J. Korean Math. Soc. **41** (2004), No. 1, pp. 107–130

STRUCTURAL PROJECTIONS ON A JBW*-TRIPLE AND GL-PROJECTIONS ON ITS PREDUAL

Remo V. Hügli

Abstract. A JB*-triple is a Banach space A on which the group $\mathrm{Aut}(B)$ of biholomorphic automorphisms acts transitively on the open unit ball B of A. In this case, a triple product $\{\cdots\}$ from $A \times A \times A$ to A can be defined in a canonical way. If A is also the dual of some Banach space A_*, then A is said to be a JBW*-triple. A projection R on A is said to be structural if the identity $\{Ra,\ b,\ Rc\} = R\{a,\ Rb,\ c,\}$ holds. On JBW*-triples, structural projections being algebraic objects by definition have also some interesting metric properties, and it is possible to give a full characterization of structural projections in terms of the norm of the predual A_* of A. It is shown, that the class of structural projections on A coincides with the class of the adjoints of neutral GL-projections on A_*. Furthermore, the class of GL-projections on A_* is naturally ordered and is completely ortho-additive with respect to L-orthogonality.

1. Introduction

The results presented here were obtained under the supervision of and in collaboration with G. T. Rüttimann (†1999) at the department of Mathematical Statistics at the University of Berne and C. M. Edwards at The Queen's College in Oxford. Their common work has ended too early, but will persist in mathematics.

We investigate certain classes of complex Banach spaces known as JB*-triples or JBW*-triples, the holomorphic properties of which entail a variety of features, including the existence of a ternary product depending canonically upon the norm on these spaces. Having its roots in

Received November 28, 2003.

2000 Mathematics Subject Classification: Primary 46L70; Secondary 17C65.

Key words and phrases: JBW*-triple, structural projection, GL-projection.

Research supported in part by a grant from Schweizerischer Nationalfonds/Fonds national suisse.

studies begun by E. Cartan [8] on bounded symmetric domains, the theory of JB*-triples has received considerable attention in the last decades, because of the deep connections that persist between them and various fields of mathematics and mathematical physics. The works of Koecher [39] and Loos [41] show that the finite dimensional JB*-triples are precisely those complex Banach spaces, the open unit balls of which are bounded symmetric domains, earlier classified by E. Cartan. A domain U in a complex Banach space is said to be *symmetric* if, for every element a in U, there exists a biholomorphic mapping of order two of U onto itself having a as an isolated fixed point. In several works, the theory of JB*-triples was generalized to infinite dimensions by Braun, Kaup and Upmeier [5], Harris and Kaup [34] and Vigué [52, 53, 54]. An early result that provided a connection between infinite dimensional holomorphy and operator algebras was obtained by Harris, who proved in [33] that the open unit ball of a C*-algebra A is a bounded symmetric domain.

The various types of operator algebras are sometimes considered as non-classical probability spaces, for instance to describe quantum mechanical systems. In this context, contractive projections represent analogs of conditional expectations in classical probability spaces. Unlike most other categories of operator algebras, the one consisting of JB*-triples is stable under contractive projections and, as a consequence, the category of JBW*-triples is stable under weak*-continuos contractive projections. This stability property, obtained by Kaup [38] and Stachò [50] from holomorphic theory, generalized earlier results by Choi and Effros [9], Effros and Størmer [26] and Friedman and Russo [28]. It is explicitly formulated for JBW*-triples in Lemma 3.2 below. For further details on these and related topics, we refer to the extended literature including [4, 8, 12, 13, 27, 31, 33, 37, 39, 41, 43, 49, 51]. An interesting survey on the holomorphic aspects of triples is provided in [3]. A detailed and more extended overview, containing also the history of Jordan structures, can be found in [47] and [48].

A particular subclass of contractive projections on JBW*-triples is given by structural projections, defined by means of the triple product. As was shown by Edwards, McCrimmon and Rüttimann [18], the set of structural projections on a JBW*-triple forms a complete lattice, isomorphic to the complete lattice of weak*-closed inner ideals of A. For the algebraic structure theory of Jordan*-triples and Jordan*-algebras, investigated by McCrimmon [42], Neher [45] and others, inner ideals play a pivotal role, similar to that of ideals in the theory of associative algebras. Further studies by Edwards and Rüttimann [21, 22] indicate that structural projections and weak*-closed inner ideals of JBW*-triples are essential also with regard to the aforementioned connections with quantum mechanics. Using the generalized Mackey-Gleason theorem of Bunce and Wright [6, 7], they showed that quantum decoherence functionals on a W*-algebra A, previously discussed in [55], can be identified with bounded measures on the complete lattice of weak*-closed inner ideals of A as a JBW*-triple, thereby showing that the properties of A relevant for physics depend only upon the triple structure of A.

This article contains a part of the main results of [14] and [16]. We also attempt to illustrate the connections that persist between these results and their precursors provided in [18, 19] upon which they heavily rely, as well as some general aspects concerning JBW*-triples. In Section 4, various characterizations of structural projections and weak*-closed inner ideals, involving uniqueness of weak*-continuous Hahn-Banach extensions and neutral projections, are presented. This culminates in Theorem 4.8 in which it is shown that a contractive projection on a JBW*-triple A is structural if and only if it is the adjoint of a neutral GL-projection. The class of GL-projections can be seen as a generalization of L-projections. Whilst L-projections and neutral projections have been investigated to some extend in the past [10, 11, 36, 40, 46], GL-projections appear to be a new concept, introduced in [14]. Both neutrality and the GL-property can be interpreted physically. The predual A_* of A is considered to be the normal state space of a statistical physical system, and a contractive linear projection P on A_* represents a repeatable operation on the state space [1, 2, 30, 44]. In this situation, P is neutral if and only if a state is unchanged by the operation whenever the transition probability of the state under the corresponding operation is one, and P is a GL-projection if and only if a state of the system 'orthogonal' to the set of states which remain unchanged has zero probability of being transmitted by the corresponding operation.

In Section 5, we elaborate a list of properties, characterizing the set $\mathcal{GL}(A_*)$ of GL-projections on the predual of a JBW*-triple, which, thereby are of independent interest. Section 6 provides investigations into the order structure and further specific features of $\mathcal{GL}(A_*)$. Examples of GL-projections showing that $\mathcal{GL}(A_*)$ encompasses well known classes of contractive projections can be found in [14, 16]. However, we confine ourselves here to constructing a set of particular examples arising in a simple manner from the theory developed.

2. JB*- and JBW*-triples

Let A be a complex Banach space with open unit ball D_A and denote by $\mathrm{Aut}(D_A)$ the group of all biholomorphic automorphisms of D_A. If $\mathrm{Aut}(D_A)$ acts transitively on D_A, the Banach space A can be shown to be a *JB*-triple*. If, furthermore, A is the dual of some Banach space, denoted A_*, then A is said to be a *JBW*-triple*. By arguments involving holomorphy, it can be shown that every JB*-triple A has a canonically determined algebraic structure, given by a triple product $\{\cdots\}$ from $A \times A \times A$ to A. By the results of [37], this allows a characterization of JB*-triples in an axiomatic setting, using only algebraic properties of the triple product and its relations to the intrinsic Banach space structure of A. A complex Banach space A is a JB*-triple if and only if there exists a mapping $\{\ldots\}$ from $A \times A \times A$ to A satisfying the following conditions.

(i) The expression $\{a,\ b,\ c\}$ is linear in the variables a and c and conjugate linear in the variable b.

(ii) For all elements a, b, c, d and e in A, the following identity, referred to as the *Jordan triple identity* is valid.

$$D(a,b)\{c,\ d,\ e\} \ = \ \{D(a,b)c,\ d, e\}+\{c,\ d,\ D(a,b)e\}-\{c,\ D(b,a)d,\ e\},$$

where $D(a,b)$ denotes the linear operator on A, defined by

$$D(a,b)c \ = \ \{a,\ b,\ c\}.$$

(iii) For every element a of A, the operator $D(a,a)$ is hermitian in that, for every real t, the linear operator $\exp(itD(a,a))$ is an isometry of A.

(iv) The spectrum $\sigma(D(a,a))$ of $D(a,a)$ is non-negative, and the norm $\|D(a,a)\|$ of $D(a,a)$ is equal to $\|a\|^2$.

The properties (i) and (ii) can be used to develop a purely algebraic theory of triples, which, in this case are referred to as *Jordan*-triples*. For more details on the algebraic features of Jordan*-triples see for example [41, 42, 45]. However, since we are interested also in analytic structure, we will always assume here that A is a JB*-triple or a JBW*-triple. Notice that, if two Banach spaces with a triple product, verifying the above axioms, are isomorphic as JB*-triples, then they are isometrically isomorphic as Banach spaces and vice versa. Furthermore, a JB*-triple is *anisotropic* in that zero is the only element a of A for which $\{a,\ a,\ a\}$ is zero. For these and further results on the connections of JB*-triples and JBW*-triples to holomorphy of Banach spaces and related topics the reader is referred to the extended literature, e.g. to [3, 13, 17, 20, 32, 33, 34, 41, 52, 53, 54].

The results in [33] imply that every C*-algebra A is an example of a JB*-triple, the holomorphic automorphisms of which are essentially Möbius transformations. The triple product is defined, for elements a, b and c in A, by

$$\{a,\ b,\ c\} \ = \ \frac{1}{2}(ab^*c + cb^*a).$$

Similarly, if A is a JB*-algebra with Jordan product $\circ : A \times A \to A$ and involution $^* : A \to A$, then it is a JB*-triple, equipped with the triple product

$$\{a,\ b,\ c\} \ = \ \frac{1}{2}(a \circ (b^* \circ c) + c \circ (b^* \circ a) - b^* \circ (a \circ c)).$$

Note that the passage from JB*-triples to JBW*-triples is analogous to that from C*-algebras to W*-algebras, or from JB*-algebras to JBW*-algebras. Therefore, if in addition the space A, described in these examples, is dual to some Banach space, then A is a JBW*-triple. If a, b and c are elements of of a complex Hilbert space H with hermitian scalar product $< .,\ . >$, then H is a JBW*-triple, with the triple product defined by

$$\{a,\ b,\ c\} \ = \ \frac{1}{2}(< a, b > c+ < c, b > a).$$

There are JB*-triples which are not isomorphic to subtriples of neither of the aforementioned examples. Such *exceptional* triples are provided by the *Cartan factors* M_3^8, the space of 3×3 hermitian matrices over the Cayley numbers $\mathbb{O}$ and $M_{1,2}^8$, the space of 1×2 matrices over $\mathbb{O}$, sometimes referred to as the *bi-Cayley triple*. For more details see for example [23, 25, 45].

3. Weak*-closed inner ideals and structural projections

A subspace B of a JB*-triple A is said to be a *subtriple* of A if B is closed under the triple product, i.e. if

$$\{B,\ B,\ B\} \ \subseteq \ B,$$

and B is said to be an *inner ideal* of A if

$$\{B,\ A,\ B\} \ \subseteq \ B.$$

A *projection* P is an idempotent linear mapping on a normed vector space E. When P is continuous and of norm one, it is said to be *contractive*. Recall that, in this case, the dual $(PE)^*$ of the range PE of

P is isometrically isomorphic to the range P^*E^* of the adjoint P^* of P. Therefore, whenever P is a contractive projection, $(PE)^*$ and P^*E^* will be identified. A projection R on a JB*-triple A is said to be *structural* if, for all elements a, b and c of A, the following relation in connection with the triple product is satisfied.

$$R\{a,\ Rb,\ c\} \ = \ \{Ra,\ b,\ Rc\}.$$

Obviously, the range RA of a structural projection R is necessarily an inner ideal of A. As a deep result, elaborated in [18], a converse holds for weak*-closed inner ideals of JBW*-triples.

THEOREM 3.1. *Every weak*-closed inner ideal I of a JBW*-triple A is the range of a unique structural projection R on A.*

By this theorem, the set $\mathcal{S}(A)$ of structural projections on A can be directly identified with the set $\mathcal{I}(A)$ of weak*-closed inner ideals of A. Observe, that the intersection of a family of inner ideals is again an inner ideal, and that $\mathcal{I}(A)$ is naturally ordered by set inclusion, with least element $\{0\}$ and greatest element A. Therefore, $\mathcal{I}(A)$ and $\mathcal{S}(A)$ form complete lattices which are isomorphic.

It was shown in [38] and [50] that on the range RA of a contractive projection R on a JB*-triple A, a triple product can be defined in a canonical way, such that RA becomes a JB*-triple in is own right, and, hence, that the category of JB*-triples is stable under contractive projections. By standard arguments of functional analysis, this result translates immediately to the case of JBW*-triples and contractive projections which are also weak*-continuous. This is described in the following lemma.

LEMMA 3.2. *Let P be a contractive projection on the predual A_* of the JBW*-triple A, and let P^* be the adjoint of P. Then, with respect to the triple product $\{\cdots\}_{P^*A}$ from $P^*A \times P^*A \times P^*A$ to P^*A, defined, for elements a, b, and c in P^*A, by*

$$\{a,\ b,\ c\}_{P^*A} = P^*\{a,\ b,\ c\},$$

the range P^*A of P^* is a JBW*-triple with predual PA_*.

It has to be pointed out here that, by the results in [4], the predual of a JBW*-triple is unique up to isometry, and, furthermore, that the triple product is separately weak*-continuous, a fact which is of great importance whenever a property specific to JBW*-triples and their triple product is to be verified.

In what follows, some algebraic and analytic relations on a JB*-triple A are presented, which, when A is a JBW*-triple, appear to be in close connection with the geometry of its predual A_*. A pair a, b of elements of A is said to be *orthogonal*, denoted $a \perp b$ if the linear operator $D(a, b)$ is identically zero on A. It can be shown that this relation is symmetric and remains valid when passing to a JB*-triple containing A as a norm closed subtriple. The *algebraic annihilator* $H^\perp$ and the *kernel* $\mathrm{Ker}(H)$ of a non-empty subset H of A are defined, respectively, to be the sets

$$\begin{aligned} \mathrm{Ker}(H) &= \{a \in A : \{H,\ a,\ H\} = \{0\}\}, \\ H^\perp &= \{a \in A : a \perp b \ \forall b \in H\} = \bigcap_{b \in H} \{b\}^\perp. \end{aligned}$$

An element u in A is said to be a *tripotent* if $\{u,\ u,\ u\}$ equals u. Every non-zero tripotent is of norm one. Furthermore, the annihilator $\{u\}^\perp$ of a tripotent u is an inner ideal of A. Hence, the annihilator $H^\perp$ of any subset H consisting of tripotents is an inner ideal [37, 41]. If a further tripotent v of A is such that

$$u \perp (v - u),$$

then, u is said to be less than or equal to v, denoted $u \leq v$. This relation provides a partial order on the set $\mathcal{U}(\mathcal{A})$ of all tripotents in A. A tripotent u in a JBW*-triple A is *σ-finite*, if any set of pairwise orthogonal tripotents all of which are less than or equal to u is of countable cardinality. The set of all σ-finite tripotents of A is denoted by $\mathcal{U}_\sigma(A)$. By the results of [35] Lemma 3.11, a JBW*-triple A is the norm-closure of the linear span of the set $\mathcal{U}(A)$, and by [27], for each element x in the predual A_* of A, there exists a tripotent $e^A(x)$ of A that is the smallest of all elements u of $\mathcal{U}(A)$ such that $u(x)$ equals $\|x\|$. The tripotent $e^A(x)$ is said to be the *support tripotent* of x. Furthermore, by [24] a tripotent u of A is σ-finite if and only if it is equal to $e^A(x)$ for some element x in A_*.

In the subsequent considerations, a linearized, weak*-closed version of the concept of support tripotent, will be of importance. For a non-empty subset G of A_*, the *support space* $s(G)$ of G is defined to be the smallest weak*-closed subspace of A containing the support tripotent $e^A(x)$ of all elements x in G. Some general properties of the support space $s(G)$ of G can be found in [14]. However, the case when G is the range PA_* of a contractive projection P is of particular interest. This situation has been investigated in [19], the results of which are based on similar considerations in [29]. For a proof of (i), (ii) and (iii) in the following lemma, see [19] Lemma 5.1 and [29] Proposition 2.2, and for (iv) see [14] Lemma 3.2.

LEMMA 3.3. *Under the conditions of Lemma 3.2, the following results hold.*

(i) *The support space $s(PA_*)$ of the norm-closed subspace PA_* of A_* is a weak*-closed subtriple of A.*
(ii) *The space $s(PA_*) \oplus_\infty s(PA_*)^\perp$ is a weak*-closed subtriple of A in which $s(PA_*)$ and $s(PA_*)^\perp$ are weak*-closed ideals.*
(iii) *The weak*-closed subspace P^*A of A is contained in $s(PA_*) \oplus_\infty s(PA_*)^\perp$, and the restriction ϕ of the M-projection from $s(PA_*) \oplus_\infty s(PA_*)^\perp$ onto $s(PA_*)$ is a weak*-continuous isometric triple isomorphism from the JBW*-triple P^*A endowed with the triple product $\{\cdots\}_{P^*A}$ onto the sub-JBW*-triple $s(PA_*)$ of A.*
(iv) *The inverse ϕ^{-1} of ϕ is the restriction of P^* to $s(PA_*)$, the predual of which can be identified with PA_*, and the pre-adjoints ϕ_* and $(\phi^{-1})_*$ are the identity mappings on PA_*.*

COROLLARY 3.4. *Under the conditions of Lemma 3.3, if either P^*A is contained in $s(PA_*)$ or $s(PA_*)$ is contained in P^*A then P^*A and $s(PA_*)$ coincide.*

Proof. This is immediate from Lemma 3.3, (iii) and (iv). □

4. Geometric structure and GL-projections

Being defined algebraically, structural projections have also interesting geometric and topological properties. Before stating further algebraic results, some generalities concerning a normed vector space E and its dual E^* and projections on these spaces are to be described. Let E_1 and E_1^* be the closed unit balls in E and E^*, respectively. For subsets G of E and H of E^*, let G° and $H_\circ$ denote the topological annihilators of G in E^* and H in E, respectively. When G and H are $\mathbb{R}$-homogeneous let $G^\sharp$ and $H_\sharp$ be the subsets of E^* and E consisting of elements that attain their norms on G and H, respectively. To be precise,

$$\begin{aligned} G^\sharp &= \{a \in E^* : \|a\| = \sup\{|a(x)| : x \in G \cap E_1\}\}, \\ H_\sharp &= \{x \in E : \|x\| = \sup\{|a(x)| : a \in H \cap E_1^*\}\}. \end{aligned}$$

A contractive projection P on E is said to be *neutral* if, whenever an element x of E has the property that

$$\|Px\| = \|x\|,$$

then x equals Px. A proof of the following lemma, characterizing neutrality, is given in [14], some parts of which can also be found in [19] and [46].

Lemma 4.1. *Let E be a complex Banach space and let P be a contractive projection on E. Then, the following conditions on P are equivalent.*

(i) *The projection P is neutral.*
(ii) *Every weak*-continuous linear functional on the range P^*E^* of the adjoint P^* of P has a unique weak*-continuous Hahn-Banach extension to E^*.*
(iii) *Every contractive projection S on E having the property that the range S^*E^* of its adjoint S^* coincides with P^*E^* is neutral.*
(iv) *The set $(P^*E^*)_\sharp$ coincides with the range PE of P.*

Furthermore, if the contractive projection P is neutral and S is a further contractive projection such that P^*E^* and S^*E^* coincide then P and S coincide.

The results presented next, which predates Theorem 3.1, were proved in [19].

Theorem 4.2. *Let A be a JBW*-triple with predual A_* and let R be a structural projection on A. Then, the following results hold.*

(i) *The projection R is contractive and weak*-continuous. In particular, there exists a contractive projection P on A_*, such that R equals the adjoint P^* of P.*
(ii) *The projection P on A_*, as in* (i), *is neutral.*

The neutrality of the pre-adjoint P of R is therefore necessary for R to be structural. On the other hand, it is easy to construct an example of a neutral projection P on a the predual of a JBW*-triple, the adjoint P^* of which is not structural. This raises the question of whether there are further non-trivial conditions upon P which, together with neutrality, would ensure that P^* is structural. Such a condition, in terms of the algebraic structure of A and P^*, is given in the following theorem from [19].

Theorem 4.3. *Let P be a neutral projection on the predual A_* of a JBW*-triple A, and let P^* be the adjoint of P. If the range P^* of P^* is a subtriple of A, then P^* is structural.*

Theorem 4.3 and Theorem 4.2 combine to give the following result.

Corollary 4.4. *Let A be a JBW*-triple with predual A_*. Let $\mathcal{I}(A)$ denote the family of weak*-closed inner ideals in A, let $\mathcal{S}(A)$ denote the family of structural projections on A, and let $\mathcal{N}(A_*)$ denote the family of neutral projections on A_* with the property that P^*A is a subtriple of A. Then the following results hold.*

(i) *The mapping $P \mapsto P^*$ is a bijection from $\mathcal{N}(A_*)$ onto $\mathcal{S}(A)$.*
(ii) *The mapping $R \mapsto RA$ is a bijection from $\mathcal{S}(A)$ onto $\mathcal{I}(A)$.*

In proving the above theorems, and in view of Lemma 4.1, the proofs of the following two lemmas, which can be found in [18, 19], become evident.

