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Abstract

We define the domain of a linear fractional transformation in a space of operators and show that both the affine automorphisms and the compositions of symmetries act transitively on these domains. Further, we show that Liouville's theorem holds for domains of linear fractional transformations and, with an additional trace class condition, so does the Riemann removable singularities theorem. We also show that every biholomorphic mapping of the operator domain $I < Z^*Z$ is a linear isometry when the space of operators is a complex Jordan subalgebra of $\mathcal{L}(H)$ with the removable singularity property and that every biholomorphic mapping of the operator domain $I + Z_1^*Z_1 < Z_2^*Z_2$ is a linear map obtained by multiplication on the left and right by J-unitary and unitary operators, respectively.

0. - Introduction

This paper introduces a large class of finite and infinite dimensional symmetric affinely homogeneous domains which are not holomorphically equivalent to any bounded domain. These domains are subsets of spaces of operators and include domains as diverse as a closed complex subspace of the bounded linear operators from one Hilbert space to another, the identity component of the group of invertible operators in a C^*-algebra and the complement of a hyperplane in a Hilbert space. Each of our domains may be characterized as a component of the domain of definition of some linear fractional transformation which maps a neighborhood of a point in the component biholomorphically onto an open set in the same space. Thus we refer

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to our domains as domains of linear fractional transformations. We show that at any point of such a domain, there exists a biholomorphic linear fractional transformation of the domain onto itself which is a symmetry at the point. This linear fractional transformation is a generalization of the Potapov-Ginzburg transformation. Moreover, for any two points Z_0 and W_0 in the domain, there exists an affine automorphism of the domain of the form $\phi(Z) = W_0 + A(Z - Z_0)B$ and ϕ is a composition of the above symmetries. In fact, we exhibit many different formulae for linear fractional transformations that are automorphisms of domains of linear fractional transformations.

We show that any bounded holomorphic function on the domain of a linear fractional transformation is constant and, more generally, that the Kobayashi pseudometric vanishes identically on such domains. We also show that if the domain of a linear fractional transformation satisfies a trace class condition, then a holomorphic function on the domain which is locally bounded with respect to points outside the domain has a holomorphic extension to these points. Further, we give a sufficient condition for two domains of linear fractional transformations to be affinely equivalent when the subspace of operators considered is the full space or a C^*-algebra. As expected, our unbounded symmetric homogeneous domains fail to have many of the properties of bounded symmetric domains. For example, Cartan's uniqueness theorem may fail, symmetries may not be unique and automorphisms may not be a composition of a linear fractional transformation and a linear map.

Next we turn to a discussion of the characterization of the automorphisms of non-homogeneous domains which are operator analogues of the domain $1+|z_1|^2 < |z_2|^2$ in $\mathbb{C}^2$ (which is included). The domains we consider are circular domains in a space of operators in the sense discussed previously by the author in [14] and are holomorphically equivalent to the open unit ball of the space with certain singular points omitted. We show that with some restrictions, these points are removable singularities and we deduce that all biholomorphic mappings of the domains are linear. In the case where the domain and range spaces of the operators considered have different dimensions, these linear maps are given by operator matrices which are the coefficient matrices of linear fractional transformations of the unit ball of the space of operators on the domain space. In the case where the operators considered have the same domain and range, our domains reduce to analogues of the exterior of the unit disc in spaces of operators and the linear maps are invertible isometries of these spaces. A result of Arazy and Solel [2] allows us to treat spaces of operators which are not necessarily closed under adjoints.

A main lemma is the result that every J^*-algebra with identity which is closed in the weak operator topology has what we call the removable singularity property, i.e., the singular operators in the open unit ball are removable singularities for bounded holomorphic functions. This is rather surprising since these singular operators may have non-empty interior in the infinite dimensional case. We also give two types of homogeneous circular domains where the group of linear automorphisms of the domains acts transitively.

Unbounded homogeneous domains in $\mathbb{C}^n$ have been considered previously by Penney [22] and Winkelmann [28]. In particular, Penney obtains a classification of the rationally homogeneous domains in $\mathbb{C}^n$ analogous to the classification theorem for bounded homogeneous domains given by Vinberg, Gindikin and Piatetskii-Shapiro. See also Hua [18].

Our discussion emphasizes the explicit construction of domains and mappings using ideas from operator theory and functional analysis. We have considered bounded symmetric homogeneous domains from this point of view previously in [10]. See [20] and [25] for an exposition of the theory of bounded symmetric domains in infinite dimensions.

Let H and K be complex Hilbert spaces and let $\mathcal{L}(H,K)$ denote the Banach space of all bounded linear transformations from H to K with the operator norm. We write $\mathcal{L}(H)$ for $\mathcal{L}(H,H)$. Throughout, $\mathcal{U}$ and $\mathcal{B}$ denote any closed complex subspaces of $\mathcal{L}(H,K)$. (The reader interested only in the finite dimensional case may take $H = \mathbb{C}^n$, $K = \mathbb{C}^m$ and identify $\mathcal{L}(H,K)$ with the vector space of all $m \times n$ matrices of complex numbers.)

1. - Domains of Linear Fractional Transformations

Let $C \in \mathcal{L}(K,H)$, $D \in \mathcal{L}(H)$ and suppose $(CZ_0 + D)^{-1}$ exists for some $Z_0 \in \mathcal{U}$. Put $X_0 = (CZ_0 + D)^{-1}C$. If $ZX_0Z \in \mathcal{U}$ for all $Z \in \mathcal{U}$, we define

$$\mathcal{D} = \mathrm{Comp}_{Z_0}\{Z \in \mathcal{U} : (CZ + D)^{-1} \text{ exists}\} \tag{1}$$

to be the *domain of a linear fractional transformation on* $\mathcal{U}$. (Here and elsewhere, Comp_{Z_0} denotes the connected component containing Z_0.) It is easy to show that any finite or infinite product of domains of linear fractional transformations is the domain of a linear fractional transformation. To justify this terminology, we observe that by [14, Prop. 5], if the linear fractional transformation

$$T(Z) = (AZ + B)(CZ + D)^{-1} \tag{2}$$

maps a domain in $\mathcal{U}$ containing Z_0 onto a subset of some $\mathcal{B}$ containing an interior point of $\mathcal{B}$ and if the coefficient matrix of T is invertible, then $ZX_0Z \in \mathcal{U}$ for all $Z \in \mathcal{U}$ and T is a biholomorphic mapping of the domain $\mathcal{D}$ above onto a similar domain in $\mathcal{B}$. We consider components because a linear fractional transformation may be a biholomorphic mapping of one component of the open set where it is defined while mapping the other components outside the space. (See Example 1 below.)

THEOREM 1. *The domain $\mathcal{D}$ of a linear fractional transformation on $\mathcal{U}$ is a symmetric affinely homogeneous domain. In particular, if $\mathcal{D}$ is given by* (1), *then for each $Y \in \mathcal{D}$,*

$$U_Y(Z) = Y - (Z - Y)(CZ + D)^{-1}(CY + D) \tag{3}$$

is a biholomorphic mapping of $\mathcal{D}$ onto itself satisfying $U_Y^2 = I$, $U_Y(Y) = Y$ and $DU_Y(Y) = -I$. Moreover, for any point $W_0 \in \mathcal{D}$, there is an invertible affine linear fractional transformation ϕ of $\mathcal{D}$ onto itself with $\phi(Z_0) = W_0$ and ϕ can be chosen to be a composition of mappings U_Y given above.

Winkelmann [28, Prop. 1] has proved a related result for the complement of an algebraic subvariety in $\mathbb{C}^n$. It is easy to show that U_Y is the only linear fractional transformation (2) with the mapping property described there which satisfies $T(Y) = Y$ and $DT(Y) = -I$ on $\mathcal{U}$. The transformations U_Y are used in [15] to obtain factorizations of certain operator matrices.

EXAMPLE 0. Any $\mathcal{U}$ is the domain of a linear fractional transformation on $\mathcal{U}$ when $C = 0$ and $D = I$ since then $X_0 = 0$ for any choice of Z_0. In this case, $U_Y(Z) = 2Y - Z$.

EXAMPLE 1. Let $\mathcal{U}$ be a power algebra [9], i.e., $\mathcal{U}$ is a closed complex subspace of $\mathcal{L}(H)$ containing the identity operator I on H and the squares of each of its elements. Put

$$G_I(\mathcal{U}) = \mathrm{Comp}_I\{Z \in \mathcal{U} : Z^{-1} \text{ exists}\}. \tag{4}$$

Clearly, $G_I(\mathcal{U})$ is the domain of a linear fractional transformation on $\mathcal{U}$ since $X_0 = I$ for $Z_0 = I$. Also, $U_Y(Z) = YZ^{-1}Y$. If $\mathcal{U}$ is a W^*-algebra, the argument in [6, Cor. 5.30] shows that the invertible operators in $\mathcal{U}$ are connected so it is not necessary to take components in this case. However, if $\mathcal{U}$ is the closure of the polynomials in the bilateral shift [6, Example 4.25], then U_I is a biholomorphic mapping of $G_I(\mathcal{U})$ but U_I does not take any other invertible elements of $\mathcal{U}$ (such as the bilateral shift) to $\mathcal{U}$.

EXAMPLE 2. Let $\mathcal{U}$ be a closed complex subspace of $\mathcal{L}(H)$ containing a projection E such that $ZEZ \in \mathcal{U}$ whenever $Z \in \mathcal{U}$. Then

$$\mathcal{D} = \mathrm{Comp}_E\{Z \in \mathcal{U} : (EZ + I - E)^{-1} \text{ exists}\}$$

is the domain of a linear fractional transformation on $\mathcal{U}$ since $X_0 = E$ for $Z_0 = E$. The previous examples are the cases $E = 0$ and $E = I$. Moreover,

$$U_E(Z) = [(E - I)Z + E](EZ + I - E)^{-1}$$

is a variant form of the Potapov-Ginzburg transformation [1, §3]. (Another variant is called the Redheffer transform in [3, p. 269]. An early reference is

[19, p. 240].) A basic property is that U_E is a biholomorphic mapping of $\mathcal{D}_1 \cap \mathcal{D}$ onto $\mathcal{D}_2 \cap \mathcal{D}$, where

$$\mathcal{D}_1 = \{Z \in \mathcal{U} : \ Z^*JZ < J\}, \quad \mathcal{D}_2 = \{Z \in \mathcal{U} : \ \|Z\| < 1\},$$

and $J = I - 2E$. This follows directly from [14, Lemma 7] since the coefficient matrix M of U_E satisfies $J_2 = (M^{-1})^* J_1 M^{-1}$, where

$$J_1 = \begin{bmatrix} J & 0 \\ 0 & -J \end{bmatrix}, \quad J_2 = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}.$$

EXAMPLE 3. If $\operatorname{ran} C$ is closed or if $\operatorname{ran} D \subseteq \operatorname{ran} C$, then

$$\mathcal{D} = \{Z \in \mathcal{L}(H, K) : (CZ + D)^{-1} \text{ exists}\}$$

is the domain of a linear fractional transformation on $\mathcal{U} = \mathcal{L}(H, K)$ when $\mathcal{D}$ is not empty since $\mathcal{U}$ is obviously closed under the required products and since $\mathcal{D}$ is connected by Proposition 4 below.

EXAMPLE 4. Let $c \in H$ with $c \neq 0$ and $d \in \mathbb{C}$. Then

$$\mathcal{D} = \{z \in H : (z, c) \neq -d\}$$

is the complement of a hyperplane in H and the domain of a linear fractional transformation on H with

$$U_y(z) = y - \frac{(y, c) + d}{(z, c) + d}(z - y).$$

(Identify H with $\mathcal{L}(\mathbb{C}, H)$ and apply Example 3 with $C = c^*$.)

EXAMPLE 5. Let $\mathcal{U}$ contain all the rank one operators in $\mathcal{L}(H, K)$ and let $x \in H$, $y \in K$ be unit vectors. Then if $d \neq 0$,

$$\mathcal{D} = \{Z \in \mathcal{U} : (Zx, y) \neq -d\}$$

is the domain of a linear fractional transformation on $\mathcal{U}$. Indeed, let $C = xy^*$, $D = dI$ and note that $CZ + D = dI + x(y^*Z)$ is invertible if and only if $d + (y^*Z)x$ is invertible. Also $X_0 = xy^*$ for $Z_0 = (1 - d)yx^*$ and

$$U_Y(Z) = Y - \frac{Z - Y}{d}\left[I - \frac{xy^*Z}{d + (Zx, y)}\right](dI + xy^*Y).$$

Since C is compact, $\mathcal{D}$ is connected by Proposition 4 below.

EXAMPLE 6. Let H be a Hilbert space with conjugation $z \to \bar{z}$ and let $c \in H$ with $c \neq 0$. Then each of the domains

$$\mathcal{D}_1 = \{z \in H : 1 + 2(z, c) + (z, \bar{z})(\bar{c}, c) \neq 0\},$$

$$\mathcal{D}_2 = \{z \in H : (z, \bar{z}) \neq 0\}$$

is linearly equivalent to the domain of a linear fractional transformation. To see this, observe that by [10, § 2] there exists an invertible linear map $z \to A_z$ of H onto a Cartan factor $\mathcal{U}$ of type IV. For $\mathcal{D}_1$, take $C = A_c^*$, $Z = A_z$ and $D = I$, and note that $CZ + D$ is invertible if and only if $z \in \mathcal{D}_1$ by [10, (9)]. Also, $X_0 = A_c^*$ for $Z_0 = 0$ so $ZX_0Z \in \mathcal{U}$ for all $Z \in \mathcal{U}$ since $\mathcal{U}$ is a J^*-algebra. Moreover, $\mathcal{D}_1$ is connected by Proposition 4 below since X_0Z satisfies a quadratic polynomial equation. A similar argument with $C = I$ and $D = 0$ establishes the result for $\mathcal{D}_2$, and in that case,

$$U_y(z) = \frac{2(z, \bar{y})y - (y, \bar{y})z}{(z, \bar{z})}.$$

PROOF OF THEOREM 1. Given $Y \in \mathcal{D}$, put $X = (CY + D)^{-1}C$ and note the identities

$$CU_Y(Z) + D = (CY + D)(CZ + D)^{-1}(CY + D), \tag{5}$$

$$(CY + D)^{-1}(CZ + D) = I + X(Z - Y). \tag{6}$$

The coefficient matrix of U_Y is

$$M = \begin{bmatrix} (I - YX) & 2Y - YXY \\ X & I - XY \end{bmatrix}$$

by (6). Clearly $M^2 = I$ so M is invertible and $U_Y^2 = I$. Also, $U_Y(Y) = Y$. Hence by hypothesis and [14, Prop. 5], U_Y is a biholomorphic mapping of $\mathcal{D}$ onto itself when $Y = Z_0$ and therefore $ZXZ \in \mathcal{U}$ whenever $Z \in \mathcal{U}$. Thus [14, Prop. 5] applies again to show that U_Y is a biholomorphic mapping of $\mathcal{D}$ onto itself for arbitrary $Y \in \mathcal{D}$. Further, $DU_Y(Y) = -I$ since $DU_Y(Y)Z = \frac{d}{dt} U_Y(Y + tZ)|_{t=0}$ and $U_Y(Y + tZ) = Y - tZ(I + tXZ)^{-1}$.

Next we show that if $Z, W \in \mathcal{D}$ satisfy $\|(CZ + D)^{-1}C(W - Z)\| < 1$, then there is a $Y \in \mathcal{D}$ with $U_Y(Z) = W$. Put $X = (CZ + D)^{-1}C$ and $R = W - Z$. By the holomorphic functional calculus [17, § 5.2], the binomial series defines a $Q \in \mathcal{L}(H)$ such that $Q^2 = I + XR$, $\sigma(Q)$ does not contain -1, and $(I + Q)^{-1}$ is a limit of polynomials in XR. Take $Y = Z + R(I + Q)^{-1}$. Then $Y \in \mathcal{U}$ by [14, Lemma 6a]. Also, since $XR = Q^2 - I$, we have that

$$\begin{aligned} U_Y(Z) - Z &= Y - Z + (Y - Z)[I + X(Y - Z)] \\ &= R(I + Q)^{-1}[2(I + Q) + XR](I + Q)^{-1} \\ &= R \end{aligned}$$

so $U_Y(Z) = W$. Moreover, $(CY + D)^{-1}$ exists by this and (5). To show that $Y \in \mathcal{D}$, let $0 \le t \le 1$. The above argument applies to the points Z and $W_t = (1 - t)Z + tW$ and the corresponding Q_t is a continuous function of t in $[0, 1]$. Therefore, there is a curve γ in $\mathcal{U}$ connecting Z to Y such that $C\gamma(t)+D$ is invertible for all $0 \le t \le 1$.

Now given $W_0 \in \mathcal{D}$, there is a curve γ in $\mathcal{D}$ connecting Z_0 to W_0. By compactness, there is a number M such that $\|(C\gamma(t) + D)^{-1}C\| \le M$ for all $0 \le t \le 1$ and there is a $\delta > 0$ such that $\|\gamma(s)-\gamma(t)\| < 1/M$ whenever $|s-t| \le \delta$ and $0 \le s,\ t \le 1$. Choose $n > 1/\delta$ and put $Z_k = \gamma(k/n)$ for $0 \le k \le n$. Then for $1 \le k \le n$,

$$\|(CZ_{k-1} + D)^{-1}C(Z_k - Z_{k-1})\| < 1, \tag{7}$$

so by what we have shown there exists a U_k with $U_k(Z_{k-1}) = Z_k$. Thus $U = U_1 \cdots U_n$ satisfies $U(Z_0) = W_0$.

To complete the proof, it suffices to show that $U_W \circ U_Y$ is an affine linear fractional transformation for any W, $Y \in \mathcal{D}$ since then we can choose n to be even in the above argument and take $\phi = U$. Put $X = (CY + D)^{-1}C$ and set

$$Q = [W - U_Y(Z)][I + X(Z - Y)].$$

Then $Q = W - Y + [I + (W - Y)X](Z - Y)$ by (6). Hence by (5) and (6),

$$\begin{aligned} U_W \circ U_Y(Z) &= W + Q(CY + D)^{-1}(CW + D) \\ &= U_W(Y) + [I + (W - Y)X](Z - Y)[I + X(W - Y)], \end{aligned}$$

as required. □

There are many biholomorphic mappings of the domain of a linear fractional transformation on $\mathcal{U}$ besides (3). For example, if $W_0 \in \mathcal{D}$ and $\|X_0(W_0 - Z_0)\| < 1$, then

$$\phi(Z) = W_0 + [I + (W_0 - Z_0)X_0]^{1/2}(Z - Z_0)[I + X_0(W_0 - Z_0)]^{1/2}$$

is an invertible affine linear fractional transformation of $\mathcal{D}$ onto itself with $\phi(Z_0) = W_0$, where the square roots are defined by the binomial series. This follows from [14, Lemma 6c], (6) and the identity

$$I + X_0[\phi(Z) - Z_0] = R^{1/2}[I + X_0(Z - Z_0)]R^{1/2},$$

where $R = I + X_0(W_0 - Z_0)$. Moreover, the above proof shows that finite compositions of these mappings act transitively on $\mathcal{D}$. Note that $\phi = U_W \circ U_Y$ when $W_0 = U_W(Y)$ and $Z_0 = Y$. If $\mathcal{U} = \mathcal{L}(H, K)$ or if $\mathcal{U}$ is a C^*-algebra with C, $D \in \mathcal{U}$, a simpler formula for a transitive set of affine mappings of $\mathcal{D}$ is given by (12) below with $R = I$.

There is a general class of involutory biholomorphic mappings of $\mathcal{D}$ which contains the transformations (3). Specifically, if $W_0 \in \mathcal{D}$ and $\|X_0(W_0 - Z_0)\| < 1$, define

$$V_{W_0}(Z) = Z_0 - T_{W_0 - Z_0}(Z - Z_0),$$

where

$$T_A(Z) = (I + AX_0)^{-1/2}(Z - A)(I + X_0 Z)^{-1}(I + X_0 A)^{1/2}.$$

(Compare [11, p. 146].) Note the identities $V_{W_0} = \phi \circ U_{Z_0}$ and $U_{Z_0} = V_{W_0} \circ \phi$. It follows that V_{W_0} is a biholomorphic mapping of $\mathcal{D}$ onto itself with $V_{W_0}^2 = I$ and $V_{W_0}(Z_0) = W_0$ and that the composition $V_{W_1} \circ V_{W_0}$ of any two such mappings is affine. In particular, $V_{Z_0} = U_{Z_0}$.

The domain of a linear fractional transformation may have biholomorphic mappings which cannot be expressed as the composition of a linear fractional transformation and an invertible linear map. For example, let $\mathcal{U} = \mathcal{L}(\mathbb{C}, \mathbb{C}^2)$, $C = [0\ 1]$ and $D = 0$. Then the domain $\mathcal{D}$ of (1) is

$$\mathcal{D} = \left\{ \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \in \mathcal{U} : z_2 \neq 0 \right\}$$

and

$$h \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} z_1 + g(z_2) \\ z_2 \end{bmatrix}$$

is a biholomorphic mapping of $\mathcal{D}$ for any holomorphic function $g : \mathbb{C} \setminus \{0\} \to \mathbb{C}$. Note that Cartan's uniqueness theorem fails for $\mathcal{D}$ since if $g(z) = (z - 1)^2$ and $Z_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, then $h(Z_0) = Z_0$ and $Dh(Z_0) = I$, but $h \neq I$ on $\mathcal{D}$. Moreover, U_{Z_0} and $h^{-1} \circ U_{Z_0} \circ h$ are distinct symmetries of $\mathcal{D}$ at Z_0.

Let $\mathcal{X}$ be any Banach space.

THEOREM 2. *If $\mathcal{D}$ is the domain of a linear fractional transformation on $\mathcal{U}$ and if $h : \mathcal{D} \to \mathcal{X}$ is a bounded holomorphic function, then h is constant. In particular, $\mathcal{D}$ is not holomorphically equivalent to a bounded domain.*

PROOF. Let $Z \in \mathcal{U}$ with $\|X_0(Z - Z_0)\| < 1$. We will construct an entire function $f : \mathbb{C} \to \mathcal{D}$ with $f(0) = Z_0$ and $f(1) = Z$. Then $g(\lambda) = h(f(\lambda))$ is a bounded entire function so $h(Z) = h(Z_0)$ since g is constant [17, Th. 3.13.2]. Hence h is constant in a neighborhood of Z_0 so h is constant on $\mathcal{D}$ by the identity theorem [17, Th. 3.16.4].

Recall [16, § 6.3] that $\limsup_{n \to \infty} \left| \binom{\lambda}{n} \right|^{1/n} \leq 1$ and that the binomial series

$$b_\lambda(z) = \sum_{n=0}^{\infty} \binom{\lambda}{n} z^n$$

satisfies $b_\lambda(z) b_{-\lambda}(z) = 1$ for all $\lambda \in \mathbb{C}$ and $|z| < 1$. Put $W = X_0(Z - Z_0)$. By the holomorphic functional calculus, $b_\lambda(W) b_{-\lambda}(W) = I$ and hence $b_\lambda(W)^{-1}$ exists for all $\lambda \in \mathbb{C}$. Define

$$f(\lambda) = Z_0 + (Z - Z_0) \sum_{n=1}^{\infty} \binom{\lambda}{n} W^{n-1}.$$

Then f is entire and $f(\lambda) \in \mathcal{U}$ for all $\lambda \in \mathbb{C}$ by [14, Lemma 6a]. Moreover, if $\lambda \in \mathbb{C}$, then

$$(CZ_0 + D)^{-1}[Cf(\lambda) + D] = I + X_0[f(\lambda) - Z_0] = b_\lambda(W),$$

so $f(\lambda) \in \mathcal{D}$. □

Note that it follows from the properties of $f(\lambda)$ given above and (7) that the pseudometric ρ assigned to $\mathcal{D}$ by any Schwarz-Pick System [12] satisfies $\rho \equiv 0$. Thus, in particular, the Caratheodory and Kobayashi pseudometrics for $\mathcal{D}$ vanish identically. (Compare [28, Cor. 1].)

The next result gives a class of domains where all points outside the domain are removable singularities for locally bounded holomorphic functions.

THEOREM 3. *Suppose the set* $\mathcal{D} = \{Z \in \mathcal{U} : (CZ+D)^{-1} \text{ exists}\}$ *is non-empty and* C *is a trace class operator. Then* $\mathcal{U}\backslash\mathcal{D}$ *is an analytic set. Moreover, if* $\mathcal{E}$ *is a domain in* $\mathcal{U}$ *and if* $h : \mathcal{D} \cap \mathcal{E} \to \mathfrak{X}$ *is a holomorphic function which is locally bounded in the sense that each operator in* $\mathcal{E}\backslash\mathcal{D}$ *has a neighborhood* $\mathcal{N}$ *such that* h *is bounded on* $\mathcal{N} \cap \mathcal{D}$*, then* h *extends to a holomorphic function* $\hat{h} : \mathcal{E} \to \mathfrak{X}$.

It is easy to show that the conclusions of Theorem 2 hold for any $\mathcal{D}$ satisfying the hypotheses of Theorem 3. See [23] for the definition and properties of analytic sets in Banach spaces. Without some restriction on C or $\mathcal{U}$, the complement of $\mathcal{D}$ may have non-empty interior and thus is not an analytic set. For example, if there is a $Z_1 \in \mathcal{U}$ such that $CZ_1 + D$ has only a one-sided inverse Q, then an elementary argument shows that $CZ + D$ is not invertible for any $Z \in \mathcal{U}$ with $\|Z - Z_1\| < 1/r$, where $r = \|C\|\,\|Q\|$.

PROOF. By (6) and a translation, we may suppose that $D = I$. It suffices to show that there is a holomorphic function $f : \mathcal{U} \to \mathbb{C}$ satisfying

$$\mathcal{D} = \{Z \in \mathcal{U} : f(Z) \neq 0\} \tag{8}$$

since then $\mathcal{U}\backslash\mathcal{D}$ is an analytic set and the remainder of the theorem follows from the extension of the Riemann removable singularities theorem given in [23, Th. II.1.1.5]. It is easy to construct such an f when C has finite rank. Indeed, put $H_1 = \operatorname{ran} C$, $K_1 = (\ker C)^\perp$, $H_2 = H_1^\perp$ and $K_2 = K_1^\perp$. Since these spaces are closed, the operators in $\mathcal{L}(H,K)$ and $\mathcal{L}(K,H)$ can be written as associated 2×2 operator matrices. In particular, if $Z \in \mathcal{L}(H,K)$,

$$Z = \begin{bmatrix} Z_1 & Z_2 \\ Z_3 & Z_4 \end{bmatrix}, \quad C = \begin{bmatrix} Y & 0 \\ 0 & 0 \end{bmatrix}, \tag{9}$$

where $Z_1 \in \mathcal{L}(H_1, K_1)$, $Z_2 \in \mathcal{L}(H_2, K_1)$, $Z_3 \in \mathcal{L}(H_1, K_2)$, $Z_4 \in \mathcal{L}(H_2, K_2)$ and $Y \in \mathcal{L}(K_1, H_1)$. Hence

$$I + CZ = \begin{bmatrix} I + YZ_1 & YZ_2 \\ 0 & I \end{bmatrix}, \tag{10}$$

where the successive appearances of I denote the identity operator on H, H_1 and H_2, respectively. Thus $Z \in \mathcal{D}$ if and only if $(I + YZ_1)^{-1}$ exists. Since H_1 is finite dimensional, we may take $f(Z) = \det(I + YZ_1)$ to obtain (8).

If C is a trace class operator, there exists a sequence $\{F_n\}$ of finite rank operators in $\mathcal{L}(H, K)$ with $\|F_n - C\|_1 \to 0$, where $\|\ \|_1$ is the trace class norm [8, Th. VI.4.1]. Let f_n be the function defined above where C is replaced by F_n. Then by [8, p. 115-117], the sequence $\{f_n\}$ is locally uniformly bounded on $\mathcal{U}$ and converges pointwise to a function f on $\mathcal{U}$ which is denoted by $f(Z) = \det(I + CZ)$. Hence f is holomorphic on $\mathcal{U}$ by an extension of Vitali's theorem [17, Th. 3.18.1] and (8) holds by [8, VII.3(13)]. □

In some cases it is not necessary to take components in the definition of the domain of a linear fractional transformation.

PROPOSITION 4. *The set*

$$\mathcal{D} = \{Z \in \mathcal{U} : (CZ + D)^{-1} \text{ exists}\} \tag{11}$$

is connected if at least one of the following conditions holds:

a) $C\mathcal{U}$ *contains only compact operators,*

b) *there is some* $Z_0 \in \mathcal{D}$ *such that each operator in* $X_0\mathcal{U}$ *satisfies a non-trivial polynomial equation,*

c) $\mathcal{U} = \mathcal{L}(H, K)$ *and* $\operatorname{ran} C$ *is closed,*

d) $\mathcal{U} = \mathcal{L}(H, K)$ *and* $\operatorname{ran} D \subseteq \operatorname{ran} C$.

Here ran denotes the range of operators. It is not difficult to deduce from (6) and [14, Lemma 18] that the domain $\mathcal{D}$ above is a circular domain in the sense of [14] when (a) or (b) holds. (Take $J = -[C\ D]^*[C\ D]$.)

PROOF. Suppose that (a) holds and that $\mathcal{D}$ is not empty. As in the proof of Theorem 3, we may suppose that $D = I$. Let $Z \in \mathcal{D}$ and put $\Omega = \{\mu \in \mathbb{C} : \mu Z \in \mathcal{D}\}$. Clearly Ω contains 0 and 1, and the set $\mathbb{C} \backslash \Omega$ is discrete by hypothesis. Hence there is a curve ω in Ω connecting 0 and 1 so $\gamma = \omega Z$ is a curve in $\mathcal{D}$ connecting 0 and Z. Therefore, $\mathcal{D}$ is connected. A similar argument shows that $\mathcal{D}$ is connected when (b) holds.

Suppose that (c) holds. As in the proof of Theorem 3, we may suppose that $D = I$. Moreover, the decompositions (9) hold, where $Y \in \mathcal{L}(K_1, H_1)$ is invertible and H_1 is a closed subspace of H. Let $Z \in \mathcal{D}$. Then $(I + YZ_1)^{-1}$ exists so by the connectedness [6, Cor. 5.30] of the set G of invertible elements of $\mathcal{L}(H_1)$, there exists a curve ω in G which connects I to $I+YZ_1$. Define

$$\gamma(t)=\begin{bmatrix} Y^{-1}\omega(t)-Y^{-1} & tZ_2 \\ tZ_3 & tZ_4 \end{bmatrix}, \quad 0\le t\le 1.$$

Then $I+C\gamma(t)$ is invertible for $0\le t\le 1$ by (10) so $\gamma(t)$ is a curve in $\mathcal{D}$ which connects 0 to Z. Thus $\mathcal{D}$ is connected.

Suppose that (d) holds and that $Z_0\in\mathcal{D}$. By [5], there is a $Z_1\in\mathcal{L}(H,K)$ with $CZ_1=D$ so $C(Z_1+Z_0)$ is invertible. Then $\operatorname{ran} C=H$ and hence $\mathcal{D}$ is connected by part (c). □

The following gives a condition under which two domains of the form (11) are affinely equivalent.

PROPOSITION 5. *Let C_1, $C_2\in\mathcal{L}(K,H)$, D_1, $D_2\in\mathcal{L}(H)$ and put*

$$\mathcal{D}_1=\{Z\in\mathcal{L}(H,K):(C_1Z+D_1)^{-1}\text{ exists}\},$$
$$\mathcal{D}_2=\{Z\in\mathcal{L}(H,K):(C_2Z+D_2)^{-1}\text{ exists}\}.$$

Suppose there exists an invertible $R\in\mathcal{L}(K)$ with $C_2=C_1R$. Then for each $Z_1\in\mathcal{D}_1$ and $Z_2\in\mathcal{D}_2$ there is an invertible affine linear fractional transformation ϕ of $\mathcal{D}_1$ onto $\mathcal{D}_2$ with $\phi(Z_1)=Z_2$.

PROOF. Observe that

$$\phi(Z)=Z_2+R^{-1}(Z-Z_1)(C_1Z_1+D_1)^{-1}(C_2Z_2+D_2) \tag{12}$$

is the required affine mapping since

$$C_2\phi(Z)+D_2=(C_1Z+D_1)(C_1Z_1+D_1)^{-1}(C_2Z_2+D_2).$$

□

Note that Proposition 5 and its proof also hold when all the spaces of operators mentioned are replaced by the same C^*-algebra.

2. - Linear Automorphisms of Domains

The main results of this section are that certain unbounded circular domains (in the sense of [14]) have only linear automorphisms. In the finite dimensional case, the domains we consider are incomplete matrix Reinhardt domains and extensions of these to non-square matrices. (See [24].) Throughout, we let $\mathcal{U}_0$ denote the open unit ball of $\mathcal{U}$ in the operator norm. Thus

$$\mathcal{U}_0=\{Z\in\mathcal{U}:\|Z\|<1\}.$$

THEOREM 6. *Suppose H is finite dimensional and $K \neq \{0\}$. Let $\mathcal{U} = \mathcal{L}(H, K \times H)$ and put*

$$\mathcal{D} = \{Z \in \mathcal{U} : I + Z_1^* Z_1 < Z_2^* Z_2\}, \text{ where } Z = \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix}$$

is the decomposition of Z with $Z_1 \in \mathcal{L}(H, K)$ and $Z_2 \in \mathcal{L}(H)$. Then h is a biholomorphic map of $\mathcal{D}$ onto itself if and only $h(Z) = LZU$, where L is an invertible linear map in $\mathcal{L}(K \times H)$ satisfying $L^* J L = J$, $J = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}$ and $U \in \mathcal{L}(H)$ is unitary.

COROLLARY 7. *Suppose $K \neq \{0\}$. Then $h \begin{bmatrix} z \\ \lambda \end{bmatrix} = \begin{bmatrix} A & b \\ c^* & d \end{bmatrix} \begin{bmatrix} z \\ \lambda \end{bmatrix}$ is a biholomorphic mapping of*

$$\mathcal{D} = \left\{ \begin{bmatrix} z \\ \lambda \end{bmatrix} \in K \times \mathbb{C} : 1 + \|z\|^2 < |\lambda|^2 \right\}$$

if and only if $T(z) = (Az + b)/[(z, c) + d]$ is a biholomorphic mapping of $K_0 = \{z \in K : \|z\| < 1\}$, where the coefficients of T have been multiplied by a positive number so that $|d|^2 - \|b\|^2 = 1$. Moreover, there are no other biholomorphic mappings of $\mathcal{D}$ or K_0.

Corollary 7 follows from Theorem 6, the form of biholomorphic mappings of K_0, and the uniqueness (up to a complex scalar) of the coefficients of linear fractional transformations. (See (13) below and extend [21, Th. 3].) Note that Theorem 6 is no longer true when $K = \{0\}$ (i.e., Z_1 does not appear) since $h(Z) = Z^t$ is a biholomorphic mapping of $\mathcal{D}$ which is not of the specified form. We consider the case $K = \{0\}$ next.

Let $\mathcal{U}$ be a power algebra. We say that $\mathcal{U}$ has the *removable singularity property* if every bounded holomorphic function $h : G_I(\mathcal{U}) \cap \mathcal{U}_0 \to \mathfrak{X}$ extends to a holomorphic function on $\mathcal{U}_0$ for any Banach space $\mathfrak{X}$. It follows from [4] that there is no loss of generality if one takes $\mathfrak{X} = \mathbb{C}$. Thus the classical Riemann removable singularities theorem for several complex variables shows that $\mathcal{U}$ has the removable singularity property whenever $\dim \mathcal{U} < \infty$. Recall [13] that a J^*-algebra with identity may be characterized as a power algebra which contains the adjoints of each of its elements.

PROPOSITION 8. *Every J^*-algebra with identity which is closed in the weak operator topology has the removable singularity property.*

THEOREM 9. *Let $\mathcal{U}$ be a power algebra with the removable singularity property and put*

$$\mathcal{D} = \{Z \in G_I(\mathcal{U}) : I < Z^* Z\}.$$

Then h is a biholomorphic mapping of $\mathcal{D}$ onto itself if and only if $h = L$, where L is a linear isometry of $\mathcal{U}$ onto itself with $L(I) \in G_I(\mathcal{U})$.

Thus by Proposition 8, the conclusions of Theorem 9 hold, for example, for $\mathcal{L}(H)$, any W^*-algebra, any finite rank J^*-algebra having a unitary element, any Cartan factor of type II or IV and any Cartan factor of type III where the dimension of the underlying Hilbert space is even or infinite. Note that the condition $I < Z^*Z$ in the definition of $\mathcal{D}$ above can also be written as $I < ZZ^*$ since for $Z \in G_I(\mathcal{U})$, the identity $\|(Z^{-1})^*\| = \|Z^{-1}\|$ holds and $I < Z^*Z$ if and only if $\|Z^{-1}\| < 1$. The domains $\mathcal{D}$ of Theorems 6 and 9 are not homogeneous. To see this when $\mathcal{D}$ is as in Theorem 6, observe that $Z_r = \begin{bmatrix} 0 \\ rI \end{bmatrix}$ is in $\mathcal{D}$ for $r > 1$. By Theorem 6, if h is a biholomorphic mapping of $\mathcal{D}$ with $h(Z_r) = Z_s$, then $I + Z_s^* J Z_s = U^*(I + Z_r^* J Z_r)U$ so $s = r$. The case of Theorem 9 is similar.

PROOF OF THEOREM 6. Suppose h has the given form $h(Z) = LZU$. Then $h(Z)^* J h(Z) = U^* Z^* J Z U$ and

$$\mathcal{D} = \{Z \in \mathcal{U} : I + Z^* J Z < 0\},$$

so $h(\mathcal{D}) \subseteq \mathcal{D}$. Since $h^{-1}(Z) = L^{-1} Z U^{-1}$ has the same form as h, it follows that h is a biholomorphic mapping of $\mathcal{D}$.

Now suppose h is a biholomorphic mapping of $\mathcal{D}$ and put $\mathcal{E} = \{Z \in \mathcal{U} : Z_2^{-1} \text{ exists}\}$. Clearly $\mathcal{D} \subseteq \mathcal{E}$ since $\dim H < \infty$ and note that $Z \in \mathcal{U}_0$ if and only if $Z \in \mathcal{U}$ and $Z^*Z < I$. Hence

$$T \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} = \begin{bmatrix} Z_1 Z_2^{-1} \\ Z_2^{-1} \end{bmatrix}$$

is a biholomorphic mapping of $\mathcal{D}$ onto $\mathcal{U}_0 \cap \mathcal{E}$ with $T^{-1} = T$. Therefore, $g = T \circ h \circ T$ is a biholomorphic mapping of $\mathcal{U}_0 \cap \mathcal{E}$. By Theorem 3 with $C = [0 \ I]$, g extends to a holomorphic mapping $\hat{g} : \mathcal{U}_0 \to \mathcal{U}$. To show that $\hat{g}(\mathcal{U}_0) \subseteq \mathcal{U}_0$, suppose this is false. Then by the maximum principle [17, Th. 3.18.4], there is a $Z_0 \in \mathcal{U}_0$ with $\|\hat{g}(Z_0)\| > 1$. Let $u, v \in H$ be unit vectors with $(\hat{g}(Z_0)u, v) > 1$ and define

$$f(Z) = \frac{1}{([\hat{g}(Z) - \hat{g}(Z_0)]u, v)}.$$

Then f is holomorphic on $\mathcal{U}_0 \cap \mathcal{E}$ but f does not have a holomorphic extension to $\mathcal{U}_0$, contradicting Theorem 3. Similarly, g^{-1} extends to a holomorphic map of $\mathcal{U}_0$ into $\mathcal{U}_0$ which is $\hat{g}^{-1}$ by the identity theorem [17, Th. 3.16.4]. Hence $\hat{g}$ is a biholomorphic mapping of $\mathcal{U}_0$ which takes $\mathcal{U}_0 \backslash \mathcal{E}$ onto itself. (Compare [27, Lemma 2.4.4].)

By [7, Th. 5.3], we may write

$$\hat{g}(Z) = (AZ + B)(CZ + D)^{-1}, \quad M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}, \tag{13}$$

where M is an invertible operator in $\mathcal{L}((K \times H) \times H)$ satisfying $M^* J_1 M = J_1$ and $J_1 = \begin{bmatrix} I_{K\times H} & 0 \\ 0 & -I_H \end{bmatrix}$. Here $A \in \mathcal{L}(K \times H)$, $B \in \mathcal{U}$, $C \in \mathcal{L}(K \times H, H)$ and $D \in \mathcal{L}(H)$. Let

$$A = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \quad C = [C_1 \ C_2]$$

be the corresponding decompositions. It suffices to show that $A_2 = 0$, $A_3 = 0$, $B_2 = 0$, $C_2 = 0$ and A_4^{-1} exists; for then the identity $h = T \circ g \circ T$ gives $h(Z) = LZU$, where $L = \begin{bmatrix} A_1 & B_1 \\ C_1 & D \end{bmatrix}$ and $U = A_4^{-1}$, and it follows from $M^* J_1 M = J_1$ that L and U are as asserted.

Pick an r with $0 < r < 1$ and fix $Z_1 \in \mathcal{L}(H, K)$ with $\|Z_1\| \le \sqrt{1 - r^2}$. Define $\phi(Z_2) = A_3 Z_1 + A_4 Z_2 + B_2$ for $Z_2 \in \mathcal{L}(H)$. By the mapping properties of g, if $\|Z_2\| < r$, then $\phi(Z_2)$ is invertible whenever Z_2 is invertible and $\phi(Z_2)$ is singular whenever Z_2 is singular. Since the determinant is analytic, it follows from the identity theorem that $\phi(Z_2)$ is singular for all singular Z_2.

If A_4 is not invertible, there is a unit vector $u \in H$ with $A_4 u = 0$. Let Z_2 be invertible with $\|Z_2\| < r$ and let $Z_2' = (I - uu^*)Z_2$. Then Z_2' is singular since $u^* Z_2' = 0$, and $\phi(Z_2') = \phi(Z_2)$, a contradiction. Thus A_4 is invertible. Put $B = A_4^{-1}(A_3 Z_1 + B_2)$. Then $Z_2 + B$ is singular whenever Z_2 is singular. To show that $B = 0$, let $u \in H$ be a unit vector and take $Z_2 = (I - B)(I - uu^*)$. Then Z_2 is singular since $Z_2 u = 0$ so $Z_2 + B = I + (Bu - u)u^*$ is singular. