac{3}{2}\right)z = 0.$$

Hence, $y = z = 0$. Thus we have proved $\{a(A_{1/2} \oplus A_0)a\} = 0$. Since $P = P_1$, we have $\overline{P} : A \to A_{1/2} \oplus A_0$, so $\{a\overline{P}(A)a\} = 0$. This yields

$$\{axa\} = \{aP(x)a\} = \overline{f(x)}\{aaa\} = \overline{f(x)}a \quad (x \in A).$$

Hence, $\{aAa\} = \mathbb{C}a$, thus a is an atom.

In the same way as in [26, Theorem 1 (Step 5)] one can prove that H is isometrically isomorphic to a Hilbert space.

(iii) $\Longrightarrow$ (ii): Let f be a linear functional on A such that $P(x) = f(x)a$. Since

$$\{axa\} = \{aP(x)a\} = \overline{f(x)}\{aaa\} = \overline{f(x)}a,$$

we have

$$\|P(x)\| = \|f(x)a\| = |f(x)| = \|\{axa\}\| \leqslant \|x\|,$$

so P is bounded. Hence, the restriction of P to H is a bounded idempotent from H to H, thus selfadjoint as well. This implies, for every $x \in H$,

$$\|\overline{P}(x)\|^2 = \langle x - P(x), x - P(x)\rangle = \|x\|^2 - \|P(x)\|^2 \leqslant \|x\|^2,$$

hence the restriction of P to H is a bicontractive projection.

By (1), for every $y \in H$ and every $z \in J$ we have

$$\{y+z, y+z, y+z\} = \{yyy\} + \{zzz\}.$$

This implies

$$\begin{aligned}\|y+z\|^3 &= \|\{y+z, y+z, y+z\}\| = \|\{yyy\} + \{zzz\}\| \\ &\leqslant \|\{yyy\}\| + \|\{zzz\}\| = \|y\|^3 + \|z\|^3.\end{aligned}$$

Then

$$\|y+z\|^3 \leqslant 2(\max\{\|y\|, \|z\|\})^3.$$

Assume that for some $n \in \mathbb{N}$ and all $y \in H, z \in J$ we have

$$\|y+z\|^{3^n} \leqslant 2(\max\{\|y\|, \|z\|\})^{3^n}.$$

Then

$$\begin{aligned}\|y+z\|^{3^{n+1}} &= \|\{y+z,\ y+z,\ y+z\}\|^{3^n} = \|\{yyy\} + \{zzz\}\|^{3^n} \\ &\leqslant 2(\max\{\|\{yyy\}\|, \|\{zzz\}\|\})^{3^n} = 2(\max\{\|y\|^3, \|z\|^3\})^{3^n} \\ &= 2(\max\{\|y\|, \|z\|\})^{3^{n+1}}.\end{aligned}$$

By induction we conclude

$$\|y+z\| \leqslant \sqrt[3^n]{2} \cdot \max\{\|y\|, \|z\|\} \quad (n \in \mathbb{N},\ y \in H,\ z \in J).$$

Hence,

$$\|y+z\| \leqslant \max\{\|y\|, \|z\|\} \quad (y \in H,\ z \in J).$$

On the other hand,

$$\begin{aligned}\max\{\|y\|, \|z\|\} &\leqslant \frac{1}{2}(\|y+z\| + \|y-z\|)\\ &\leqslant \frac{1}{2}(\|y+z\| + \max\{\|y\|, \|z\|\}),\end{aligned}$$

which implies

$$\max\{\|y\|, \|z\|\} \leqslant \|y+z\| \qquad (y \in H,\ z \in J).$$

Hence, for every $x \in A$,

$$\|x\| = \max\{\|y\|, \|z\|\},$$

where $y \in H$ and $z \in J$ are such that $x = y + z$.