Lemma 4.5. *Let A be a JBW*-triple, with predual A_*, and let J be a subtriple of A. Then, the following conditions are equivalent.*

(i) *J is a weak*-closed inner ideal in A.*
(ii) *The set $J_\sharp$ of elements of A_* attaining their norm on J is a subspace of A_*.*
(iii) *Every element of the predual J_* of J has a unique weak*-continuous Hahn-Banach extension as a weak*-continuous linear functional on A.*

Lemma 4.6. *Let A be a JBW*-triple, with predual A_*, let R be a structural projection on A, and let J be the weak*-closed inner ideal RA in A. Then, the following results hold.*

(i) *R is the unique structural projection with range J.*
(ii) *R is contractive and weak*-continuous.*
(iii) *The kernel $\mathrm{Ker}(J)$ of J coincides with the kernel $\mathrm{Ker}(R)$ of R.*
(iv) *The predual J_* of J coincides with $J_\sharp$ which consists of the elements x in A_* the support tripotent $e^A(x)$ of which lies in J.*
(v) *J coincides with the support space $s(J_*)$ of J_*.*

Hence, the uniqueness of weak*-continuous linear functionals on J has a precise expression in terms of algebraic conditions upon J. Recall that, in contrast to structurality, neutrality is a purely metric property of a projection P on A_*. Therefore, it would be desirable to replace the condition that P^*A be a subtriple by a condition using only the norm of A_*. A solution of this problem involves the relation of L-orthogonality between elements of A_*.

A pair x, y of elements in a normed vector space E is said to be *L-orthogonal*, denoted $x \diamond y$, if

$$\|x+y\| \;=\; \|x-y\| = \|x\| + \|y\|.$$

The *L-complement* $F^\diamond$ of a non-empty subset G of E is defined by

$$G^\diamond \ = \ \{x \in E \ : \ x \diamond y \ \forall y \in G\}.$$

A contractive projection P on E is said to be a *GL-projection* if the L-complement $(PE)^\diamond$ of its range is a subset of the kernel Ker$P$ of P. Hence, GL-projections provide a generalisation of L-projections which are defined to be those linear projections P for which $(PE)^\diamond$ coincides with KerPE.

By [29] Lemma 2.5, for every tripotent u of A, there exists a family $\{x_i\}_{i\in I}$ of pairwise L-orthogonal elements in A_*, such that u equals the weak*-limit $\sum_{i\in I} e^A(x_i)$. The relations between support tripotents and L-orthogonality, described in the following lemma, can be found in [24] and [27]. See also [36].

LEMMA 4.7. *Let A be a JBW*-triple with predual A_*. Then, for elements x and y in A_*, the conditions $x\diamond y$ and $e^A(x)\perp e^A(y)$ are equivalent. Furthermore, in this case, $e^A(x)y$ equals zero.*

We now state a main result, established in [14], which resolves the problem posed above. The proof requires a certain characterization of GL-projections on A_*, provided only in Theorem 5.3, and is therefore postponed to the end of Section 5.

THEOREM 4.8. *Let A be a JBW*-triple, with predual A_*, and let R be a linear projection on A. Then R is a structural projection if and only if there exists a neutral GL-projection P on A_* with adjoint equal to R.*

5. Characterizations of GL-projections on A_*

The set $\mathcal{GL}(A_*)$ of GL-projections on the predual A_* of a JBW*-triple A not only proves to be useful for characterizing structural projections, but has some further interesting properties of its own. As a first goal of this section, Theorem 5.3 provides several characterizations of GL-projections among the set of contractive projections on A_*. Its proof is based on the following technical lemma.

LEMMA 5.1. *Let A be a JBW*-triple, with predual A_*, and let G be a non-empty subset of A_*, having L-orthogonal complement $G^\diamond$ and support space $s(G)$. Then, the following results hold.*

(i) *The support space $s(G^\diamond)$ of $G^\diamond$ coincides with the weak*-closed inner ideal $s(G)^\perp$, and the predual $(s(G)^\perp)_\sharp$ of $s(G)^\perp$ coincides with $G^\diamond$.*

(ii) *The kernel* $\mathrm{Ker}(s(G)^\perp)$ *of* $s(G)^\perp$ *coincides with the topological annihilator* $(G^\diamond)^\circ$ *of* $G^\diamond$*.*
(iii) *The L-orthogonal complement* $G^\diamond$ *of* G *is contained in the topological annihilator* $s(G)_\circ$ *of* $s(G)$*.*

Proof. (i) Let H denote the subset $\{e^A(x) : x \in G\}$ of $\mathcal{U}_\sigma(A)$. Since taking the annihilator reverses the order of set inclusion, it is clear that $s(G)^\perp$ is a subset of $H^\perp$. Conversely, let b be an element in $H^\perp$, i.e. $\{e^A(x),\ b,\ a\} = 0$ for all $x \in G$ and $c \in A$. Linearity and the weak*-continuity of the triple product implies that, for all $c \in s(G)$, the product $\{c,\ b,\ a\}$ is also zero. Therefore, b lies in $s(G)^\perp$. This shows that the annihilator $s(G)^\perp$ of the weak*-closed subtriple $s(G)$ is given by

$$s(G)^\perp \ = \ H^\perp \ = \ \bigcap_{x\in G} \{e^A(x)\}^\perp, \tag{5.1}$$

and, in particular, that $s(G)^\perp$ is a weak*-closed inner ideal of A. These results hold more generally for any weak*-closed subtriple B of A and H equal to $\mathcal{U}_\sigma(B)$. Here, only the special case when B is equal to $s(G)$ is needed. By Lemma 4.7, an element x of A_* lies in $G^\diamond$ if and only if, for all elements y in G, the tripotent $e^A(x)$ lies in $\{e^A(y)\}^\perp$, which by equation (5.1), is true if and only if $e^A(x)$ lies in the weak*-closed inner ideal $s(G)^\perp$, and by Lemma 4.6 (iv), if and only if x lies in $(s(G)^\perp)_\sharp$, completing the proof of (i).

(ii) By Lemma 4.6 (iii), the kernel $\mathrm{Ker}(s(G)^\perp)$ of the weak*-closed inner ideal $s(G)^\perp$ coincides with the kernel of the structural projection onto $s(G)^\perp$, which, by Lemma 4.6 (iv), itself coincides with $((s(G)^\perp)_\sharp)^\circ$. The result follows from (i).

(iii) Observe that $s(G)$ is contained in $\mathrm{Ker}(s(G)^\perp)$. Using (i) and Lemma 4.6 (iv),

$$G^\diamond = (s(G)^\perp)_\sharp = \mathrm{Ker}(s(G)^\perp)_\circ \subseteq s(G)_\circ,$$

as required. □

Corollary 5.2. *Under the conditions of Lemma 5.1, the following results hold.*

(i) *The L-orthogonal complement* $G^\diamond$ *of a non-empty subset* G *of* A_* *is a norm-closed subspace of* A_**.*
(ii) *Let* F*,* G *and* H *be mutually L-orthogonal subspaces of* A_**. Then, the subspace* $F \oplus G$ *is L-orthogonal to* H*.*

Proof. The proof of (i) follows from Lemma 5.1(i). To prove (ii), observe that both F and G are contained in the closed subspace $H^\diamond$, from which it follows that the subspace $F \oplus G$ is contained in $H^\diamond$. $\square$

It is now possible to establish various characterizations of GL-projections among the contractive projections using the support space of their ranges.

THEOREM 5.3. *Let A be a JBW*-triple, with predual A_*, let P be a contractive projection on A_*, with adjoint P^*, and let $s(PA_*)$ be the support space of the range PA_* of P. Then, the following conditions are equivalent.*

(i) *P is a GL-projection.*
(ii) *The range P^*A of P^* is contained in the kernel $\mathrm{Ker}(s(PA_*)^\perp)$ of the weak*-closed inner ideal $s(PA_*)^\perp$.*
(iii) *$s(PA_*)$ is contained in P^*A.*
(iv) *$s(PA_*)$ coincides with P^*A.*
(v) *$s(PA_*)$ contains P^*A.*
(vi) *The topological annihilator $s(PA_*)_\circ$ of $s(PA_*)$ is contained in the kernel $\mathrm{Ker}(P)$ of P.*
(vii) *$s(PA_*)^\perp$ is contained in the weak*-closed inner ideal $(P^*A)^\perp$.*
(viii) *$s(PA_*)^\perp$ coincides with $(P^*A)^\perp$.*

Proof. (i) $\Leftrightarrow$ (ii) This follows from Lemma 5.1.

(iii) $\Leftrightarrow$ (iv) $\Leftrightarrow$ (v) This follows from Corollary 3.4.

(ii) $\Rightarrow$ (v) By Lemma 3.3 (iii), for each element a in P^*A, there exist elements b in $s(PA_*)$ and c in $s(PA_*)^\perp$ such that a is equal to $b+c$. It follows from (ii) that

$$0 = \{c,\ a,\ c\} = \{c,\ b+c,\ c\} = \{c,\ b,\ c\} + \{c,\ c,\ c\} = \{c,\ c,\ c\},$$

and, by the anisotropy of A, c is equal to 0. It follows that a lies in $s(PA_*)$, as required.

(v) $\Rightarrow$ (vi) By taking topological annihilators this is immediate.

(vi) $\Rightarrow$ (i) By Lemma 5.1(iii),

$$(PA_*)^\diamond \subseteq s(PA_*)_\circ \subseteq \mathrm{Ker}(P),$$

as required.

(v) $\Rightarrow$ (vii) By taking algebraic annihilators this is immediate.

(vii) $\Rightarrow$ (ii) Observing that $s(PA_*)^{\perp\perp}$ is contained in $\mathrm{Ker}(s(PA_*)^\perp)$,

$$P^*A \subseteq (P^*A)^{\perp\perp} \subseteq s(PA_*)^{\perp\perp} \subseteq \mathrm{Ker}(s(PA_*)^\perp),$$

as required.

(iv) $\Rightarrow$ (viii) $\Rightarrow$ (vii) These trivially hold. $\square$

We are finally in the position to establish a proof of Theorem 4.8.

Proof. If P is a neutral GL-projection, then Theorem 5.3 and Lemma 3.3 show that the range P^*A of P^* is a subtriple of A, and by Corollary 4.4, the adjoint P^* of P is structural.

Conversely, let R be a structural projection on A. Then, by Corollary 4.4 there exists a unique neutral projection P on A_*, such that P^* is equal to R. By Theorem 5.3 it is enough to show that $s(PA_*)$ is a subset of P^*A. That this is indeed the case can be seen either from Lemma 4.6 (v), or from the following, perhaps more illustrative calculations. Firstly, since P^* is structural, P^*A is a subtriple, even an inner ideal of A. Consider an element x in the range PA_* of P. By Lemma 3.3 it can be seen that

$$\begin{aligned} P^*e^A(x) &= \{P^*e^A(x),\ P^*e^A(x),\ P^*e^A(x)\} \\ &= P^*\{P^*e^A(x),\ P^*e^A(x),\ P^*e^A(x)\}, \end{aligned}$$

and, furthermore, that the element $P^*e^A(x)$ is equal to $e^A(x)+v$ for some v in the annihilator $(s(PA_*))^\perp$ of the support space $s(PA_*)$ of PA_*. Using the structurality of P^* and the orthogonality of v and $e^A(x)$ it can be seen that

$$\begin{aligned} P^*e^A(x) &= \{P^*e^A(x),\ e^A(x),\ P^*e^A(x)\} \\ &= \{e^A(x)+v,\ e^A(x),\ e^A(x)+v\} \\ &= \{e^A(x),\ e^A(x),\ e^A(x)\} + 2\{v,\ e^A(x),\ e^A(x)\} \\ &\quad + \{v,\ e^A(x),\ v\} \\ &= e^A(x). \end{aligned}$$

Hence, the support tripotent $e^A(x)$ of every x in PA_* is contained in the range P^*A of P^* and, consequently, the support space $s(PA_*)$ of PA_* is a subspace of P^*A, and the proof is complete. □

6. Order structure of GL-projections

The set of all projections on a vector space E is ordered in a natural way. For projections P and Q on E the relation $P \leq Q$, is said to hold if $PE \subseteq QE$. This relation is not a partial order, although it is transitive and reflexive, but not in general antisymmetric, i.e. the conditions $P \leq Q$ and $Q \leq P$ do not imply that P and Q coincide. When P and Q are confined to a subset $\mathcal{F}$ of all projections on E, the antisymmetric property of $\leq$ restricted to $\mathcal{F}$ is equivalent to the condition that every subspace F of E is the range of at most one element of $\mathcal{F}$. A first consequence of Theorem 6.1, a main result of [16], is that this occurs when $\mathcal{F}$ is chosen to be equal to $\mathcal{GL}(A_*)$.

THEOREM 6.1. *Let A be a JBW*-triple with predual A_*, and let P be a contractive projection on A_*. Then, there exists a unique GL-projection S on A_* such that the range SA_* of S coincides with the range PA_* of P, and the range S^*A of its adjoint S^* coincides with the support space $s(PA_*)$ of PA_*. Furthermore, S is given by $P\phi_*$, where ϕ is the contractive, weak*-continuous projection from $s(PA_*) \oplus s(PA_*)^\perp$ to $s(PA_*)$.*

Proof. Let ϕ be the isometric triple isomorphism from P^*A onto $s(PA_*)$ defined in Lemma 3.3. Then, the mapping ϕP^* is a weak*-continuous linear mapping from A onto $s(PA_*)$. Observe that, by Lemma 3.3 (iv), for all elements a in A,

$$(\phi P^*)^2 a = \phi(P^*\phi)P^*a = (\phi P^*)a,$$

and ϕP^* is a contractive projection onto P^*A. Let S be the contractive projection on A_* such that S^* coincides with ϕP^*. Then, again using Lemma 3.3 (iv),

$$SA_* = (\mathrm{Ker}(S^*))_\circ = (\mathrm{Ker}(\phi P^*))_\circ = (\mathrm{Ker}(P^*))_\circ = PA_*,$$

as required. It remains to show that S is a GL-projection. However, from Lemma 3.3 (iii) it can be seen that

$$S^*A = s(PA_*) = s(SA_*) \subseteq \mathrm{Ker}(s(SA_*)^\perp),$$

and it follows from Lemma 5.1(ii) that

$$(SA_*)^\diamond = (\mathrm{Ker}(s(SA_*)^\perp))_\circ \subseteq (S^*A)_\circ = \mathrm{Ker}(S),$$

as required. In order to obtain uniqueness, suppose that Q and S are GL-projections on A_* such that QA_* and SA_* coincide. It follows from Theorem 5.3 that

$$Q^*A = s(QA_*) = s(SA_*) = S^*A,$$

and, hence, that

$$\mathrm{Ker}(Q) = (Q^*A)_\circ = (S^*A)_\circ = \mathrm{Ker}(S).$$

Since Q and S have the same range and kernel, Q and S coincide. $\square$

COROLLARY 6.2. *The set $\mathcal{GL}$ of all GL-projections on the predual A_* of a JBW*-triple A is a partially ordered set, the least element of which is the zero projection and the greatest element of which is the identity projection on A_*.*

Proof. The relation $\leq$ is clearly transitive and reflexive, that $\leq$ is also antysimmetric follows immediately from Theorem 6.1. $\square$

It is not known wether $\mathcal{GL}(A_*)$ is a lattice, let alone a complete lattice. Nevertheless it is possible to provide criteria for the existence of suprema and infima of subsets of $\mathcal{GL}(A_*)$ under certain conditions, one of which is mutual orthogonality of the elements in the subset considered. For a result concerning the supremum and infimum of pairs of commuting GL-projections, see [16]. The set of all linear projections on a normed vector space E inherits naturally any relation which may, or may not, persist between the ranges PE and QE of linear projections P and Q. In this sense, the relations $P \diamond Q$ and $P \,\square\, Q$ are said to hold if $PE \diamond QE$ and $PE \,\square\, QE$ respectively. The relation $x \,\square\, y$ denotes *M-orthogonality* between elments x and y of E, which is defined to hold if $\|x \pm y\|$ equals $\max\{\|x\|,\ \|y\|\}$. The following theorem connects geometric and algebraic orthogonality of GL-projections on A_* and their adjoints.

