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Transitivity of inner automorphisms in infinite dimensional Cartan factors

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Abstract Let C be a Cartan-factor having arbitrary dimension $\dim C$. It is shown that the group $\mathrm{Inn}(C)$ of inner automorphisms of C acts transitively on the manifold $\mathcal{U}_r(C)$ of tripotents with finite rank r in C. This extends results by Loos (Bounded Symmetric Domains and Jordan Pairs. Mathematical Lectures. University of California, Irvine, 1977) valid in finite dimensions, and similar findings by Isidro et al. (Math Z 233(4):741–754, 2000; Acta Sci Math (Szeged) 66(3–4), 2000; Expo Math 20(2):97–116, 2002; Q J Math 57(4):505–525, 2006). Hence, the results presented here close a significant gap concerning the transitivity property of the general infinite-dimensional case. The proofs given here are based on new methods, independent of those used for the finite-dimensional cases.

1 Introduction

The set $\mathcal{U}_r(\mathcal{B}(H, K))$ of partial isometries of a fixed rank r in $\mathcal{B}(H, K)$ provides a well known example of a Riemannian symmetric space. That $\mathcal{U}_r(\mathcal{B}(H, K))$ is connected was shown by Halmos and McLaughlin [6]. A more general approach to Riemannian symmetric spaces is provided by Jordan structures, such as JB*-algebras, JB*-triples and their weak*-closed versions, JBW*-triples, of which $\mathcal{B}(H, K)$ is but one of six fundamental types, the Cartan factors. The partial isometries of $\mathcal{B}(H, K)$ are precisely the tripotent elements. In [15] Loos showed that when C is a finite-dimensional Cartan factor, then the group $\mathrm{Inn}(C)$ acts transitively on the set $\mathcal{U}_r(C)$ of tripotents of rank r in C. An analogous statement for Hilbert spaces of arbitrary dimension was proved in [10]. The goal of the present article is to

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generalize that result to all Cartan-factors. The methods of our proofs are independent of those of Loos, and may be applied to the finite dimensional Cartan-factors, in particular to those of type V and VI, which have dimensions 16 and 27, respectively, though these require more extensive calculations than the other types. In this paper we focus on the infinite dimensional cases.

The categories of JB*-triples and JBW*-triples have gained significance for their profound connections with several fields in mathematics and mathematical physics. The recent book [5] by Friedman and Scarr presents applications, mainly of rectangular factors and spin factors, in relativity theory and quantum mechanics. The Riemannian structure of manifolds of tripotents and of their generalizations in Jordan-algebras was further investigated in recent works by Isidro et al. [12–14] and Nomura [17]. Precursors of their results were obtained by Hirzebruch [8]. The above-cited works show that the geodesics in those manifolds are given by paths of inner automorphisms of the corresponding JBW*-triple. The arguments presented in [10,11] show that the transitivity of $\mathrm{Inn}(C)$ on $\mathcal{U}_1(C)$ is closely connected to the problem of the existence of contractive projections onto subtriples of JB*-triples or JBW*-triples.

This article is organized as follows. In Sect. 2 we present some general facts and definitions of the theory of JB*-triples and JBW*-triples. Section 3 is devoted to some details concerning Cartan-factors, which represent an important class of JBW*-triples. They arise as the irreducible components of atomic JBW*-triples. In this section we also study two examples which illustrate the transitivity property, and which serve as key tools to proving the general cases. Section 4 contains the main results. Theorem 4.1 shows that any two minimal tripotents in an (infinite-dimensional) Cartan-factor C are connected by an element φ of the group $\mathrm{Inn}(C)$, that is $\mathrm{Inn}(C)$ acts transitively on the set $\mathcal{U}_1(C)$ of minimial tripotents in C. As a corollary, the result holds for any two tripotents of equal finite rank (Theorem 4.3). In the last section, we also investigate some consequences of the main result to weak*-operator limits of inner automorphisms and to the σ-finite elements of C. However, those results do not provide a straightforward generalization of the main result, and are therefore to be regarded as tentative.

The techniques depend to a great extent on the coordinatization of the Cartan-factors by grids, a method introduced by Neher [16]. The strategy of our proof is to use appropriate transformations, reducing the general case to two special cases, namely that of the Jordan algebra $\mathbb{S} = \mathbb{M}_2^s(\mathbb{C})$ of symmetric complex 2×2-matrices, and that of complex Hilbert spaces.

2 Preliminaries

Recall that a *JB*-triple* is a complex Banach space A, equipped with a triple product $(a, b, c) \mapsto \{a\ b\ c\}$ from $A \times A \times A$ to A having the properties that the expression $\{a, b, c\}$ is symmetric and linear in a and c and conjugate linear in b, the *Jordan triple identity* holds, that is

$$[D(a, b), D(c, d)] = D(\{a\ b\ c\}, d) - D(c, \{d\ a\ b\}), \tag{2.1}$$

where $[\ \ ,\ \ ]$ denotes the commutator, and $D(a, b)$ is the linear mapping on A defined by $D(a, b)c = \{a\ b\ c\}$. Moreover, the mapping $(a, a) \mapsto D(a, a)$ is continuous from $A \times A$ to the Banach space $B(A)$ of bounded linear operators on A, for each element a in A, $D(a, a)$ is hermitian in the sense of [1, Definition 5.1], with non-negative spectrum and has norm $\|D(a, a)\| = \|a\|^2$. If A is also the dual of some Banach space A_*, then A is said to be a

JBW-triple*, and A_* is referred to as the *predual* of A. A subspace B of A is said to be a *subtriple* if $\{B\ B\ B\}$ is contained in B.

An element u of A is said to be a *tripotent* if $\{u,\ u,\ u\} = u$. The set of tripotents of A is denoted by $\mathcal{U}(A)$. Let j, k and l be equal to 0, 1 or 2. For each tripotent u of A, the norm- and weak*-continuous projections

$$\begin{aligned} P_2(u) &= Q(u)^2, \\ P_1(u) &= 2(D(u,u) - Q(u)^2), \\ P_0(u) &= \mathrm{id_A} - 2D(u,u) + Q(u)^2 \end{aligned}$$

are referred to as the *Peirce projections* corresponding to u. It can be seen that $P_0(u) + P_1(u) + P_2(u)$ equals the identity id_A on A and that if $j \neq k$, then $P_j(u)P_k(u)$ equals zero. The ranges, $A_k(u)$ of $P_k(u)$ are weak*-closed subtriples of A, referred to as the *Peirce spaces* of u. Moreover, for all elements a of A,

$$a \in A_k(u) \quad \text{if and only if} \quad D(u,u)a = \frac{k}{2}a. \tag{2.2}$$

Hence, the Peirce spaces are the eigenspaces of $D(u,u)$, with eigenvalues 0, 1/2 and 1, respectively. Using these properties, the algebraic relations $u \perp v$ (u and v are *orthogonal*), $u \top v$ (u and v are *collinear*) and $u \vdash v$ (u *governs* v) are defined for elements u and v of $\mathcal{U}(A)$ as follows:

$$u \perp v :\Leftrightarrow u \in A_0(v) \quad \text{and } v \in A_0(u), \tag{2.3}$$

$$u \top v :\Leftrightarrow u \in A_1(v) \quad \text{and } v \in A_1(u), \tag{2.4}$$

$$u \vdash v :\Leftrightarrow u \in A_1(v) \quad \text{and } v \in A_2(u). \tag{2.5}$$

If the tripotents u and v are orthogonal then $u + v$ is a tripotent. Moreover, the conditions $u \in A_0(v)$ and $v \in A_0(u)$ (used in 2.3) are equivalent. A non-zero tripotent u is said to be *minimal* if it is not the sum of non-zero orthogonal tripotents. If A is a JBW*-triple, then u is minimal if and only if $Q(u)A = \mathbb{C}u$ [16]. A JBW*-triple is said to be *atomic* if it is the weak*-closed span of its minimal tripotents. In this case, each element a of A can be written as a (possibly infinite) weak*-convergent linear combination $a = \sum_{i \in I} \alpha_i u_i$ of pairwise orthogonal minimal tripotents u_i. If a itself is a tripotent, then the (non-zero) coefficients α_i are of unit modulus. The cardinality $|I|$ of the index set I depends only on the element a, and is referred to as the *rank* of a. Hence, the minimal tripotents are precisely those of rank one. The rank of the JBW*-triple A is the maximal rank of any of its elements. Let $\mathcal{U}_r(A)$ denote the set of tripotents having rank r.