Let λ be an arbitrary modulus one complex number and let $T_\lambda = P + \lambda\overline{P}$. Let us note that the restriction of T_λ to H is an isometry from H to H by Remark 2.1, and let us also note that $T_\lambda(z) = \lambda z$ for every $z \in J$. Let $x \in A$ and let $y \in H$ and $z \in J$ be such that $x = y + z$. Then $T_\lambda(y) \in H$ and $T_\lambda(z) \in J$, so

$$\|T_\lambda(x)\| = \|T_\lambda(y) + T_\lambda(z)\| = \max\{\|T_\lambda(y)\|, \|T_\lambda(z)\|\} = \max\{\|y\|, \|z\|\} = \|x\|.$$

Hence, T_λ is an isometry for every modulus one complex number λ, thus P is a bicircular ($\equiv$ hermitian) projection. □

From Theorem 2.2 we immediately get (cf. [26, Corollary 1]):

Corollary 2.3. *The only prime JB*-triples admitting rank one bicontractive projections are Hilbert spaces.*

Regarding Remark 2.1 and Theorem 2.2, it would be interesting to establish the structure of a JB*-triple A with the property that every bicontractive projection on A is hermitian.

3. Remarks on applications to C*-algebras

Recall that the structure of hermitian ($\equiv$ bicircular) projections on a C*-algebra is determined in [15, Theorem 3.3].

Remark 3.1. If A is $\mathcal{K}(H)$ or $\mathcal{B}(H)$ (the algebra of all compact linear operators on a complex separable infinite dimensional Hilbert space H and the algebra of all bounded linear operators on H, respectively), then Theorem 2.1, [5, Lemma 2.1] and [15, Corollary 3.6] yield the structure of generalized bicircular projections on A (cf. [5], as well as [19]).

Remark 3.2. Let Ω be a locally compact Hausdorff space and let $C_0(\Omega)$ be the algebra (with usual pointwise operations) of all continuous complex-valued functions on Ω vanishing at infinity. Define an involution by $f^*(w) = \overline{f(w)}$ for every $w \in \Omega$. Then $C_0(\Omega)$, equipped with the supremum norm, is a commutative C*-algebra. (According to Gelfand–Naimark theorem, e.g. [1, Theorem 1.2.1] or [13, Theorem 1.3.1], every commutative C*-algebra is isometrically $*$-isomorphic to $C_0(\Omega)$ for some, up to homeomorphism unique, locally compact Hausdorff space Ω.) By [15, Corollary 3.4], the only nonzero hermitian projection on $C_0(\Omega)$ is the identity operator, since the multiplier algebra of $C_0(\Omega)$ is the algebra of all bounded continuous complex-valued functions on Ω (e.g. [1, Example 1.1.8]). According to Theorem 2.1, for the complete description of generalized bicircular projections on $C_0(\Omega)$, it remains to determine the structure of isometric reflections on $C_0(\Omega)$.

If $T : C_0(\Omega) \to C_0(\Omega)$ is a surjective linear isometry, the Banach-Stone theorem (e.g. [2, Theorem 7.1]) implies the existence of a homeomorphism $\phi : \Omega \to \Omega$ and a continuous function $u : \Omega \to \mathbb{C}$, with $|u(w)| = 1$ for every $w \in \Omega$, such that

$$T(f)(w) = u(w)f(\phi(w)) \quad (f \in C_0(\Omega), w \in \Omega). \tag{5}$$

Assume that T is an isometric reflection. Then (5) yields

$$\begin{aligned} f(w) = &\ T(T(f))(w) = u(w)T(f)(\phi(w)) \\ = &\ u(w)u(\phi(w))f(\phi^2(w)) \quad (f \in C_0(\Omega), w \in \Omega). \end{aligned} \tag{6}$$

If there exists $w_0 \in \Omega$ such that $\phi^2(w_0) \neq w_0$, then there exists $f_0 \in C_0(\Omega)$ such that $f_0(w_0) = 1$ and $f_0(\phi^2(w_0)) = 0$ (the existence of such a function, the so-called Urysohn function, follows from the fact that every locally compact Hausdorff space is a $T_{3\frac{1}{2}}$ space; see e.g. [27, p. 25]). However, this is in contradiction with (6). Therefore, ϕ^2 is the identity operator. Taking this into account in (5), we get $u(\phi(w)) = \overline{u(w)}$ for every $w \in \Omega$.

Hence, if $P : C_0(\Omega) \to C_0(\Omega)$ is a projection (which is not zero nor the identity operator), then $P + \lambda\overline{P}$ is an isometry for some modulus one complex number $\lambda \neq 1$ if and only if $\lambda = -1$ and there exist a homeomorphism $\phi : \Omega \to \Omega$ satisfying $\phi^2(w) = w$ for every $w \in \Omega$, and a continuous function $u : \Omega \to \mathbb{C}$ satisfying $|u(w)| = 1$ and $u(\phi(w)) = \overline{u(w)}$ for every $w \in \Omega$, such that

$$P(f)(w) = \frac{1}{2}(f(w) + u(w)f(\phi(w))) \quad (f \in C_0(\Omega),\ w \in \Omega).$$

(Some of the related papers are [9,4,14]).

Remark 3.3. Let A be a C*-algebra. In the sequel we use terminology, notations and the results from [15]. It turns out that the existence of nontrivial hermitian projections (that is, other than zero and the identity) in the setting of C*-algebras is closely connected with the existence of nontrivial selfadjoint projections in the multiplier algebra (more details on multiplier algebras can be found in e.g. [1]). It also turns out that A admits rank one bicontractive projections if and only if it admits central minimal selfadjoint projections.

(a) Assume that, for every $*$-ideal I of A, the only selfadjoint projections in $M(I^\perp \oplus I^{\perp\perp})$ are zero and the identity. Let $P : A \to A$ be a hermitian projection. By [15, Theorem 3.3], the restriction of P to $I^\perp \oplus I^{\perp\perp}$ is a trivial projection, that is either zero, or the identity. Every hermitian projection on A satisfies (see e.g. [15, Lemma 3.2] which is based on [25, Proposition 3.4], that largely motivated the proof of Theorem 2.1)

$$P(xyx) = P(x)yx - xP(y^*)^*x + xyP(x) \quad (x, y \in A). \tag{7}$$

If P is zero on $I^\perp \oplus I^{\perp\perp}$, then (7) implies

$$xP(y^*)^*x = 0 \quad (x \in I^\perp \oplus I^{\perp\perp}, y \in A),$$

so P is zero on A. If P is the identity on $I^\perp \oplus I^{\perp\perp}$, then (7) implies

$$xP(y^*)^*x = xyx \quad (x \in I^\perp \oplus I^{\perp\perp}, y \in A),$$

thus P is the identity on A.

Hence, on the assumption that, for every $*$-ideal I of A, the only selfadjoint projections in $M(I^\perp \oplus I^{\perp\perp})$ are zero and the identity, the preceding discussion together with Theorem 2.1 yields the following conclusion: if $P : A \to A$ is a nontrivial linear projection, then $P + \lambda\overline{P}$ is an isometry for some modulus one complex number $\lambda \neq 1$ if and only if $\lambda = -1$ and P is the average of the identity with an isometric reflection. An example of a C*-algebra satisfying the assumed property is any simple unital C*-algebra without selfadjoint projections except zero and the identity (the first one was constructed by Blackadar; see [3] and also e.g. [13, IV.8]).

(b) Let us emphasize that, if for every ideal I of A we have $I^{\perp} \oplus I^{\perp\perp} = A$ (this is the case, for example, when A is a prime C*-algebra, a C*-subalgebra of $\mathcal{K}(H)$, or a von Neumann algebra [15, Corollaries 3.8 and 3.9]) and if there exists a selfadjoint projection in $M(A)$ which is not zero nor the identity, then there exists a nontrivial hermitian projection on A [15, Example 3.1].

(c) Let us also note that, for every noncentral selfadjoint projection $p \in M(A)$, the mapping $P : A \to A$ defined by

$$P(x) = 2pxp - xp - px + x \quad (x \in A),$$

is a nonhermitian generalized bicircular projection on A, which can be verified via [15, Theorem 3.3]. Namely, if we assume that P is hermitian, then [15, Theorem 3.3] implies the existence of a $*$-ideal I of A and a selfadjoint projection $q \in M(I^{\perp} \oplus I^{\perp\perp})$ such that $P(x) = qx$ for every $x \in I^{\perp}$ and $P(x^*)^* = qx$ for every $x \in I^{\perp\perp}$. Since $P(px) = P((px)^*)^* = pxp$ for every $x \in A$, we have $pxp = qpx$ for every $x \in I^{\perp} \oplus I^{\perp\perp}$. This implies

$$(px - xp)y(px - xp) = 0 \quad (x, y \in I^{\perp} \oplus I^{\perp\perp}).$$

Since $I^{\perp} \oplus I^{\perp\perp}$ is the essential ideal of A, this yields $px = xp$ for every $x \in I^{\perp} \oplus I^{\perp\perp}$, and p commutes with all elements in A as well. Thus it also commutes with all elements in $M(A)$; a contradiction.

(d) Let $p \in A$ be a central minimal selfadjoint projection. Then the mapping $P : A \to A$ defined by $P(x) = px$, for every $x \in A$, is a rank one bicontractive projection.

Conversely, let $P : A \to A$ be a rank one bicontractive projection. By Theorem 2.2, P is a hermitian projection. Then, according to [15, Theorem 3.3], there exist a $*$-ideal I of A and a selfadjoint projection $p \in M(I^{\perp} \oplus I^{\perp\perp})$ such that $P(x) = px$ for every $x \in I^{\perp}$ and $P(x) = xp$ for every $x \in I^{\perp\perp}$. By Theorem 2.2, there exists an atom $a \in A$ such that $P(A) = \mathbb{C}a$ and $\{a\overline{P}(A)a\} = 0$. Let f be a bounded linear functional on A satisfying $P(x) = f(x)a$. If $f(I^{\perp}) = f(I^{\perp\perp}) = 0$, then $P(I^{\perp} \oplus I^{\perp\perp}) = 0$, so $P(A) = 0$; a contradiction. Hence, $f(I^{\perp}) \neq 0$ or $f(I^{\perp\perp}) \neq 0$.

Assume that $f(I^{\perp}) \neq 0$. Let $x_0 \in I^{\perp}$ be such that $f(x_0) \neq 0$. Then $f(x_0)a = P(x_0) = px_0$ implies $a = \frac{1}{f(x_0)}px_0 \in I^{\perp}$. Hence, $pa = a$. Now we have

$$f(aa^*)a = P(aa^*) = p(aa^*) = aa^*.$$

Let $e = aa^* \in I^{\perp}$. Then e is a minimal selfadjoint projection in A such that $pe = ep = e$. Since $|f(aa^*)| = 1$, we have

$$P(x) = f(x)a = f(x)\overline{f(aa^*)}e.$$

From $\{a\overline{P}(A)a\} = 0$ we get

$$ax^*a = aP(x)^*a = \overline{f(x)}f(aa^*)aea,$$

which implies

$$ex^*e = \overline{f(x)}f(aa^*)e = P(x)^*$$

and finally

$$P(x) = exe \quad (x \in A).$$

In particular,

$$px = exe \quad (x \in I^{\perp}),$$
$$xp = 0 \quad (x \in I^{\perp\perp}).$$

This implies

$$(e - p)x(e - p) = 0 \quad (x \in I^{\perp} \oplus I^{\perp\perp}),$$

hence $e = p$. Then $p \in I^{\perp}$ and

$$P(x) = px = pxp \quad (x \in I^{\perp}).$$

This yields $px = xp$ for every $x \in I^{\perp} \oplus I^{\perp\perp}$. Thus p commutes with all elements in A. Hence, p is a central minimal selfadjoint projection such that $P(x) = px$.

The case $f(I^{\perp\perp}) \neq 0$ can be discussed in an analogous way.

Acknowledgement

The author wishes to express her thanks to Tomislav Berić and Stanislav Jabuka for helpful suggestions, and the referee for useful comments.

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