Periodica Mathematica Hungarica Vol. 56 (1), 2008, pp. 71–82
DOI: 10.1007/s10998-008-5071-y

STABILITY PROBLEMS FOR THE MATHEMATICAL PENDULUM

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Dedicated to the memory of Professor Miklós Farkas

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(*Received August 9, 2007; Accepted September 23, 2007*)

Abstract

The first part of this review paper is devoted to the simple (undamped, unforced) pendulum with a varying coefficient. If the coefficient is a step function, then small oscillations are described by the equation

$$\ddot{x} + a^2(t)x = 0,\ a(t) := a_k \quad \text{if} \quad t_{k-1} \le t < t_k, \quad k = 1, 2, \ldots.$$

Using a probability approach, we assume that $(a_k)_{k=1}^{\infty}$ is given, and $\{t_k\}_{k=1}^{\infty}$ is chosen at random so that $t_k - t_{k-1}$ are independent random variables. The first problem is to guarantee that all solutions tend to zero, as $t \to \infty$, provided that $a_k \nearrow \infty$ $(k \to \infty)$. In the problem of swinging the coefficient a^2 takes only two different values alternating each others, and $t_k - t_{k-1}$ are identically distributed. One has to find the distributions and their critical expected values such that the amplitudes of the oscillations tend to ∞ in some (probabilistic) sense. In the second part we deal with the damped forced pendulum equation

$$\ddot{x} + 10^{-1}\dot{x} + \sin x = \cos t.$$

In 1999 J. Hubbard discovered that some motions of this simple physical model are chaotic. Recently, using also the computer (the method of interval arithmetic), we gave a proof for Hubbard's assertion. Here we show some tools of the proof.

Mathematics subject classification number: 34C11, 34C28, 65G30, 70K40.

Key words and phrases: random coefficients, parametric resonance, forced damped pendulum, computer-aided proof, interval arithmetic.

[1] Supported by the Hungarian NFSR (OTKA T49516) and by the Analysis and Stochastics Research Group of the Hungarian Academy of Sciences.

0031-5303/2008/$20.00
Akadémiai Kiadó, Budapest
Springer, Dordrecht

1. Introduction

The simple mathematical plane pendulum is a mass m attached to a rigid massless rod with length ℓ. According to Newton's Second Law, the motion is described by the nonlinear second order differential equation

$$\ddot{\varphi} + \frac{g}{\ell} \sin \varphi = 0, \tag{1.1}$$

where φ denotes the angle from the direction of the gravity to the rod measured counterclockwise, and g is the gravitational constant. If friction and drag are taken into account, then the pendulum is damped, whose motions are described by the equation

$$\ddot{\varphi} + b\dot{\varphi} + \frac{g}{\ell} \sin \varphi = 0 \qquad (b > 0), \tag{1.2}$$

where the positive constant b is the damping coefficient. We will consider also the case when an external periodic force also acts:

$$\ddot{\varphi} + b\dot{\varphi} + \frac{g}{\ell} \sin \varphi = A \cos \kappa t, \tag{1.3}$$

where A and κ denotes the amplitude and the frequency of the external force, respectively.

In this paper we give a brief review on our results obtained for the simple pendulum with varying length and for the forced damped pendulum. In Section 2 we consider the special case of the linearized version of equation (1.1) when ℓ is a step function. We use a probability approach. It will be pointed out that $\lim_{t\to\infty} \varphi(t) = 0$ almost surely provided $\lim_{t\to\infty} \ell(t) = 0$. We consider also the problem of swinging, i.e., the case when ℓ takes only two different values, and the distances between the consecutive jump points are independent identically distributed random variables with expected value T. We find the values of T for which the so-called *parametric resonance* happens, i.e., the lower equilibrium is unstable in some sense. In Section 3 we show that the motion of the forced damped pendulum (1.3) can be unpredictable or chaotic.

2. Small oscillations of a pendulum with varying length

"Small oscillations" of the simple pendulum (1.1) are described by the linear equation

$$\ddot{\varphi} + \omega_0^2 \varphi = 0, \qquad \left(\omega_0 = \sqrt{\frac{g}{\ell}}\right). \tag{2.1}$$

As is well known, all motions are periodic with the same period $2\pi/\omega_0$. If the length of the pendulum varies in time ($\ell = \ell(t)$), then (2.1) is of the form

$$\ddot{x} + a^2(t)x = 0, \tag{2.2}$$

where $a(t) := \sqrt{g/\ell(t)}$. At first, imagine the situation when one has to lift a weight by a pulley and a rope through a gap. Then it can be assumed that ℓ is a decreasing step function and $\lim_{t\to\infty} \ell(t) = 0$. If the gap is narrow, then it is natural to want all solutions to have the property

$$\lim_{t\to\infty} x(t) = 0, \tag{2.3}$$

in other words, one has to stabilize the equilibrium $x = \dot{x} = 0$ *with respect to* x. (The equilibrium is unstable with respect to the vector $(x, \dot{x})$ because a is unbounded!)

It can happen that one wants to destabilize the lower equilibrium with respect to x, e.g., in the problem of swinging (see [1]). The swinger changes the height of his/her center of gravity periodically (at least nearly periodically). This results in a periodic change of the length of the corresponding mathematical pendulum. The goal is to find the values of the period (the values of the approximate period) for which the zero solution of (2.2) is unstable.

2.1. Increasing coefficient a

Consider equation (2.2) and suppose that $a\colon [0,\infty) \to [0,\infty)$ is non-decreasing and $a(t) \nearrow \infty$ as $t \to \infty$. By the Polya–Sonin theorem the solutions are not periodic any more, they oscillate, the amplitudes of the oscillations decrease, and the frequencies increase. We are concerned to guarantee that the amplitudes tend to zero, as $t \to \infty$, i.e., (2.3) is satisfied. H. Milloux [4] proved that such a (non-trivial) solution always *exists*, provided a is differentiable. We extended this result to step function coefficients [14]. The Armellini–Tonelli–Sansone theorem (see, e.g., [13]) says that *all* solutions satisfy (2.3) if $a(t) \nearrow \infty$ "regularly", as $t \to \infty$, which briefly means that the growth of the function a cannot be located to a set of small measure.

Equation (2.2) with a step function coefficient a^2 often serves as a mathematical model in applications [21]. For example, functions in the problems of lifting a weight and swinging mentioned in the Introduction may be step functions. In this section we restrict ourselves to this kind of coefficients.

Given two sequences of positive numbers $(a_k)_{k=1}^{\infty}$, $(t_k)_{k=1}^{\infty}$ ($t_k \le t_{k+1}$, $k = 1, 2, \ldots$) and $t_0 := 0$, we consider the second order linear differential equation

$$\ddot{x} + a^2(t)x = 0,\ a(t) := a_k \quad \text{if} \quad t_{k-1} \le t < t_k, \quad k = 1, 2, \ldots. \tag{2.4}$$

A function $x\colon [0,\infty) \to \mathbb{R}$ is called a *solution* of (2.4) if it is continuously differentiable on $[0,\infty)$ and it solves the equation on every (t_{k-1}, t_k) for $k = 1, 2, \ldots$.

There are different approaches to study equation (2.4). It is possible to apply the results of the general theory to this special case [11], [15], [16], [25]. Á. Elbert [7], [8], [9] initiated the method of step-by-step integration. We suggested a new geometric approach [14], [19], [18], [17], [20], [12]. Introducing the new state variable $y := \dot{x}/a_k$ we rewrote equation (2.4) into the 2-dimensional system

$$\dot{x} = a_k y, \quad \dot{y} = -a_k x \quad \text{if} \quad t_{k-1} \le t < t_k, \quad k = 1, 2, \dots. \tag{2.5}$$

The dynamics turns the x-y plane uniformly around the origin. Since $\dot{x}(t)$ has to be continuous, the initial value $y(t_k)$ on the interval $[t_k, t_{k+1})$ is defined by

$$y(t_k) := \frac{a_k}{a_{k+1}} y(t_k - 0), \quad k = 1, 2, \dots, \tag{2.6}$$

which means a contraction along the y-axis of measure a_k/a_{k+1} at $t = t_k$.

Let us turn to the stabilization of the origin with respect to x. First of all, one has to observe that the Armellini–Tonelli–Sansone theorem and its improvements do not work here because the increase of the coefficient a is located to the enumerable set $\{t_k\}_{k=1}^{\infty}$. Integrating step-by-step, Á. Elbert [7], [8], [9] obtained a very sharp (but sophisticated) sufficient condition for the asymptotic stability with respect to x in a form of the divergence of an infinite series containing a_k and $\tau_k := t_k - t_{k-1}$. It would be very hard to control motions by the use of this formula even if one could observe and measure the state variables during the motions, what, in general, cannot be assumed. On the other hand, analyzing this formula (see also counterexamples in [10], [25], one conjectures that equation (2.2) with coefficient of the property $a(t) \nearrow \infty$ $(t \to \infty)$ can have solutions not vanishing asymptotically only for *exceptional* coefficients a^2. For this reason, we formulated the following practical problem [14]: How often does it happen that *all* solutions tend to zero? To be more precise, what is the probability that all solutions tend to zero, provided that the heights of the steps in the step function a are given, but the distances between the consecutive time points of jump are independent random variables?

In [17] we answered this question in the case when the differences $t_k - t_{k-1}$ $(k = 1, 2, \dots)$ are totally independent random variables uniformly distributed on the interval $[0, 1]$. We proved that if $a_k \nearrow \infty$ $(k \to \infty)$, then it is almost sure that all solutions of (2.4) tend to zero as $t \to \infty$. In [19] this result was extended to non-monotonous sequences $(a_k)_{k=1}^{\infty}$. Recently we could generalize the result to random sequences $\{\tau_k = t_k - t_{k-1}\}_{k=1}^{\infty}$ of *arbitrary* distributions [6].

Let F_k and ϕ_k denote the distribution function and the characteristic function of the random variable τ_k, respectively, i.e.,

$$F_k(x) := \mathcal{P}(\tau_k \le x), \quad \phi_k(t) := \int_0^{\infty} e^{itx}\, dF_k(x),$$

and suppose that $\tau_1, \tau_2, \dots, \tau_k, \dots$ are totally independent.

Theorem 2.1. *If* $a_k \nearrow \infty$ $(k \to \infty)$, *and*

$$\limsup_{k\to\infty} |\phi_k(2a_k)| < 1, \tag{2.7}$$

then for every solution x *of equation* (2.4)

$$\lim_{t\to\infty} x(t) = 0 \qquad \textit{almost surely.}$$

Let us apply this theorem to the case where τ_k is uniformly distributed on $[0, T_k]$. Then

$$\phi_k(t) = \frac{\sqrt{2}\sqrt{1-\cos T_k t}}{T_k t} \qquad (t \geq 0),$$

and the stability condition (2.7) has the form

$$\limsup_{k\to\infty} |\phi_k(2a_k)| = \limsup_{k\to\infty} \frac{|\sin T_k a_k|}{T_k a_k} < 1. \tag{2.8}$$

Corollary 2.2. *Suppose* $a_k \nearrow \infty$ $(k \to \infty)$, *and* $\tau_k = t_k - t_{k-1}$ *is uniformly distributed on the interval* $[0, T_k]$. *If*

$$\liminf_{k\to\infty} \left\{\frac{T_k}{2} a_k\right\} = \liminf_{k\to\infty} \{\mathcal{E}(\tau_k) a_k\} > 0$$

($\mathcal{E}\{\cdot\}$ *denotes the expected value), then* $\lim_{t\to\infty} x(t) = 0$ *almost surely. In particular, if* $T_k \geq T_* > 0$ $(k = 1, 2, \ldots)$, *then the assertion holds.*

This is a generalization of the main result in [17], where the t_k's were identically distributed ($T_k = 1$, $k = 1, 2, \ldots$).

Applying Elbert's sufficient condition [7], [8], [9] to the problem of lifting a weight by a pulley and a rope through a gap, the worker has to keep the members of the sequence $\{\tau_k a_k\}_{k=1}^{\infty}$ away from the set $\{j\pi\}_{j=0}^{\infty}$, which is not an easy instruction. At the same time, Corollary 2.2 yields the simple instruction: "Do not hurry!".

2.2. The problem of swinging

If the swinger changes the height of his/her center of gravity periodically with period $2T$, then the equation of small oscillations of the equivalent mathematical pendulum is of the form (2.4) with

$$t_{2n} = 2nT,\ t_{2n+1} = (2n+1)T,\ a_{2n+1} = \pi + \varepsilon,\ a_{2(n+1)} = \pi - \varepsilon \quad (n = 0, 1, 2, \ldots),$$

where $\varepsilon > 0$ can be arbitrarily small (if the swinger is a child, then ε is small). This is the Hill–Meissner equation [1], [21]. The problem is to find the values of T for which the amplitudes of all nontrivial motions tend to infinity as $t \to \infty$ at arbitrarily small values of ε ("parametric resonance"). It can be proved [1], [21] that the appropriate values are $T = m/2$ $(m = 1, 2, \ldots)$.

To make the model more realistic, it is natural to assume that a is not a deterministic periodic function; namely, the time intervals between the consecutive jump points are independent identically distributed random variables with expected value T. This model is of the form (2.4) with

$$a(t) := \begin{cases} \pi + \varepsilon & \text{if } t_{2n} \le t < t_{2n+1}, \\ \pi - \varepsilon & \text{if } t_{2n+1} \le t < t_{2(n+1)} \end{cases} \qquad n = 0, 1, 2, \ldots \tag{2.9}$$

At first we formulate a general instability theorem for equation (2.4), then we apply the result to the problem of swinging (2.9).

THEOREM 2.3. *Suppose that* $(\tau_k = t_k - t_{k-1})_{k=1}^{\infty}$ *are independent identically distributed random variables with expected value* T *and characteristic function* ϕ.

If

$$\begin{aligned} \beta = \beta(\varepsilon, T, \phi) := -(\pi^2 + \varepsilon^2)\{|\phi\,(2(\pi+\varepsilon))\,| + |\phi\,(2(\pi-\varepsilon))\,|\} + \\ +2\varepsilon\pi\{1 + |\phi\,(2(\pi+\varepsilon))\,||\phi\,(2(\pi-\varepsilon))\,|\} > 0, \end{aligned} \tag{2.10}$$

then for every solution of equation (2.4) *with coefficient* (2.9) *we have*

$$\lim_{k\to\infty} \mathcal{E}\left\{x^2(t_k) + \frac{(\dot{x}(t_k))^2}{a_{k+1}}\right\} = \infty.$$

The problem of *random parametric resonance* requires to determine the distributions (characteristic functions ϕ) and the expected values T satisfying (2.10). It is easy to see that $\phi(2\pi) = 0$ is necessary for (2.10).

At first let us consider the uniform distribution on the interval $[0, 2T]$. (Actually, this does not seem to be a good tactic for swinging!) The necessary condition gives

$$\phi(2\pi) = \frac{1}{2T\pi}\sin(2T\pi) = 0 \quad \Longrightarrow \quad T = \frac{m}{2} \qquad (m = 1, 2, \ldots),$$

which coincides with the solution of the problem of the deterministic (periodic) parametric resonance [1]. However, setting $T = m/2 + \delta$ we get

$$\lim_{\delta\to 0} \beta(\varepsilon, \frac{m}{2} + \delta, \phi) = -\frac{2\varepsilon^2}{\pi^2 - \varepsilon^2} + O(\varepsilon^3) < 0,$$

i.e., (2.10) is not satisfied in accordance with our expectations. But it is rather surprising that we get the same result for the uniform distribution on the interval $[T-\mu, T+\mu]$, with arbitrarily small $\mu > 0$. This means that decreasing the standard deviation cannot produce random parametric resonance.

Our next try was the sum of n independent random variables uniformly distributed on $[0, 2T]$. It is interesting that $n = 2$ already results in parametric resonance. The appropriate values are

$$T \in \left\{ \frac{n}{2}, 2\frac{n}{2}, 3\frac{n}{2}, \dots, m\frac{n}{2}, \dots \right\}.$$

For example, if $n = 2$, then $T \in \mathbb{N}$, and the domain of instability (so-called Arnold's tongues) are described by

$$|T - n| < \frac{n}{\pi}\sqrt{\varepsilon} - \frac{n}{\pi}\varepsilon + O(\varepsilon^{3/2}) \qquad (\varepsilon \to 0)$$

(compare with [1]).

3. Chaotic motions of the forced damped pendulum

3.1. Description of the chaos

Consider a special case of the forced damped pendulum equation (1.3):

$$\ddot{x} + 10^{-1}\dot{x} + \sin x = \cos t. \tag{3.1}$$

R. Borelli and C. Coleman [5] observed that numerical solutions of this equation were very sensitive to the integration method, step-length, and initial conditions near some points of the plane $(x, \dot{x})$. For example, the t-x graphs of the solutions starting from the points $P_1(0, 1.98)$, $P_2(0, 2.00)$, and $P_3(0, 2.01)$ can be seen on Figure 1.

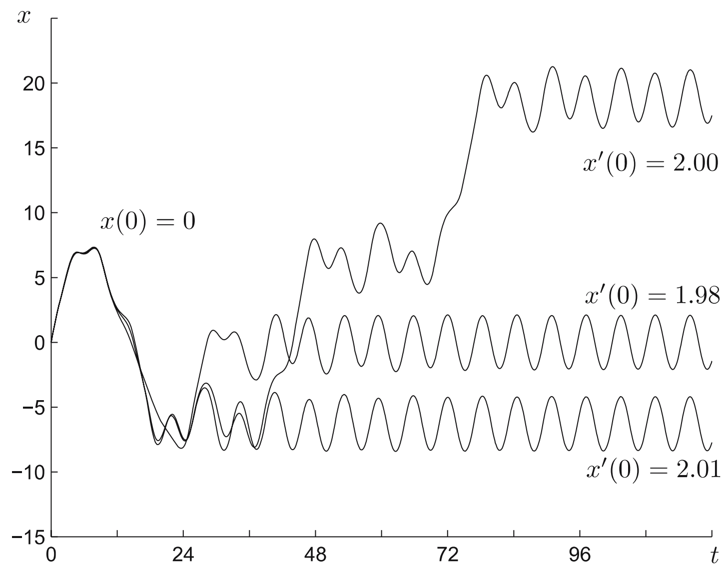

FIGURE 1. *Sensitivity to initial conditions*

One has to observe that the solutions are asymptotically periodic with period 2π (period of the external force). This experiment suggests that there exists a stable 2π-periodic motion around the downward position, which ultimately attracts all three solutions. A superficial observer could think that this 2π-periodic solution attracts all solutions. But this is not true. J. Hubbard [22] claimed the existence of uncountably many "strange" motions of the damped forced pendulum (3.1) whose asymptotic behavior is unpredictable; in other words, these motions are *chaotic* in the sense defined via natural mechanical terms in the following way. Let us divide the time line $\mathbb{R}$ into the intervals $I_k := [2k\pi, 2(k+1)\pi]$ $(k \in \mathbb{Z})$, and define ε_k in the following way:

$$\varepsilon_k := \begin{cases} \text{-1} & \text{if the pendulum crosses the bottom position exactly once,} \\ & \text{namely clockwise, during } I_k; \\ 1 & \text{if the pendulum crosses the bottom position exactly once,} \\ & \text{namely counterclockwise, during } I_k; \\ 0 & \text{if the pendulum does not cross the bottom position at all} \\ & \text{during } I_k. \end{cases}$$

If something else happens during I_k, then ε_k is not defined. It is surprising that the ε_k's may be prescribed arbitrarily, independently of one another for both the past and the future.

ASSERTION 3.1. (J. Hubbard) *Given any bi-infinite sequence of events* $\{\dots, \varepsilon_{-1}, \varepsilon_0, \varepsilon_1, \dots\}$ $(\varepsilon_k \in \{-1, 0, 1\},\ k \in \mathbb{Z})$, *called "itinerary", there exists a motion of the forced damped pendulum* (3.1) *that during each time interval* $[2k\pi, 2(k+1)\pi]$ *will "do"* ε_k.

For example, during the motion corresponding to the itinerary $\{\dots, 0, 0, 0, \dots\}$ the pendulum *never* crosses the bottom position. In fact, we can prove [4] that there exists an unstable 2π-periodic solution around the upper position not touching the bottom position.

In [22] Hubbard did not prove Assertion 3.1. In [4] we give a general theorem for detecting chaos in systems of differential equations, which can be applied to prove Assertion 3.1. In the same paper we also prove the existence of exactly two 2π-periodic solutions to (3.1). The application of the general chaos theorem to (3.1) needs rigorous methods of computations, which is done by interval arithmetic.

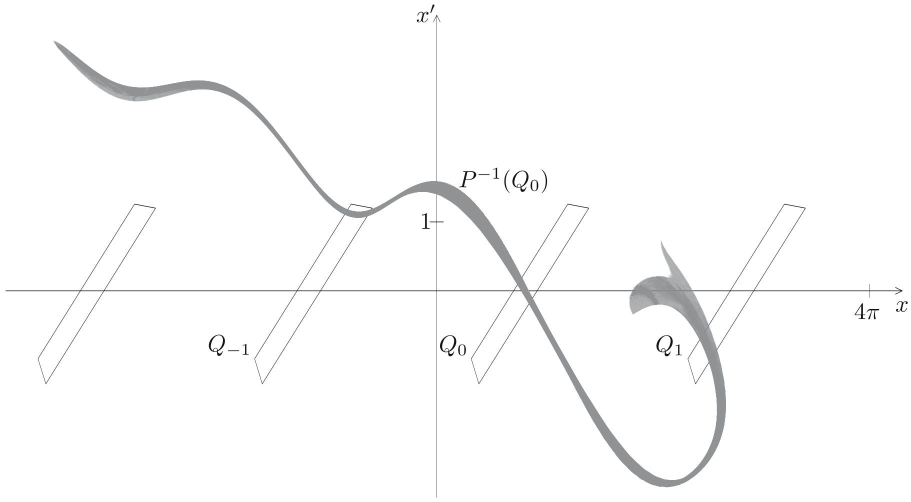

FIGURE 2A. *Backward image of* Q_0

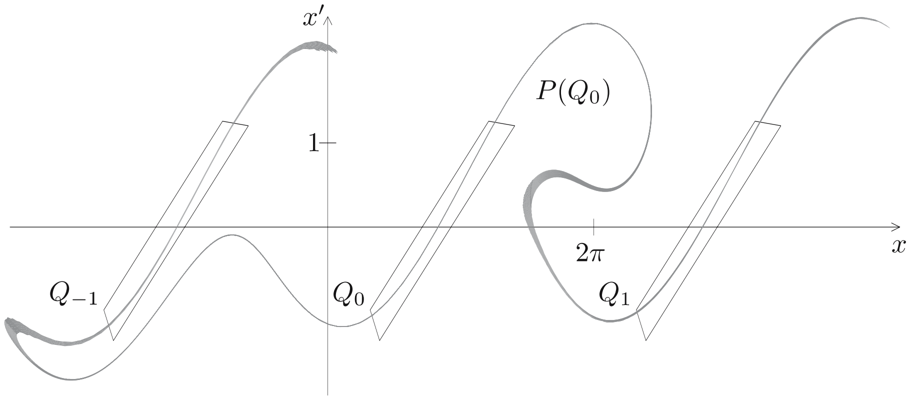

FIGURE 2B. *Forward image of* Q_0

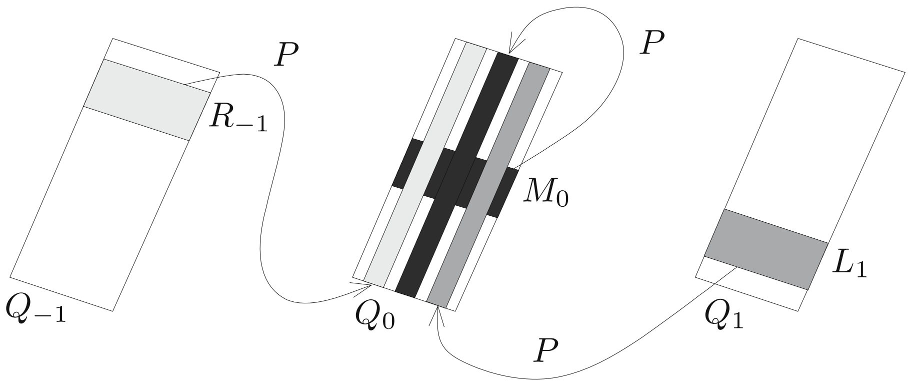

FIGURE 2C. *Schematic* $P(\bigcup_{k\in\mathbb{Z}} Q_k) \cap Q_0$

3.2. The tools of the proof of Assertion 3.1

Let $x(\cdot;t_0,x_0,\dot{x}_0)$ denote the solution of (3.1) satisfying the initial condition $x(t_0;t_0,x_0,\dot{x}_0)=x_0,\ \dot{x}(t_0;t_0,x_0,\dot{x}_0)=\dot{x}_0$. The mapping

$$P\colon \mathbb{R}^2 \to \mathbb{R}^2, \qquad P\colon (x_0,\dot{x}_0) \mapsto (x(2\pi;0,x_0,\dot{x}_0),\ \dot{x}(2\pi;0,x_0,\dot{x}_0))$$

is called the *period mapping* or *Poincaré mapping* to equation (3.1). If we are interested in stability properties of solutions of (3.1), then, instead of the *differential equation* (3.1), we can investigate the *discrete dynamical system*

$$P^k := \underbrace{P \circ P \circ \cdots \circ P}_{k \text{ times}}\colon \mathbb{R}^2 \to \mathbb{R}^2 \quad (k \in \mathbb{Z}). \tag{3.2}$$

An *orbit* of (3.2) is a bi-infinite sequence $\{P^k(x_0,\dot{x}_0)\}_{k\in\mathbb{Z}}$ $((x_0,\dot{x}_0) \in \mathbb{R}^2)$. The solution $x(\cdot;0,x_0,\dot{x}_0)$ of (3.1) is 2π-periodic if and only if $(x_0,\dot{x}_0)$ is a fixed point of P. A 2π-periodic solution of (3.1) is stable if and only if the corresponding fixed point of P is stable in the discrete dynamical system (3.2).

We prove [4] that P has exactly two fixed points in the region $(0,2\pi) \times (-\infty,\infty)$: a sink $s_0(4.236\ldots,0.392\ldots)$ and a saddle $u_0(2.634\ldots,0.026\ldots)$. The function $x \mapsto \sin x$ is 2π-periodic, so a horizontal 2π-shift of a fixed point of P is a fixed point, too. This means that we have infinitely many sinks and saddles:

$$s_k := s_0 + (2k\pi,0), \quad u_k := u_0 + (2k\pi,0) \quad (k \in \mathbb{Z}).$$

The basins of the sinks are of a very sophisticated structure. They are tangled; every basin meanders around the plane. To be more precise: the basins seem to have the Wada property, i.e., every point of the boundary of any basin belongs to the boundaries of all the others [22]. This is the root of the chaotic behaviour formulated in Assertion 3.1.

In the proof of Assertion 3.1 we need certain quadrilaterals $\{Q_k\}_{k\in\mathbb{Z}}$ around the saddles: "long" in the unstable and "short" in the stable directions so that there are so-called *"exceptional" orbits* of the Poincaré mapping P with the following properties:

- an exceptional orbit is contained in $\bigcup_{k\in\mathbb{Z}} Q_k$;
- an exceptional orbit visits the quadrilaterals consecutively: if $P^n(x_0,\dot{x}_0) \in Q_k$ for some $k,n \in \mathbb{Z}$, then either $P^{n+1}(x_0,\dot{x}_0) \in Q_{k-1}$ or $P^{n+1}(x_0,\dot{x}_0) \in Q_k$ or $P^{n+1}(x_0,\dot{x}_0) \in Q_{k+1}$.

It can be seen that to realize an itinerary $(\varepsilon_k)_{k=1}^{\infty}$ is equivalent to have an exceptional orbit visiting an appropriate sequence of quadrilaterals $(Q_{i_k})_{k=1}^{\infty}$. In the main step of the proof of Assertion 3.1 we show that for an arbitrary consecutive order $(Q_{i_k})_{k\in\mathbb{Z}}$ of quadrilaterals there is an exceptional orbit visiting the quadrilaterals in the prescribed order. To this end we have to know forward images $P(Q_k)$ and backward images $P^{-1}(Q_k)$. Thanks to the horizontal 2π-periodicity of the discrete dynamical system (3.2) it is enough to know the images $P(Q_0)$ and $P^{-1}(Q_0)$. For suitably chosen quadrilaterals the forward image $P(Q_0)$ crosses Q_{-1}, Q_0, Q_1 in long and thin "vertical strips", and the backward image $P^{-1}(Q_0)$ crosses Q_{-1}, Q_0, Q_1 in short and flat "horizontal strips" (see Figure 2). Let us denote these horizontal strips by R_{-1}, M_0, L_1, respectively. (P moves $R_{-1} \subset Q_{-1}$ to the *right*, it leaves the *middle* strip $M_0 \subset Q_0$ in Q_0, and it moves $L_1 \subset Q_1$ to the *left*.) The same connection is true for any triple Q_{k-1}, Q_k, Q_{k+1} ($k \in \mathbb{Z}$) with $R_{k-1} \subset Q_{k-1}$, $M_k \subset Q_k$, $L_{k+1} \subset Q_{k+1}$. Using the method of interval arithmetic we can prove by reliable computer simulations [24] that such $\{Q_k, R_k, M_k, L_k\}_{k\in\mathbb{Z}}$ exist.

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