Let $\mathrm{Aut}(A)$ be the group of all triple automorphisms of A. The subgroup of $\mathrm{Aut}(A)$ generated by the exponentials of $itD(a,a)$ is called the *inner automorphism group* denoted $\mathrm{Inn}(A)$. It is known that if $\dim(A) < \infty$ then $\mathrm{id}_A \in \mathrm{sp}_{\mathbb{R}}\{D(a\ a) : a \in A\}$, hence the torus $\mathbb{T} = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$ is a subgroup of $\mathrm{Inn}(A)$ [18]. This may not be the case in infinite dimensions. However, the orbit of a tripotent u is the same under $\mathbb{T}\mathrm{Inn}(A)$ as it is under $\mathrm{Inn}(A)$, i.e.,

$$\mathbb{T}\mathrm{Inn}(A)(u) = \mathrm{Inn}(A)(u). \tag{2.6}$$

Indeed, from (2.2) we see that $\mathbb{C}u \subseteq A_2(u)$ and that $D(u,u)|_{A_2(u)} = \mathrm{id}_{A_2(u)}$. Therefore $\exp itD(uu)(u) = e^{it}u$, which implies (2.6). Hence, for our purpose, using $\mathrm{Inn}(A)$ will not restrict the generality of the arguments. Observe that if B is a subtriple of A, then $\mathrm{Inn}(B)$ is a subgroup of $\mathrm{Inn}(A)$. Clearly $\varphi \in \mathrm{Aut}(A)$ preserves the set $\mathcal{U}(A)$ of tripotents and any relation between them which is defined in terms of the triple product. It also preserves the rank of tripotents.

For tripotents $u, v \in \mathcal{U}(A)$ we define the equivalence relation $u \sim v$ to hold if there exists $\varphi \in \mathrm{Inn}(A)$ such that $\varphi(u) = v$.

The following example illustrates the transitivity property of inner automorphisms. It will also be important in proving the main result.

Example 2.1 The Cartan factor $\mathbb{S} := \mathbb{M}_2^s(\mathbb{C})$ of symmetric 2×2-matrices is a unital Jordan-algebra of dimension 3, spanned by the minimal tripotents a, $b = I - a$, and the tripotent u given by

$$a = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad u = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

It is known that, since $\mathbb{S}$ is a finite dimensional triple factor, the inner automorphisms of $\mathbb{S}$ act transitively on the minimal tripotents [15]. Let's look at the details. Consider the inner derivation $G(a, u) := 2(D(u, a) - D(a, u))$ on $\mathbb{S}$, given by

$$G(a, u)\left(\begin{bmatrix} x & y \\ y & z \end{bmatrix}\right) = \begin{bmatrix} -2y & x - z \\ x - z & 2y \end{bmatrix}.$$

To find its exponential, identify G with the linear map on $\mathbb{C}^3$ by

$$G\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2y \\ x - y \\ 2y \end{bmatrix} = \begin{bmatrix} 0 & -2 & 0 \\ 1 & 0 & -1 \\ 0 & 2 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

Then, we obtain

$$\exp(tG) = \frac{1}{2}\begin{bmatrix} 1 + \cos(2t) & -2\sin(2t) & 1 - \cos(2t) \\ \sin(2t) & 2\cos(2t) & -\sin(2t) \\ 1 - \cos(2t) & \sin(2t) & 1 + \cos(2t) \end{bmatrix}.$$

Thus

$$\exp(tG)\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} \cos^2(t) & \sin(t)\cos(t) \\ \sin(t)\cos(t) & \sin^2(t) \end{bmatrix}. \tag{2.7}$$

This is a well known one-parameter family of inner automorphisms, with $\exp((\pi/2)G)a = b$. This shows that $a \sim b$. Since $\mathbb{S}$ is finite dimensional, Corollary 5.9 in [15] implies that any two tripotents of rank one are equivalent. We provide an explicit proof for this case. For elements x, y and z of $\mathbb{C}^2$, regarded as a Hilbert space with the usual inner product, the elementary operator $x \otimes y$ is defined by $x \otimes y(z) = \langle z, y\rangle x$ (see also the next section). It is enough to show that $a \sim x \otimes \bar{x}$, for any unit vector $x = \lambda e_1 + \mu e_2 \in H$. Hence, we need to show that

$$a = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \sim \begin{bmatrix} \lambda^2 & \lambda\mu \\ \lambda\mu & \mu^2 \end{bmatrix} = x \otimes \bar{x}.$$

When λ and μ are real then this is achieved by taking $\lambda = \cos t$ and $\mu = \sin t$. For complex λ, μ, notice that

$$D(a, a)\begin{bmatrix} x & y \\ y & z \end{bmatrix} = \begin{bmatrix} x & y/2 \\ y/2 & 0 \end{bmatrix}.$$

It follows that

$$\exp(it\ D(a,a))\begin{bmatrix} x & y \\ y & z \end{bmatrix} = \begin{bmatrix} e^{it}x & e^{it/2}y \\ e^{it/2}y & z \end{bmatrix}.$$

Choosing t such that $\exp(it/2) = \frac{\bar{\lambda}}{\lambda}$, we see that

$$x \otimes \bar{x} = \begin{bmatrix} \lambda^2 & \lambda\mu \\ \lambda\mu & \mu^2 \end{bmatrix} \sim \begin{bmatrix} |\lambda^2| & |\lambda|\mu \\ |\lambda|\mu & \mu^2 \end{bmatrix}.$$

A similar argument shows that

$$\begin{bmatrix} |\lambda^2| & |\lambda|\mu \\ |\lambda|\mu & |\mu^2| \end{bmatrix} \sim \begin{bmatrix} |\lambda^2| & |\lambda\mu| \\ |\lambda\mu| & |\mu^2| \end{bmatrix},$$

and so we can conclude that

$$x \otimes \bar{x} \sim \begin{bmatrix} |\lambda^2| & |\lambda\mu| \\ |\lambda\mu| & |\mu^2| \end{bmatrix} = \begin{bmatrix} \cos^2(t) & \sin(t)\cos(t) \\ \sin(t)\cos(t) & \sin^2(t) \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

A first step into the case of infinite dimensional factors is to deal with Hilbert spaces.

Example 2.2 A complex Hilbert space H with inner product $\langle .,.\rangle$ is a Cartan-factor of type I. The triple product is defined for $a, b, c \in H$ by

$$\{a,\ b,\ c\} = \frac{1}{2}\left(\langle a,\ b\rangle c + \langle c,\ b\rangle a\right). \tag{2.8}$$

It is easily seen that the non-zero tripotents are precisely the elements of norm one, and these are also minimal tripotents, i.e., of rank 1 in H. The relation of Hilbert-orthogonality is that of collinearity in terms of triple structure.

To make the arguments of this presentation more self-contained we give a proof of the transitivity property of Hilbert spaces. Let $S_1(H)$ denote the the unit-sphere of H. We may first assume that $\dim H = 2$. We may again refer to Loos' result in [15] to obtain the transitivity property for the finite dimensional case. The following calculations, like those in Example 2.1, are more elementary. Consider the basis vector b_1, b_2 and an arbitrary vector c of norm one, given by

$$b_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix},\quad b_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix},\quad c = \begin{bmatrix} \gamma_1 \\ \gamma_2 \end{bmatrix}$$

Let $\mathcal{B}_s = \{b_1, b_2\}$ be the standard basis of H. For any complex number λ of modulus one, let $a = a(\lambda)$ be the element of H, defined by

$$a = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ \lambda \end{bmatrix}.$$

Denote the operator $D(a,a)$ by D_λ, to indicate its dependence on λ. It follows from (2.8) that a is an eigenvector of D_λ. Any vector orthogonal to a is also an eigenvector of D_λ. It is easy to calculate the matrix of D_λ and its exponential from the corresponding diagonal forms. The operator $\exp itD_\lambda$, is explicitly given by

$$\exp itD_\lambda = \frac{1}{2}\begin{bmatrix} e^{it} + e^{it/2} & \bar{\lambda}(e^{it} - e^{it/2}) \\ \lambda(e^{it} - e^{it/2}) & e^{it} + e^{it/2} \end{bmatrix}.$$

In particular, the unit vector b_1 is mapped to

$$(\exp itD_\lambda)(b_1) = (\exp itD_\lambda)\begin{bmatrix}1\\0\end{bmatrix} = \frac{1}{2}\begin{bmatrix}e^{it}+e^{it/2}\\ \lambda(e^{it}-e^{it/2})\end{bmatrix}. \tag{2.9}$$

This vector is of norm one, for all reals t. The modulus of its first component is $|e^{it}+e^{\frac{i}{2}t}|/2$ and attains all values between 0 and 1 when t runs through $\mathbb{R}$. If c is any vector with components γ_1 and γ_2, and if c has norm one in H, then $|\gamma_2|$ equals $\sqrt{1-|\gamma_1|^2}$ and t can be chosen such that

$$\frac{1}{2}\left|e^{it}+e^{it/2}\right| = |\gamma_1|, \quad \text{and} \quad \frac{1}{2}\left|e^{it}-e^{it/2}\right| = |\gamma_2|. \tag{2.10}$$

If $\gamma_1 = 0$, then $\sigma(\exp 2\pi i D_\lambda)b_1$ equals c, for some $\sigma \in \mathbb{T}$. Otherwise set $\lambda = \gamma_2(e^{it}+e^{it/2})/\gamma_1(e^{it}-e^{it/2})$. The equations (2.10) and (2.9) imply that

$$|\lambda| = 1 = \left|\frac{2\gamma_1}{e^{it}+e^{it/2}}\right|, \quad \text{and} \quad \frac{2\gamma_1}{e^{it}+e^{it/2}}(\exp itD_\lambda)\begin{bmatrix}1\\0\end{bmatrix} = c.$$

Since a two-dimensional subspace of H is a subtriple, the above result shows the desired equivalence $b_1 \sim c$ in H.

The following theorem summarizes some results obtained in [10].

Theorem 2.3 *Let $\mathcal{C}$ be a collinear system in a JB*-triple A, and let H be the subspace $H = \overline{\mathrm{sp}\mathcal{C}}$ of A. Then, H is a subtriple of A if and only if, either $|\mathcal{C}| \leq 2$ or for any three distinct elements u, v, w of $\mathcal{C}$, the product $\{u\ v\ w\}$ vanishes. If this is the case, then the following results hold.*

(1.) *The subtriple H is a Hilbert space with orthonormal basis $\mathcal{C}$, and the triple product on H given by (2.8) coincides with the restriction of the triple product of A to H.*
(2.) *The set $\{\sigma\ \exp\ itD(a,a) : \ \sigma \in \mathbb{T},\ a \in H\}$ consists of linear isometric triple isomorphisms on the whole space A and acts transitively on the unit sphere $S_1(H)$ of H.*
(3.) *If A is a JBW*-triple, then H is also weak*-closed in A, and therefore, is a JBW*-subtriple of A.*

Since the set $\{\sigma\ \exp\ itD(a,a) : \ \sigma \in \mathbb{T},\ a \in H\}$ generates $\mathbb{T}\mathrm{Inn}(H)$, Theorems 2.3(2.) and (2.6) show that $\mathrm{Inn}(H)$ acts transitively on $S_1(H)$.

3 Cartan-factors

There are six types of simple, or irreducible atomic JBW*-triples, known as the Cartan-factors. They are refered to, respectively, as the *rectangular factors* which are of the form $\mathcal{B}(G,H)$ for Hilbert spaces G and H, the *hermitean factors* consisting of the elements of $\mathcal{B}(H)$ that are symmetric with respect to transposition, the *symplectic factors*, i.e., the space of the anti-symmetric elements of $\mathcal{B}(H)$, the *spin factors*, constructed from H, the *bi-Cayley triple* which is the space of 2×2-matrices with entries in the split Octonions $\mathbb{O}_s$, as well as the *Albert triple* consisting of the symmetric 3×3-matrices with entries in $\mathbb{O}_s$. Since the bi-Cayley triple and the Albert triple have (complex) dimension 16 and 27, respectively we will not include them in our current considerations. Instead we focus on the remaining four types, all of which include infinite-dimensional examples. In [16] the methods of coordinatization of JBW*-triples by grids was developed, and a classification of atomic JBW*-triples was given.

We briefly discuss some important, well known details concerning the infinite-dimensional cases. For general information on the subject we refer to the books [5,15,16]. Given complex Hilbert spaces H and K, the space $\mathcal{B}(H, K)$ of bounded operators from H to K is a JBW*-triple when equipped with the triple product

$$\{a, b, c\} = \frac{1}{2}(ab^*c + cb^*a). \tag{3.1}$$

Here, a^* denotes the usual adjoint of the element $a \in \mathcal{B}(H, K)$ [5,7]. A JBW*-triple of this form is a Cartan factor of type I, also referred to as rectangular type. Recall that for $x \in H$ and $y \in K$, the elementary operator $y \otimes x$, seen as an element of $\mathcal{B}(H, K)$ acts by contraction with the inner product of H by $(y \otimes x)(z) = \langle z, x\rangle y$. In particular if $a := y_1 \otimes x_1$ and $b := y_2 \otimes x_2$, for $x_1, x_2 \in H$ and $y_1, y_2 \in K$, then

$$ab^* = \langle x_2, x_1\rangle y_1 \otimes y_2, \quad b^*a = \langle y_1, y_2\rangle x_1 \otimes x_2. \tag{3.2}$$

Let $(h_i)_{i\in I}$ be an orthonormal basis of H and $(k_j)_{j\in J}$ an orthonormal basis of K. Then, for all $(i, j) \in I \times J$, the element $e_{i,j} := k_j \otimes h_i$ is a minimal tripotent of C, and $(e_{i,j})_{i,j\in I\times J}$ is a rectangular grid that spans C up to weak*-closure. The Cartan-factors of type II and III are subtriples of the type I factors $\mathcal{B}(H)$, and are described by a fixed conjugation $x \mapsto \bar{x}$ on H. For $a \in \mathcal{B}(H)$ we set $a^t = \overline{a^*\bar{x}}$. Type II and III factors are the subspaces of $\mathcal{B}(H)$ consisting of elements with the properties that $a = a^t$ and $a = -a^t$, respectively. An element a of A is said to be *minimal* if $\{a\ A\ a\} = \mathbb{C}a$. In [7] the minimal elements of Cartan-factors were described in terms of the elementary operators. In our notation, the minimal elements of type I, II, and III factors are the sets $\{x \otimes y : x, y \in H,\ \ y \in K\}$, $\{x \otimes \bar{x} : x \in H\}$ and $\{x \otimes \bar{y} - y \otimes \bar{x} : x, y \in H\}$, respectively. For more details see [7].

For any non-zero cardinal number c a spin factor $S(c)$ is obtained from a Hilbert space H with dimension c, by setting $S = H \oplus H'$ or $S = \mathbb{C} \oplus H \oplus H'$. The former are the *even spin factors*, the later are the *odd spin factors*. The spaces H and H' are of equal dimension and are related by conjugations described below. The inner product $\langle ., .\rangle$ of H extends to S by setting

$$\langle(\alpha, x_1, y_1), (\beta, x_2, y_1)\rangle = \alpha\overline{\beta} + \langle x_1, x_2\rangle + \langle y_1, y_2\rangle.$$

For the purpose of coordinatization, we use an orthonormal basis $B = \{e_k\}_{k\in K}$ of H. The mapping $e_j \mapsto e_i'$ is a symmetry between H and H', and we define $e_0' = e_0$. The involution $* : S \to S$ is the conjugate-linear involutive map determined by

$$e_j^* = e_j', \quad e_0^* = e_0 \quad (i \in I). \tag{3.3}$$

The triple product of S is given by

$$2\{a\ b\ c\} = \langle a, b\rangle_S c + \langle c, b\rangle_S a - \langle a, c^*\rangle_S b^* \tag{3.4}$$

It will be convenient to write an element a of S as a complex number γ, combined with a column of elements x and y of H, which will be written as rows, i.e., $x = (\alpha_1, \alpha_2, \ldots)$ and $y = (\beta_1, \beta_2, \ldots)$, for $\alpha_k, \beta_k \in \mathbb{C}$, $k \in K$. Then, $a \in S$ is given by

$$a = \gamma \oplus \begin{bmatrix} x \\ y \end{bmatrix} = \gamma \oplus \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \ldots \\ \beta_1 & \beta_2 & \beta_3 & \ldots \end{bmatrix}.$$

This representation has the advantage that the orthogonal matrix units appear in the same column. Any other pairing of matrix units provides a collinear pair, that is $e_k \top \{e_j, e'_k\}$, for $j \neq k$. We define e_0 to be zero if S is an even spin factor, otherwise e_0 is the unit $e_0 = (1, 0, 0) \in S$. Note that the index 0 is not an element of K.

The Cartan-factor $\mathbb{S}$ presented in Example 2.1 is both a hermitean Cartan-factor and a spin factor. As a spin factor, it is obtained by setting $H = \mathbb{C}$, and $\mathbb{S} = \mathbb{C}e_0 \oplus H \oplus H$. Then $e_0 = (1, 0, 0)$, $e_1 = (0, 1, 0)$ and $e'_1 = (0, 0, 1)$. It is easily seen that $S(3)$ is isomorphic to $\mathbb{S}$ presented in Example 2.1. An application of Theorem 2.3 gives the following result.

Lemma 3.1 *Let a be an element of the spin factor $S(3)$ as defined above. Then there exists an inner automorphism $\varphi \in \mathrm{Inn}(S(3))$ such that $\varphi(a) = 0e_0 + \alpha e_1 + \beta e'_1$, i.e., the component in e_0 vanishes. Moreover, φ can be chosen such that $\alpha = 1$, or such that $\beta = 1$.*

Proof Since a non-trivial spin factor has rank two, the element $a \in S(3)$ can be written as a linear combination of two orthogonal minimal tripotents, i.e., $a = \alpha u_1 + \beta u_2$, for elements $u_1, u_2 \in \mathcal{U}_1(S(3))$ which are such that $u_1 \perp u_2$, and $\alpha, \beta \in \mathbb{C}$. Since $S(3)$ is a finite dimensional triple factor, Example 2.1 shows that $\mathrm{Inn}(S(3))$ acts transitively on $\mathcal{U}_1(S(3))$. In particular, there exists $\varphi \in \mathrm{Inn}(S(3))$, such that $\varphi(u_1) = e_1$. Since the set of all tripotents w satisfying $e_1 \perp w$ is $\mathbb{T}e'_1$, and φ preserves orthogonality, it follows that $\varphi(u_2) = \lambda e'_1$, for some $\lambda \in \mathbb{T}$. The inner automorphism $\psi = \exp itD(e'_1, e'_1)$ is such that $\psi(e_0) = e^{it/2} e_0$, $\psi(e_1) = e_1$, and $\psi e'_1 = e^{it} e'_1$. Hence, for an appropriate choice of $t \in \mathbb{R}$, the inner automorphism $\theta = \psi \circ \varphi$ has all the required properties. □

4 Main results

With the preparations given in the previous sections, we can now proceed to establish the announced main results of this article. We treat each case of infinite dimensional Cartan-factor separately. The general strategy of the proof is the same in each case. The result in the special cases $\mathbb{S}$ and Hilbert-spaces is used repeatedly.

Theorem 4.1 *Let C be an infinite dimensional Cartan factor. Then the group $\mathrm{Inn}(C)$ acts transitively on the set $\mathcal{U}_1(C)$ of minimal tripotents of C.*

Proof **Type I:** Let the Cartan-factor C be represented on Hilbert spaces H and K. We assume that the dimensions of H and K are at least 3. The proof for the cases in which $\dim H \leq 2$ or $\dim K \leq 2$ is a simplifed version of this proof. Choose orthonormal bases $\{h_i\}_{i \in I}$ and $\{k_j\}_{j \in J}$ of H and K, respectively. It is sufficient to fix an index $(i_0, j_0) \in I \times J$ and to provide an inner automorphism φ which sends the element $v := e_{i_0, j_0} = k_{j_0} \otimes h_{i_0}$ to an arbitrary rank one tripotent u. It is known that the tripotents of $\mathcal{B}(H, K)$ are precisely the partial isometries from H to K. Hence $p = u^* u$ and $q = uu^*$ are orthoprojections on H and K respectively. By Harris [7], there are elements x in H and y in K with $\|x\| = \|y\| = 1$, such that

$$u = y \otimes x = \sum_{i \in I} \langle h_i, x \rangle \; y \otimes h_i .$$

Consider distinct indices r and s in I. The triple products of these elements are determined by the relations (3.1) and (3.2). From these we obtain,

$$\begin{aligned}\{y \otimes h_r, y \otimes h_r, y \otimes h_s\} &= \frac{1}{2}(y \otimes h_r \cdot h_r \otimes y \cdot y \otimes h_s + y \otimes h_s \cdot h_r \otimes y \cdot y \otimes h_r) \\ &= \frac{1}{2} y \otimes h_s .\end{aligned}$$

A similar calculation shows that, for distinct r and s in J and i in I we have,

$$\{k_r \otimes h_i,\ k_r \otimes h_i,\ k_s \otimes h_i\} = \frac{1}{2} k_s \otimes h_i.$$

It can be seen that for three distinct indices, we have

$$\{y \otimes h_r,\ y \otimes h_r,\ y \otimes h_s\} = 0,$$
$$\{k_r \otimes h_i,\ k_r \otimes h_i,\ k_s \otimes h_i\} = 0.$$

Hence, the collections $(y \otimes h_i)_{i \in I}$ and $(k_j \otimes h_i)_{j \in J}$ are collinear systems in C and, by Theorem 2.3, their respective closed spans are Hilbert spaces and subtriples of A. In particular $u = y \otimes x$ is a norm one element in the span of $(y \otimes h_i)_{i \in I}$, and $y \otimes h_{i_0}$ is a norm one element in the span of $(k_j \otimes h_{i_0})_{j \in J}$. Hence by Theorem 2.3 (2.), there are mappings φ_1 and φ_2 in $\mathrm{Inn}(C)$ such that $\varphi_1(e_{i_0,\, j_0}) = y \otimes h_{i_0}$, and $\varphi_2(y \otimes h_{i_0}) = u$. This proves that the relation $v \sim u$ holds.

Type II: Suppose that C is a Cartan-factor of type II with minimal tripotents $a = x \otimes \bar{x}$ and $b = y \otimes \bar{y}$, where $x, y \in H$ have unit norm. Let $\perp_H$ denote the usual orthogonality in a Hilbert-space. Assume first that $x \perp_H y$, which is equivalent to saying that a and b are orthogonal tripotents. It is easily checked that $v = x \otimes \bar{y} + y \otimes \bar{x}$ is a tripotent that lies in the Peirce-$\frac{1}{2}$ space of both a and b. Moreover, the space $\mathrm{sp}\{a, b, v\}$ is closed under triple products, as is verified by the identities

$$\begin{aligned}
\{a,\ a,\ a\} &= \ a, & \{a,\ v,\ a\} &= \ 0,\\
\{a,\ a,\ v\} &= \frac{1}{2} v, & \{a,\ v,\ v\} &= \frac{1}{2} v,\\
\{a,\ a,\ b\} &= \frac{1}{2} b, & \{a,\ v,\ b\} &= \frac{1}{2} b,
\end{aligned} \tag{4.1}$$

From the relations (4.1) it is seen that the linear map $J : C \to \mathbb{S}$ determined by

$$J(a) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad J(b) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad J(v) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{4.2}$$

is a triple-isomorphism. This together with the transitivity property of $\mathbb{S}$ guarantees that $a \sim b$. What if x and y are not orthogonal? If $\dim H \geq 3$, then there exists $z \in x^{\perp_H} \cap y^{\perp_H}$, and, by the aforesaid it follows that $a \sim z \otimes \bar{z} \sim b$. We are left with the case when $\dim H = 2$ and $C = \mathbb{S}$, which has been dealt with in Example 2.1.

Type III: Unlike in Type II factors, minimal tripotents in Type III factors may be collinear. This allows us to employ arguments similar to those used for Type I. We assume the Cartan factor C to be modeled on the Hilbert space H which has orthonormal basis $\{e_i\}_{i \in I}$, and that for each $i \in I$, $\bar{e}_i = e_i$. Although we may denominate elements as e_1 and e_2, there is no assertion that H is separable. It is easily checked that for any fixed $i \in I$, the set $\mathcal{C}_i := \{e_i \otimes e_j - e_j \otimes e_i :\ j \in I \backslash \{i\}\}$ of minimal tripotents is a collinear system that satisfies the assumption of Theorem 2.3. Remark that, by Harris [7], the minimal elements in a type III factor are of the form $x \otimes y - \bar{y} \otimes \bar{x}$. Furthermore, an easy calculation shows that $x \otimes y - \bar{y} \otimes \bar{x}$ is a tripotent if and only if

$$\|x\|^2 \|y\|^2 - |\langle \bar{x}, y \rangle|^2 = 1. \tag{4.3}$$

If $\bar{x}\perp_H y$ then we can and will assume that $\|x\| = \|y\| = 1$. Note also that if $c = x\otimes y-\bar{y}\otimes\bar{x}$ is a minimal tripotent then by writing $y = \alpha\bar{x} + z$, with $z\perp_H\bar{x}$, we see that

$$\begin{aligned} c = x \otimes y - \bar{y} \otimes x &= x \otimes (\alpha\bar{x} + z) - (\bar{\alpha}x + \bar{z}) \otimes \bar{x} \\ &= x \otimes z - \bar{z} \otimes \bar{x}. \end{aligned} \tag{4.4}$$

Therefore, c can be represented by orthogonal vectors, i.e.,

$$\bar{x} \perp_H z. \tag{4.5}$$

Using (4.3) and (4.5) we assume that the elements $x, y \in H$, which represent the minimal tripotent $c = x \otimes y - \bar{y} \otimes \bar{x}$ have unit norm and are such that $\bar{x} \perp_H y$.

It needs to be shown that

$$c = x \otimes y - \bar{y} \otimes \bar{x} \sim e_1 \otimes e_2 - e_2 \otimes e_1.$$

First, we know that $\mathcal{C}_1 = \{e_1 \otimes e_j - e_j \otimes e_1 : j \in I\backslash\{1\}\}$ is a collinear system which satisfies the assumptions of Theorem 2.3. Write $y = \alpha e_1 + z$, with $z \perp_H e_1$. Suppose that $z \neq 0$. Then the element $(e_1 \otimes z - \bar{z} \otimes e_1)/\|z\|$ is a minimal tripotent contained in $\overline{\mathrm{sp}}\mathcal{C}_1$. Theorem 2.3 and (4.4) imply that

$$e_1 \otimes \frac{y}{\|z\|} - \frac{\bar{y}}{\|z\|} \otimes e_1 = e_1 \otimes \frac{z}{\|z\|} - \frac{\bar{z}}{\|z\|} \otimes e_1 \sim e_1 \otimes e_2 - e_2 \otimes e_1. \tag{4.6}$$

If $z = 0$, then $|\alpha| = 1$. Notice that $c \sim -c$, and use the same argument to conclude that

$$c \sim -c = \bar{\alpha}e_1 \otimes \bar{x} - x \otimes \alpha e_1 = e_1 \otimes \alpha\bar{x} - \bar{\alpha}x \otimes e_1 \sim e_1 \otimes e_2 - e_2 \otimes e_1.$$

This finishes the proof in case $z = 0$.

Since $\bar{x} \perp_H y$ we can set $h_1 = x$, $h_2 = \bar{y}$ and extend the set $\{h_1, h_2\}$ to an orthonormal basis $\{h_i\}_{i\in I}$ of H. We write e_1 as $e_1 = \alpha h_2 + a$, with $a \perp_H h_2$. Suppose that $a \neq 0$. Notice that $\mathcal{D}_2 = \{h_i \otimes h_2 - \bar{h}_2 \otimes \bar{h}_i : i \in I\backslash\{2\}\}$ is a collinear system that satisfies the assumption of (2.) in Theorem 2.3, and $(a \otimes h_2 - \bar{h}_2 \otimes a)/\|a\|$ is a tripotent in $\overline{\mathrm{sp}}\mathcal{D}_2$. These facts and (4.6) imply that

$$e_1 \otimes \frac{y}{\|a\|} - \frac{\bar{y}}{\|a\|} \otimes e_1 = e_1 \otimes \frac{a}{\|a\|} - \frac{\bar{a}}{\|a\|} \otimes e_1 \sim x \otimes y - \bar{y} \otimes \bar{x}. \tag{4.7}$$

The case in which $a = 0$ is easily dealt with in a similar manner as the case $z = 0$ above. The relations (4.7) and (4.6) provide the desired equivalence.

Type IV: With the notation of Sect. 3 the minimal tripotents e_k and e'_k form an ortho-collinear standard grid $\{e_k\}_{k\in K} \cup \{e'_k\}_{k\in K}$ in S. By a standard summability argument, elements x and y of S can have at most countably many nonzero coordinates with respect to B of H. Therefore, the coordinates of x and y can be labeled by $\mathbb{N}$. The identities that define a spin triple imply that, for distinct indices j and k in K,

$$e_k \perp e'_k, \quad e_k \top e_j, \quad e_k \top e'_j. \tag{4.8}$$

Moreover, in the case when $e_0 \neq 0$, we have

$$e_0 \vdash e_k, \quad \vdash e'_k \tag{4.9}$$

Let u be an arbitrary element of $\mathcal{U}_1(S)$. It is enough to show that there is an inner automorphism φ such that $\varphi(u) = e_1$. In our matrix representation, the elements u and e_1 are given by

$$u = \delta e_0 \oplus \begin{bmatrix} \lambda_1 & \lambda_2 & \lambda_3 & \dots \\ \mu_1 & \mu_2 & \mu_3 & \dots \end{bmatrix}, \quad e_1 = 0 \oplus \begin{bmatrix} 1 & 0 & 0 & \dots \\ 0 & 0 & 0 & \dots \end{bmatrix}, \quad e'_1 = 0 \oplus \begin{bmatrix} 0 & 0 & 0 & \dots \\ 1 & 0 & 0 & \dots \end{bmatrix}.$$

As before, it is shown that the component δe_0 vanishes by applying an appropriate inner automorphism. Denote by P_0, P_1 and $P_{1'}$ the coordinate projections on S onto the one-dimensional subspaces $\mathbb{C}e_0$, $\mathbb{C}e_1$ and $\mathbb{C}e'_1$, respectively. It is straightforward from the definition of the triple product (3.4) that the subspace $T = \mathbb{C}e_0 \oplus \mathbb{C}e_1 \oplus \mathbb{C}e'_1$ of S is a subtriple of S, isomorphic to the triple factor $S(3)$ described in Lemma 3.1. Moreover, T is the range of the projection $P_T := P_0 + P_1 + P_{1'}$ on S. Let e_k be an element of $B \setminus \{e_0, e_1, e'_1\}$. For $a, b \in \{e_0, e_1, e'_1\}$, expressions for $D(a, b)e_k$ are obtained from (3.4). If $a \neq b$, then

$$D(a, b)e_k = 0. \tag{4.10}$$

If $a = b$, it can be seen from (3.4), or from (4.8) and (4.9) that,

$$D(e_0, e_0)e_k = e_k, \tag{4.11}$$

$$D(e_1, e_1)e_k = D(e'_1, e'_1)e_k = \frac{1}{2}e_k. \tag{4.12}$$

These show that $P_T D(a, b)e_k = D(a, b)P_T e_k = 0$. For $e_k \in \{e_0, e_1, e'_1\}$, we have $D(a, b)$ $e_k \in T$. It follows that P_T commutes with all elements of $\mathrm{Inn}(T)$ which is a subgroup of $\mathrm{Inn}(S)$. By Lemma 3.1, there exists an element θ in $\mathrm{Inn}(T)$ which annihilates the component $P_0(u) = \delta e_0$ of $P_T u$ in T. We conclude that

$$0 = P_0\, \theta\, P_T(u) = P_0\, P_T\, \theta(u) = P_0\, \theta(u). \tag{4.13}$$

In what follows, the spaces H and H' are identified with the subtriples $0 \oplus H \oplus 0$ and $0 \oplus 0 \oplus H$ of S, respectively. We will show that, for $a, b \in H$, the operator $D(a, b)$ commutes with the coordinate projection P_0 and $P_{H'}$ onto the spaces $\mathbb{C}e_0$ and H'. In the case of H', we can set $e_0 = 0$ and omit this component altogether. It will be assumed that $|K| \geq 3$. The calculations for $|K| \leq 2$ can be simplified in an obvious way. Consider distinct indices j, k, l in K. The definition of the triple product (3.4) and the relations (4.8) show that, for $s \in S$,

$$D(s, e_1)e'_1 = 0, \tag{4.14}$$

$$D(e_k, e_k)e'_j = \frac{1}{2}e'_j, \tag{4.15}$$

$$D(e_k, e_l)e'_j = 0. \tag{4.16}$$

For an arbitrary element a of H, the operator $D(a, a)$ is a linear combination of $D(e_m, e_n)$ $(m, n \in K)$. The above equations show that $D(a, a)b \in H'$, for all $b \in H'$. Since H' is a closed subspace, this implies that

$$\exp itD(a, a)H' \subseteq H', \tag{4.17}$$

hence, that H' is invariant under $\mathrm{Inn}(H)$. Now, suppose that e_0 is not zero. Then, for each index $k \in K$, the elements e_0 and e_k are in the relation $e_0 \vdash e_k$ for all $k \in K$. This implies that

$$D(e_k, e_k)e_0 = \frac{1}{2}e_0,$$

$$D(e_k, e_j)e_0 = 0 \quad (\text{for } k \neq j).$$

It follows that, for all $a \in H$,

$$\exp itD(a, a)e_0 \in \mathbb{C}e_0, \tag{4.18}$$

and, hence, that $\mathbb{C}e_0$ is also invariant under $\mathrm{Inn}(H)$. From (4.13) and (4.18) we see that

$$P_0 \varphi \theta(u) = 0. \tag{4.19}$$

Hence from now on, the component in e_0 is assumed to be zero, and is omited. Denote by P_H the canonical projection from S onto H. From the above arguments it can be seen that, for all elements a and b of $H \subseteq S$, and all $\varphi \in \mathrm{Inn}(H)$,

$$P_H \; D(a,b) = D(a,b) \; P_H, \tag{4.20}$$

$$P_H \; \varphi = \varphi \; P_H. \tag{4.21}$$

Theorem 2.3 and (4.21) imply that there exists $\varphi_0 \in \mathrm{Inn}(H) \subseteq \mathrm{Inn}(S)$ such that

$$\varphi_0(P_H\theta(u)) = P_H\varphi_0(\theta(u)) \in \mathbb{C}e_1.$$

From this, (4.19) and (4.21) it follows that

$$a := \varphi_0(\theta u) = \begin{bmatrix} \alpha & 0 & 0 & \dots \\ \beta_1 & \beta_2 & \beta_3 & \dots \end{bmatrix} \in \mathcal{U}_1(S).$$

To proceed, suppose that $\alpha \neq 0$. By (3.4) it can be seen that

$$a^3 = \{a\ a\ a\} = \left(\alpha\overline{\alpha} + \sum_{k\in K} \beta_k\overline{\beta}_k\right)\begin{bmatrix} \alpha & 0 & 0 & \dots \\ \beta_1 & \beta_2 & \beta_3 & \dots \end{bmatrix} - \alpha\beta_1 \begin{bmatrix} \overline{\beta}_1 & \overline{\beta}_2 & \dots \\ \overline{\alpha} & 0 & \dots \end{bmatrix} = a.$$

This equation shows that, if $\beta_1 \neq 0$ then, to match the zero-components in the top row of a it is necessary that $\beta_k = 0$, for $k \neq 1$. Matching all the remaining components of a entails that $\alpha\overline{\alpha} = \beta_1\overline{\beta}_1 = 1$. But then, $a = \alpha e_1 \oplus \beta_1 e'_1$, and, since $e_1 \perp e'_1$, the tripotent a has rank two, in contradiction to the assumption. Insisting on $\alpha \neq 0$, we conclude that $\beta_1 = 0$. Using again (3.4) it is easy to verify that $\mathcal{C} = \{e_1\} \cup \{e'_k\}_{k\in K\setminus\{1\}}$ is a collinear system such that, for any three distinct elements $u, v, w \in \mathcal{C}$, the product $\{u\ v\ w\}$ vanishes. By Theorem 2.3 there exists $\varphi_1 \in \mathrm{Inn}(S)$ such that $\varphi(u) = \varphi_1(a) = \varphi_1 \circ \varphi_0 \circ \theta(u) = e_1$, as required.

The case when $\alpha = 0$ is easily dealt with in a similar way, by applying Theorem 2.3 to the Hilbert space and subtriple H' of S. We can find $\varphi_1 \in \mathrm{Inn}(S)$ such that $\varphi_1(a) = e'_2$. Since $e_1 \top e'_2$, the subspace $\mathbb{C}e_1 \oplus \mathbb{C}e'_2$ is a Hilbert space and a subtriple of S. The previous argument finishes the proof. □

In the remainder of this section we present some corollaries and generalizations of Theorem 4.1.

Recall that the rank of a JBW*-triple A is the maximal cardinality of an orthogonal family in $\mathcal{U}(A)$, and is denoted by $\mathrm{rank}(A)$. Notice that any (non-trivial) spin factor has rank 2. If A is a subtriple of $\mathcal{B}(H, K)$ then

$$\mathrm{rank}(A) \leq \min\{\dim(H),\ \dim(K)\}.$$

A further observation concerning the Peirce-0-space $C_0(u)$ (or equivalently the orthogonal complement) of a minimal tripotent u of A is given next.

Proposition 4.2 *Let C be an Cartan-factor. Let $u \in \mathcal{U}(C)$ be a minimal tripotent. Then the Peirce-0-space $C_0(u) = u^\perp$ of u has the following properties: If C is of type I, II, or III, then $C_0(u)$ is of the same type. If C is of type IV, then $C_0(u)$ is one-dimensional.*

Proof Let u be an arbitrary minimal tripotent. Using the representations of C as subtriples of $A = \mathcal{B}(G, H)$, we may fix a particular minimal tripotent, e.g., $v = h_1 \otimes k_1$, and note that $C_0(v) = \mathcal{B}(H_1, K_1)$, where H_1 and K_1 are the ortho-complements of the basis vectors h_1 and k_1 in H and K, respectively. Since there is an automorphism φ of C with $\varphi(u) = v$, it follows that $C_0(u) = A_0(u)$ is isomorphic to $C_0(v)$. This proves the statement for the case of type I factors. The argument works for the remaining cases. For type II and type III factors (where $H = K$), we set $v = h_1 \otimes h_1$, or $v = h_1 \otimes h_2 - h_2 \otimes h_1$, and we observe that $C_0(v) = A_0(v) \cap C$. For a type IV factor $C = e_0 \oplus H \oplus H'$ we can set $v = e_1$. Then $C_0(v) = e_1' \simeq C_0(u)$. □

Theorem 4.1 holds for all finite rank-classes in $\mathcal{U}(C)$, as shown next. This result, in particular (1.), is therefore a more elaborate version of Theorem 4.1. It further generalizes the classical results in [15], as well as those in [6].

Theorem 4.3 *Let C be a Cartan-factor (of arbitrary dimension). Then the following results hold.*

(1.) *Let $U = \{u_1, \dots u_n\}$ and $V = \{v_1, \dots, v_n\}$ be orthogonal subsets of minimal tripotents in a Cartan-factor C (of arbitrary dimension). Then there exists a mapping $\varphi \in \mathrm{Inn}(C)$ such that, $\varphi(u_k) = v_k$.*

(2.) *Let u and v be tripotents of finite rank in C. Then the relation $u \sim v$ holds if and only if* $\mathrm{rank}(u) = \mathrm{rank}(v)$.

Proof **(1.):** In the case when C is finite-dimensional, the statement follows from [15], Theorem 5.9. Hence, we assume that C is infinite-dimensional. From Theorem 4.1 it can be seen that there exists $\varphi_1 \in \mathrm{Inn}(C)$ with the property that $\varphi_1(u_1) = v_1$. Since φ_1 is a triple-automorphism, it follows that $\varphi_1(u_2) \perp v_1$. Hence the elements $\varphi_1(u_2)$ and v_2 are both contained in the Peirce-0-space $A_0(v_1)$ of v_1. Since, by Proposition 4.2, $A_0(v_1)$ is itself a Cartan-factor, the argument can be repeated for the sets $\varphi_1(U \backslash \{u_1\})$ and $V \backslash \{v_1\}$. The desired automorphism is given by $\varphi = \varphi_n \varphi_{n-1} \dots \varphi_1$.

(2.): This result extends [15], Corollary 5.12 to the infinite-dimensional cases. It is clear that the condition $\mathrm{rank}(u) = \mathrm{rank}(v)$ is necessary for $u \sim v$ to hold. On the other hand, if $\mathrm{rank}(u) = \mathrm{rank}(v) =: r < \infty$ then there exists orthogonal subsets $\{u_1, \dots u_r\}$ and $\{v_1, \dots, v_r\}$ of $\mathcal{U}_1(C)$, such that $u = \sum_{n=1}^{r} u_n$ and $v = \sum_{n=1}^{r} v_n$. Applying the result (1.) shows that $u \sim v$, as required. □

5 Remarks on weak*-operator limits

Dealing with infinite JBW*-triples C, it is natural to ask how the actions of $\mathrm{Inn}(C)$ behaves in relation to the weak*-topology. We therefore conclude this paper with some observations concerning limits of sequences in $\mathrm{Inn}(C)$. It is clear that taking such limits will pose some problems regarding the regular properties of $\mathrm{Inn}(C)$. It is beyond the scope of this paper to address these problems in great depth. Instead we provide some further rather straightforward consequences of the main results. These may also indicate possible directions of further research into the connections between limits of products in $\mathrm{Inn}(C)$ and more appropriate topologies. For example, the last theorem of this article provides a ‘positive’ and a ‘negative’ result concerning the SOT-closure and the weak*-closure of $\mathrm{Inn}(C)$. It is therefore regarded as tentative, and it leads to the open problem of establishing more appropriate generalizations of the main result Theorem 4.1. The result (2.) therein is a consequence of the well known fact that the unit-ball is the weak*-closure of the unit-sphere.

Recall that any (linear) topology τ on C gives rise to the correspoding τ*-operator* topology which is defined as follows. A net $\{P_i\}_{i \in I}$ in $\mathcal{B}(C)$ is said to converge in the τ-operator topology if, for each $a \in C$, the net $\{P_i a\}_{i \in I}$ is τ-convergent. For any set I, let I^{fin} denote the set of all finite subsets of I, partially ordered by set-inclusion. Then, $\{P_i\}_{i \in I}$ is said to be τ-operator summable if the net $\{\sum_{i\in F} P_i\}_{F\in I^{fin}}$ is τ-operator convergent. We will use the weak*-topology $\tau = \sigma(C, C_*)$ on C or the norm topology on C_*. The norm-operator topology is the well known *strong operator* topology (SOT). For any set F, $l^2(F)$ denotes the Hilbert-space of l^2-summable functions $f : F \to \mathbb{C}$. Given any two index sets F and G, denote by $C_{F\times G}$ the type I factor $C_{F\times G} := \mathcal{B}(l^2(F), l^2(G))$. The next two results are of technical nature. They are obtained from standard methods in operator theory.

Lemma 5.1 *Let $C := C_{I\times J}$ be the type I Cartan-factor $\mathcal{B}(H, K)$, parametrized by the index sets I and J and with corresponding standard grid $\mathcal{G} := \{e_{i,j} := h_i \otimes k_j : (i, j) \in I \times J\}$ (as in Sect. 3). For $(i, j) \in I \times J$, let $P_{i,j}$ be the canonical projection from C to $\mathbb{C}e_{i,j}$, with pre-adjoint $P_{i,j*}$ on the predual C_* of C. Then, for subsets F of I and G of J, the family $\{P_{i,j*}\}_{(i,j)\in F\times G}$ is SOT-summable in $\mathcal{B}(C_*)$, and $\{P_{i,j}\}_{(i,j)\in F\times G}$ is $\sigma(C, C_*)$-operator summable in $\mathcal{B}(C)$.*

Proof For all (finite) subsets F of I and G of J, the projection $P_{F\times G} := \sum_{k\in F\times G} P_k$ on C is weak*-continuous and contractive. Any partition $F = F_1 \cup F_2$ (with $F_1 \cap F_2 = \emptyset$) of F provides a grading

$$C_{F\times G} = C_{F_1\times G} \oplus C_{F_2\times G}.$$

Accordingly, $b \in C_{F\times G}$ is written as $b = b_1 \oplus b_2$, for $b_1 \in C_{F_1\times G}$ and $b_2 \in C_{F_1\times G}$. Choose a norm one element $\xi = \xi_1 + \xi_2$ in $l^2(F)$, with $\xi_1 \in l^2(F_1)$, $\xi_2 \in l^2(F_2)$. Then $1 = \|\xi_1\|^2 + \|\xi_2\|^2$. Let α_j and β_j $(j = 1, \dots, |G|)$ be the standard coordinates of $b_1\xi$ and $b_2\xi$ in $l^2(G)$. Then,

$$\begin{aligned}\|b\|^2 &\le \|(b_1 + b_2)\xi\|^2 = \|b_1\xi_1 + b_2\xi_2\|^2 \\ &= \sum_{j=1}^{|G|} |\alpha_j + \beta_j|^2 \le \sum_{j=1}^{|G|} |\alpha_j|^2 + |\beta_j|^2 = \|b_1\xi_1\|^2 + \|b_2\xi_2\|^2 \\ &\le \|b_1\|^2\|\xi_1\|^2 + \|b_2\|^2\|\xi_2\|^2 \le \|b_1\|^2 + \|b_2\|^2. \end{aligned} \tag{5.1}$$

The Hahn-Banach theorem implies that for each $x \in P_{F\times G}C_*$, there exists $b_1 \in P_{F_1\times G}C$ and $b_2 \in P_{F_2\times G}C$ with $\|b_1\| = \|b_2\| = 1$ and

$$(b_1 \cdot P_{F_1\times G}x) = \|P_{F_1\times G}x\|, \quad (b_2 \cdot P_{F_2\times G}x) = \|P_{F_2\times G}x\|.$$

We can assume that not both of these expressions vanish simultaneously. Define the elements b_1' and b_2' by

$$b_1' = \frac{\|P_{F_1\times G}x\|b_1}{(\|P_{F_1\times G}x\|^2 + \|P_{F_2\times G}x\|^2)^{\frac{1}{2}}}, \quad b_2' = \frac{\|P_{F_2\times G}x\|b_2}{(\|P_{F_1\times G}x\|^2 + \|P_{F_2\times G}x\|^2)^{\frac{1}{2}}}.$$

Then, by (5.1),

$$\|b_1' + b_2'\|^2 \le \|b_1'\|^2 + \|b_2'\|^2 = 1.$$

Moreover,

$$|((b_1' + b_2') \cdot (P_{F_1\times G}x + P_{F_2\times G}x))|^2 = \|P_{F_1\times G}x\|^2 + \|P_{F_2\times G}x\|^2$$

This and the contractivity of $P_{F\times G}$ implies that

$$\|x\|^2 \ge \|P_{F\times G}x\|^2 = \|P_{F_1\times G}x + P_{F_2\times G}x\|^2 \ge \|P_{F_1}x\|^2 + \|P_{F_2}x\|^2. \tag{5.2}$$

To see that $\left\{\sum_{k\in F} R_k x\right\}_{F\in I^{fin}}$ is a Cauchy-net in the norm of C, consider any $\varepsilon > 0$, and let the mapping $f : I^{fin} \to \mathbb{R}_+$ be defined by $f(F) := \|P_{F\times G}x\|^2$. By (5.2), the constant M_x, defined by $M_x := \sup\{f(F) : F \in I^{fin}\}$ is finite. There exists $F_0 \in K^{fin}$, such that $M_x - \varepsilon^2 \leq f(F_0)$. The relations (5.2) show that, for any $G \in K^{fin}$ which is such that $F_0 \cap G = \emptyset$, we have

$$M_x - \varepsilon^2 \leq f(F_0) \leq f(F_0 \cup G) - \tilde{f}(G).$$

It follows that

$$f(G) = \left\|\sum_{k\in G} P_k x\right\|^2 \leq \varepsilon^2,$$

and hence that $\|\sum_{k\in G} P_k x\| \leq \varepsilon$. This shows the desired Cauchy property, and hence the SOT-convergence of $\{P_{F\times G}\}_{F\in I^{fin}, G\in J^{fin}}$. We define the sum P to be the corresponding SOT -limit. It follows that for all $a \in C$ and all $x \in C_*$,

$$\lim_{F\to\infty}\left(\sum_{k\in F} P_k a \cdot x\right) = \lim_{F\to\infty}\left(a \cdot \sum_{k\in F} P_{k*} x\right) = (b \cdot Px).$$

This means precisely that $(P_k a)_{k\in K}$ is $\sigma(C, C_*)$-summable, for each $a \in E^*$, hence that $(P_k)_{k\in K}$ is $\sigma(C, C_*)$-operator summable. □

Corollary 5.2 *Let $\{F_n\}_{n\in\mathbb{N}}$ and $\{G_n\}_{n\in\mathbb{N}}$ be sequences of subsets $F_n \subseteq I$ and $G_n \subseteq J$, with the property that $F_{n+1} \subseteq F_n$, $G_{n+1} \subseteq G_n$, $\bigcap_{n\in\mathbb{N}} F_n = \emptyset$, and $\bigcap_{n\in\mathbb{N}} G_n = \emptyset$. For each $n \in \mathbb{N}$, let φ_n be an element of $\mathrm{Inn}(C_{F_n\times G_n})$. Then the sequence $\{\prod_{n=1}^m \varphi_n\}_{m\in\mathbb{N}}$ is $\sigma(C, C_*)$-operator convergent, with limit φ of norm at most one.*

Proof Let x be an arbitrary element of C_*. Lemma 5.1 and its proof show that for each $\varepsilon > 0$ there exists $F_\varepsilon \in I^{fin}$ and $G_\varepsilon \in J^{fin}$ such that, for all $F \in I^{fin}$, $G \in J^{fin}$, with $F \cap F_\varepsilon = \emptyset$, $G \cap G_\varepsilon = \emptyset$, the values

$$\|x|_{C_{F\times G}}\| = \|P_{F\times G*}x\|, \quad \|x|_{C_{F_\varepsilon\times G}}\| = \|P_{F_\varepsilon\times G*}x\|, \quad \|x|_{C_{F\times G_\varepsilon}}\| = \|P_{F\times G_\varepsilon*}x\|,$$

are less than or equal to ε. By assumption there exists also $m \in \mathbb{N}$ which is such that $F_n \cap F_\varepsilon = \emptyset$ and $G_n \cap F_\varepsilon = \emptyset$, whenever $n \geq m$. This implies that, for $n \geq m$ the relation $C_{F_n\times G_n} \perp C_{F_\varepsilon\times G_\varepsilon}$ holds. It follows that $\varphi_n|_{C_{F_\varepsilon\times G_\varepsilon}} = \mathrm{id}_{C_{F_\varepsilon\times G_\varepsilon}}$, hence, for all $r \in \mathbb{N}$, that $\left(\mathrm{id}_C - \prod_{n=m+1}^{m+r} \varphi_n\right) P_{F_\varepsilon\times G_\varepsilon} = 0$. We set, for $r \in \mathbb{N}$,

$$\Delta\varphi := \prod_{n=1}^{m} \varphi_n - \prod_{n=1}^{m+r} \varphi_n = \left(\mathrm{id}_\mathrm{C} - \prod_{n=1}^{m+r} \varphi_n\right)\prod_{n=1}^{m} \varphi_n.$$

Since the mappings φ_n are isometries, we have that $\|(\mathrm{id}_\mathrm{C} - \prod_{n=1}^{m+r} \varphi_n)\| \leq 2$. Combining these results we find that

$$\|\Delta\varphi(x)\| = \left\|\left(\left(\mathrm{id}_\mathrm{C} - \prod_{n=1}^{m+r} \varphi_n\right)(P_{F_\varepsilon\times G} + P_{F\times G_\varepsilon} + P_{F\times G})\prod_{n=1}^{m} \varphi_n\right)(x)\right\| \leq 6\varepsilon.$$

This shows that the desired convergence holds. Let φ be the corresponding limit, a linear operator on C. Since each φ_n, hence all finite products $\prod_{n=1}^m \varphi_n$ are isometries, it follows that $|f \prod_{n=1}^m \varphi_n(a)| \leq \|f\|\|a\|$, for all $f \in C_*$, $a \in C$, and, hence that $\|\varphi\| \leq 1$. □

Recall that, by definition, a tripotent u of a JBW*-triple A is *σ-finite* if any orthogonal family of tripotents in $A_2(u)$ is at most countable. For characterizations of σ-finite trioptents in terms of the geometry of A, see [3,4] or [9]. If A is a Cartan-factor this is equivalent to the condition that there exists a countable orthogonal family $(u_n)_{n\in\mathbb{N}}$ of minimal tripotents such that u is the weak*-convergent sum $u = \sum_{n\in\mathbb{N}} u_n$.

Theorem 5.3 *Let C be a Cartan factor of infinite rank. Then, the following results hold.*

(1.) *Let u and v be σ-finite tripotents of C of proper infinite rank. Then there exists a sequence $\{\varphi_n\}_{n\in\mathbb{N}}$ in $\mathrm{Inn}(C)$ such that $\varphi = \prod_{n\in\mathbb{N}} \varphi_n$ exists as a weak*-operator limit, and $\varphi(u) = v$.*

(2.) *For each tripotent u of finite rank r in C, there exists a sequence $\{\varphi_n\}_{n\in\mathbb{N}}$ in $\mathrm{Inn}(C)$ such that $\varphi = \prod_{n\in\mathbb{N}} \varphi_n$ exists as a weak*-operator limit, and $\varphi(u) = 0$ in the weak*-topology.*

Proof **(1.):** Let $\mathcal{G} = \{e_{i,j}\}_{(i,j)\in I\times J}$ be the standard grid of C, described in Sect. 3. The index sets I and J are infinite. Hence, we can assume that $\mathbb{N} \subseteq I \cap J$. Then $\mathcal{G}$ contains a countable orthogonal family of minimal tripotents $F = \{e_{ii}\}_{i\in M}$. Since u is σ-finite, it can be written as the sum $u = \sum_{n=1}^{\infty}$, for some orthogonal family $(u_n)_{n\in\mathbb{N}}$ in $\mathcal{U}_1(C)$. Similarly, we can assume that $v = \sum_{i\in M} e_{ii}$. The procedure used in the proof of Corollary 4.3 is applied inductively to obtain the formal product $\varphi_u = \prod_{m\in M} \varphi_m$, which is such that $\varphi_u(u_n) = e_{n,n}$. Notice that type II and III factors are weak*-closed subtriples of type I factors. By construction, the sequence $\{\varphi_m\}_{m\in\mathbb{N}}$ satisfies the assumptions of Corollary 5.2. Hence, φ_u is well defined as an operator on C and as a limit of the finite partial products. We conclude that $\varphi_u(u) = \sum_{n\in\mathbb{N}} e_{n,n} = v$. This completes the proof of (1.).

(2.): From the classification of the Cartan-factors it can be seen that each of the corresponding grids contains a maximal orthogonal family $\mathcal{F}$ of cardinality $|\mathcal{F}| = \mathrm{rank}(C)$. Therefore, $\mathcal{F}$ contains a countable subset $(e_n)_{n\in\mathbb{N}}$. Theorem 4.3 implies the existence of a sequence (φ_n) in $\mathrm{Inn}(C)$ such that $\varphi_n(e_n) = e_{n+1}$. The argument used in the prove of (1.) shows that $\prod_{n\in\mathbb{N}} \varphi_n$ exists. Hence $(\prod_{k=1}^{n} \varphi_n(e_1))_{n\in\mathbb{N}}$ is a weak*-null sequence. □

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