THEOREM 6.3. *Let P and Q be GL-projections on the predual A_* of a JBW*-triple A. Then, the following conditions are equivalent*:

(i) $P \diamond Q$,
(ii) $P^* \perp Q^*$,
(iii) $P^* \,\square\, Q^*$.

Furthermore, if these conditions hold, then $PQ = QP = 0$.

Proof. (i)$\Leftrightarrow$ (ii) Suppose that PA_* is L-orthogonal to QA_*, i.e. every element x in PA_* is L-orthogonal to every element y in $Q(A_*)$. Since, by Lemma 4.7, the conditions $x \diamond y$ and $e^A(x) \perp e^A(y)$ are equivalent, it can be seen that

$$\{e^A(x)\ :\ x \in PA_*\} \quad \perp \quad \{e^A(y)\ :\ y \in QA_*\}.$$

By the linearity and the weak*-continuity of the triple product, it follows that

$$s(PA_*) \perp s(QA_*),$$

and, therefore, by Theorem 5.3 (iv), that

$$P^*A \perp Q^*A.$$

The converse can be proved by reversing the argument.

(ii) $\Leftrightarrow$ (iii) By the GL-property of P and Q, the subspaces P^*A and Q^*A are weak*-closed subtriples, for which, by [15] Theorem 4.4, the relations $\perp$ and $\square$ coincide. Hence (ii) and (iii) are also equivalent.

Since P and Q are GL-projections, it follows from Lemma 3.3 and Theorem 5.3 that

$$P^*A \subseteq \mathrm{Ker}Q^*, \quad Q^*A \subseteq \mathrm{Ker}P^*,$$

and, hence, that both PQ and QP are equal to zero. □

Lemma 6.4 provides a preliminary version of Theorem 6.5 for a finite family of pairwise L-orthogonal GL-projections. It also shows that, in this case, a converse of the statemtent of Theorem 6.5 holds.

Lemma 6.4. *Let A be a JBW*-triple with predual A_* and let $P_1, P_2, \ldots, P_n$ be pairwise L-orthogonal contractive projections on A_*. Then their sum $\sum_{k=1}^n P_k$, the range of which is $\bigoplus_{k=1}^n P_k A_*$, is a GL-projection if and only if P_1, P_2,...,P_n are GL-projections.*

Proof. By the mutual L-orthogonality of the projections P_1, $P_2, \ldots,$ P_n, it follows from Theorem 6.3 that

$$(\sum_{k=1}^{n} P_k)A_* \;=\; \bigoplus_{k=1}^{n} P_k A_*. \tag{6.1}$$

Suppose that $P_1, \ldots, P_n$ are GL-projections. The proof proceeds by induction. Consider the case, in which n is equal to 2. From Theorem 6.3 it can be seen that, for GL-projections P and Q, the sum

$$(P^* + Q^*)A \;=\; P^*A \oplus Q^*A.$$

is an M-sum, and, hence, that, for all elements a in A,

$$||(P^* + Q^*)a|| \;=\; \max\{\|P^*a\|, ||Q^*a||\} \;\leq\; ||a||.$$

Therefore, $(P+Q)^*$, is a contractive projection and, consequently, $P+Q$ is also contractive. Since P and Q are GL-projections, by definition

$$(PA_*)^\diamond \subseteq \mathrm{Ker}PA_*, \quad (QA_*)^\diamond \subseteq \mathrm{Ker}QA_*.$$

Therefore,

$$\begin{aligned}((P+Q)A_*)^\diamond \;&=\; (PA_* \oplus QA_*)^\diamond \;=\; (PA_*)^\diamond \cap (QA_*)^\diamond \\ &\subseteq\; \mathrm{Ker}P \cap \mathrm{Ker}Q \;\subseteq\; \mathrm{Ker}(P+Q),\end{aligned}$$

and, hence, $P + Q$ is a GL-projection. If the statement is true for GL-projections $P_1, \ldots, P_k$ and if P_{k+1} is a GL-projection, such that for all j equal to $1, \ldots, k$,

$$P_j \diamond P_{k+1},$$

then $\sum_{j=1}^{k} P_j$ is a GL-projection. By Corollary 5.2 (i), the L-complement $(P_{k+1}A_*)^\diamond$ of $P_{k+1}A_*$ is a subspace of A_*. It follows that

$$(\sum_{j=1}^{k} P_j) \diamond P_{k+1}. \tag{6.2}$$

Hence, putting P equal to $\sum_{j=1}^{k} P_j$ and Q equal to P_{k+1}, it can be seen from above that $\sum_{j=1}^{k+1} P_j$ is a GL-projection.

On the other hand, suppose that $\sum_{j=1}^{n} P_j$ is a GL-projection. This sum of projections is itself a projection, if and only if, for j equal to $1, \ldots, n$ with $j \neq k$,

$$P_j^* A \;\subseteq\; \mathrm{Ker} P_k^*. \tag{6.3}$$

Furthermore, by the mutual L-orthogonality of $P_1, \ldots, P_n$, Theorem 6.3 implies that, for distinct j and k from $1, \ldots, n$,

$$s(P_j A_*) \subseteq s(P_k A_*)^\perp. \tag{6.4}$$

Let x be an element in of $P_j A_*$. Then, (6.4) and (6.3) show that, if k is different from j,

$$e^A(x) \;\in\; s(P_j A_*) \;\subseteq\; s(P_k A_*)^\perp \;\subseteq\; \mathrm{Ker} P_k^*.$$

But by assumption and Theorem 5.3, the support tripotent $e^A(x)$ of x lies in $(\sum_{k=1}^{n} P_k^*)A$, and, hence,

$$e^A(x) = \sum_{k=1}^{n} P_k^*(e^A(x)) \;=\; P_j^*(e^A(x)).$$

Consequently, the support space $s(P_j A_*)$ of $P_j A_*$ is a subset of $P_j^* A$ and, by Theorem 5.3 (iii), P_j is a GL-projection. This completes the proof. □

THEOREM 6.5. *For every family $\{P_i\}_{i \in I}$ of pairwise L-orthogonal GL-projections on A_*, the supremum $\bigvee_{i \in I} P_i$ exists as a GL-projection, the range of which is $\overline{\bigoplus_{i \in I} P_i}^{n}$. Furthermore, $\bigvee_{i \in I} P_i$ is explicitly given, for elements x in A_* as the norm limit*

$$(\bigvee_{i \in I} P_i)x \;=\; \sum_{i \in I} P_i\, x.$$

Proof. Let I^f denote the directed set of all finite subsets of I, ordered by set inclusion. By Lemma 6.4, for every finite subset F in I^f, there exists a GL-projection P_F, given by

$$P_F = \sum_{i \in F} P_i.$$

In particular, P_F is contractive. Using the mutual L-orthogonality of $\{P_i\}_{i \in F}$, it can be seen that, for all elements x in A_*,

$$\sum_{i \in F} \|P_i x\| = \|\sum_{i \in F} P_i x\| = \|P_F x\| \leq \|x\| . \tag{6.5}$$

Let $\sup_{F \in I^f} \| \sum_{i \in F} P_i x\|$ be denoted by k. Then, for every positive number ε, there exists a finite subset F_ε of I, such that

$$k - \varepsilon \leq \|\sum_{i \in F_\varepsilon} P_i x\| \leq k. \tag{6.6}$$

Consider an element F in I^f such that $F \cap F_\varepsilon$ is empty. Using again the L-orthogonality of $\{P_i\}_{i \in I}$ and relation (6.6),

$$\begin{aligned} \|\sum_{j \in F} P_j x\| &\leq \|\sum_{i \in F_\varepsilon} P_i x\| + \|\sum_{j \in F} P_j x\| - k + \varepsilon \\ &= \|\sum_{j \in F \cup F_\varepsilon} P_j x\| - k + \varepsilon \\ &\leq k - k + \varepsilon = \varepsilon. \end{aligned}$$

It follows that $\{P_F x\}_{F \in I^f}$ is a Cauchy net with respect to the norm topology on A_*. Hence, the norm limit $\lim_{F \in I^f} P_F x$ of this net exists for every x in A_*. By (6.5), and the uniform boundedness principle, the mapping $x \mapsto Px$, defined by

$$Px = \lim_{F \in I^f} P_F x$$

is a linear contractive projection on A_*. In particular, the range PA_* of P is given by

$$PA_* = \overline{\mathrm{lin} \bigcup_{i \in I} P_i A_*}^n, \tag{6.7}$$

which, by Theorem 6.3 coincides with $\overline{\bigoplus_{i \in I} P_i A_*}^n$. It remains to verify that P is a GL-projection. Since, for each i in I, P_i is a GL-projection, the L-complement $(P_iA_*)^\diamond$ of P_i is a subset of KerP_i. It follows that

$$\begin{aligned}(PA_*)^\diamond &= (\overline{\mathrm{lin}\bigcup_{i\in I} P_iA_*}^n)^\diamond \subseteq (\mathrm{lin}\bigcup_{i\in I} P_iA_*)^\diamond \subseteq (\bigcup_{i\in I} P_iA_*)^\diamond \\ &= \bigcap_{i\in I} P_iA_*^\diamond \\ &\subseteq \bigcap_{i\in I}\mathrm{Ker}P_i = \{x\in A_* : P_i(x) = 0\,\forall i\in I\} \\ &\subseteq \mathrm{Ker}P.\end{aligned}$$

Hence P is a GL-projection. If Q is any upper bound of the family $\{P_i\}_{i\in I}$, then the range QA_* of Q must contain $\overline{\mathrm{lin}\bigcup_{i\in I} P_iA_*}^n$ as a subspace. The proof is completed by Corollary 6.2 and equation (6.7). $\square$

We close with an application of the above results, which allows us to construct examples of GL-projections on A_*.

THEOREM 6.6. *Let $\{x_i\}_{i\in I}$ be a family of mutually L-orthogonal elements in the predual A_* of a JBW*-triple A. Then, there exists a GL-projection P on the norm closed subspace $\overline{\mathrm{lin}\{x_i : i\in I\}}^n$, and P and its adjoint P^* are explicitly given, for elements y in A_* and a in A, by*

$$Py = \sum_{i\in I} e^A(x_i)(y)\,x_i, \quad P^*a = \sum_{i\in I} x_i(a)\,e^A(x_i),$$

where the sums are to be understood as norm and weak-limits respectively.*

Proof. Consider first the case in which I consists of one element. By the Hahn-Banach theorem and Theorem 6.1, every one dimensional subspace of A_* is the range of a unique GL-projection. More explicitly, it must be shown that, for an element x of norm one in A_*, with support tripotent $e^A(x)$ in A, the mappings P_x and P_x^*, which are given by

$$P_x\,y = e^A(x)(y)x, \quad P_x^*\,a = a(x)e^A(x),$$

meet our requirement in this case. Clearly P_x is a linear mapping with range $\mathbb{C}x$. This mapping is a projection since

$$\begin{aligned}P_x(P_xy) &= P_x(e^A(x)(y)x) = e^A(x)(y)P_xx \\ &= e^A(x)(y)e^A(x)(x)x = e^A(x)(y)x \\ &= P_xy.\end{aligned}$$

The projection P_x is contractive since, for all elements y in A_*,

$$\|P_xy\| = \|e^A(x)(y)x\| \le \|e^A(x)\|\,\|y\|\,\|x\| = \|y\|.$$

Furthermore, if the elements x and y in A_* are L-orthogonal, then it follows from Lemma 4.7 (ii) that

$$P_x y = e^A(x)(y)x \; = \; 0,$$

and, hence, that P_x is a GL-projection. The uniqueness of P_x is a consequence of Theorem 6.1 . The dual projection P_x^* of P_x is defined for all elements a in A and y in A_* by

$$(P_x^* a)(y) \; = \; a(x)e^A(x)(y).$$

Hence, $P_x^* a$ is equal to $a(x)e^A(x)$. When I is a general set, an application of Theorem 6.5 leads to the desired proof.

□

Further examples of GL-projections are provided in [14] and [16], where it is shown, in particular, that certain projections on a JBW*-triple A, known as Peirce projections, are the adjoints of GL-projections. For a description of Peirce projections see for example [17, 20, 41, 45].

References

[1] E. M. Alfsen and F. Schultz, *State Spaces of Jordan algebras*, Acta Math. **140** (1978), 155–190.

[2] ———, *State Spaces of Operator Algebras*, Birkhäuser, Basel, Boston 2001.

[3] J. Arazy, *An application of infinite dimensional holomorphy to the geometry of Banach spaces*, Geometrical Aspects of Functional Analysis, Lecture Notes in Math. **1267**, Springer, Berlin/Heidelberg/New York, 1980.

[4] T. J. Barton and R. M. Timoney, *Weak*-continuity of Jordan triple products and its applications*, Math. Scand. **59** (1986), 177–191.

[5] R. Braun, W. Kaup and H. Upmeier, *Holomorphic characterization of Jordan C^*-algebras*, Math. Z. **161** (1987), 277–290.

[6] L. J. Bunce and J. D. M. Wright, *Complex measures on projections in von Neumann algebras*, J. London Math. Soc. **46** (1992), 269–279.

[7] L. J. Bunce and J. D. M. Wright, *The Mackey-Gleason problem for vector measures on projections in von Neumann algebras*, J. London Math. Soc. **49** (1994), 133–149.

[8] E. Cartan, *Sur les domaines bornés homogènes de l'espace de n variables complexes*, Abh. Math. Sem. Univ. Hamburg **11** (1935), 116–162.

[9] M. D. Choi and E. G. Effros, *Injectivity and operator spaces*, J. Funct. Anal. **24** (1977), 156–209.

[10] F. Cunningham, *L-structure in L-spaces*, Trans. Amer. Math. Soc. **95** (1960), 274–299.

[11] F. Cunningham, E. G. Effros and N. Roy, *M-structure in dual Banach spaces*, Israel J. Math. **14** (1973), 304–309.

[12] S. Dineen, *Complete holomorphic vector fields in the second dual of a Banach space*, Math. Scand. **59** (1986), 131–142.

[13] ———, *The second dual of a JB*-triple system*, *Complex Analysis, Functional Analysis and Approximation Theory*, (J. Mujica ed.), North-Holland, Amsterdam, (1986), 67–69.

[14] C. M. Edwards, R. V. Hügli and G. T. Rüttimann, *A geometric characterization of structural projections on the predual of a JBW*-triple*, to appear in J. Funct. Anal.

[15] C. M. Edwards and R. V. Hügli, *M-orthogonality and holomorphic rigidity in complex Banach spaces*, to appear.

[16] ———, *Order structure of GL-projections on the predual of a JBW*-triple*, In preparation.

[17] C. M. Edwards, D. Lörch and G. T. Rüttimann, *Compatible subtriples of Jordan*-triples*, J. Algebra **216** (1999), 707–740.

[18] C. M. Edwards, K. McCrimmon and G. T. Rüttimann, *The range of a structural projection*, J. Funct. Anal. **139** (1996), 196–224.

[19] C. M. Edwards and G. T. Rüttimann, *Structural projections on JBW*-triples*, J. London Math. Soc. **53** (1996), 354–368.

[20] ———, *Peirce inner ideals in Jordan *-triples*, J. Algebra **180** (1996), 41–66.

[21] ———, *The lattice of weak*-closed inner ideals in a W*-algebra*, Commun. Math. Phys. **197** (1998), 131–166.

[22] ———, *Gleason's Theorem for rectangular JBW*-triples*, Commun. Math. Phys. **203** (1999), 269–295.

[23] ———, *Inner ideals in the bi-Cayley triple*, Atti Sem. Mat. Fis. Univ. Modena (1999), 235–259.

[24] ———, *Orthogonal faces of the unit ball in a Banach space*, Atti Sem. Mat. Fis. Univ. Modena **49** (2001), 473–493.

[25] C. M. Edwards, G. T. Rüttimann and S. Vasilovsky, *Inner ideals in exceptional JBW*-triples*, Michigan Math. J. **40** (1993), 139–152.

[26] E. G. Effros and E. Størmer, *Positive projections and Jordan structure in operator algebras*, Math. Scand. **45** (1979), 127–138.

[27] Y. Friedman and B. Russo, *Structure of the predual of a JBW*-triple*, J. Reine Angew. Math. **356** (1985), 67–89.

[28] ———, *Solution of the contractive projection problem*, J. Funct. Anal. **60** (1985), 56–79.

[29] ———, *Conditional expectation and bicontractive projections on Jordan C*-algebras and their generalisations*, Math. Z. **194** (1987), 227–236.

[30] ———, *Classification of atomic facially symmetric spaces*, Canad. J. Math. **45** (1993), no. 1, 33–87.

[31] H. Hanche-Olsen and E. Størmer, *Jordan Operator Algebras*, Pitman Advanced, Boston/London/Melbourne, 1984.

[32] L. A. Harris, *A generalisation of C*-algebras*, Proc. London Math. Soc. **42** (1981), 331–361.

[33] ———, *Bounded symmetric domains in infinite dimensional spaces.*, in T. L. Hayden and T. J. Suffridge (Eds.): *Proceedings on Infinite Dimensional Holomorphy*, Lecture Notes in Math. **364** (13-40), Springer Berlin/Heidelberg/New York, 1974.

[34] L. A. Harris and W. Kaup, *Linear algebraic groups in infinite dimensions*, Illinois J. Math. **21** (1977), 666–674.

[35] G. Horn, *Characterisation of the predual and ideal structure of a JBW*-triple*, Math. Scand. **61** (1983), 117–133.

[36] B. Iochum, *Cônes autopolaires et algèbres de Jordan*, Lecture Notes in Mathe. **1049**, Springer Berlin/Heidelberg/New York, 1984.

[37] W. Kaup, *A Riemann mapping theorem for bounded symmetric domains in complex Banach spaces*, Math. Z. **183** (1983), 503–529.

[38] ———, *Contractive projections on Jordan C*-algebras and generalisations*, Math. Scand. **54** (1984), 95–100.

[39] M. Koecher, *Positivitätsbereiche im $\mathbb{R}^m$*, Amer. J. Math. **79** (1957), 575–596.

[40] D. Li, *Espaces L-facteurs biduaux: bonne disposition*, meilleure approximation et proprieté de Radon-Nikodym, Quart. J. Math. Oxford **38** (1987), 229–243.

[41] O. Loos, *Bounded Symmetric Domains and JordanPairs*, Mathematical Lectures, University of California, Irvine, 1977.

[42] K. McCrimmon, *Inner ideals in quadratic Jordan algebras*, Trans. Amer. Math. Soc. **159** (1971), 445–468.

[43] K. Meyberg, *Jordan-Tripelsysteme und die Koecher-Konstruktion von Lie-Algebren*, Math. Z. **115** (1970), 58–78.

[44] M. Neal and B. Russo, *State Spaces on JB*-Triples*, in preparation.

[45] E. Neher, *Jordan Triple Systems by the Grid Approach*, Lecture Notes in Math. **1280**, Springer, Berlin/Heidelberg/New York, 1987.

[46] R. R. Phelps, *Uniqueness of Hahn-Banach extensions and unique best approximations*, Trans. Amer. Math. Soc. **95** (1960), 238–255.

[47] A. Rodriguez Palacios, *Jordan structures in analysis*, in W. Kaup, K. McCrimmon and H. Petersson (Eds.): Jordan Algebras, Proceedings of the Conference held in Oberwolfach, Germany, August 9-15 1992, W. de Gruyter Berlin/New York, 1994.

[48] B. Russo, *Structure of JB*-triples*, in W. Kaup, McCrimmon and H. Petersson (Eds.): Jordan Algebras, Proceedings of the Conference held in Oberwolfach, Germany, August 9-15 1992, W. de Gruyter Berlin/New York, 1994.

[49] B. Sheppard, *A Stone-Weierstrass Theorem for JB*-Triples*, J. London Math. Soc. **65** (2002), no. 2, 381–396.

[50] L. L. Stachò, *A projection principle concerning biholomorphic automorphisms*, Acta. Sci. Math. **44** (1982), 99–124.

[51] H. Upmeier, *Jordan algebras in analysis, operator theory and quantum mechanics*, Providence Amer. Math. Soc., 1986

[52] J. P. Vigué, *Le groupe des automorphismes analytiques d'un domaine borné d'un espace de Banach complexe*, Ann. Sci. École Norm. Sup. **9** (1976), no. 4, 203–282.

[53] ———, *Les domaines bornés symetriques d'un espace de Banach et les systèmes triples de Jordan*, Math. Ann. **229** (1977), 223–231.

[54] ———, *Les automorphismes analytiques isométriques d'un variété complexe normée*, Bull. Soc. Math. France **110** (1982), 49–73.

[55] J. D. M. Wright, *The structure of decoherence functionals for von Neumann quantum histories*, J. Math. Phys. **36** (1995), 5409–5413.

School of Mathematics
Trinity College
Dublin 2, Ireland
E-mail: hugli@maths.tcd.ie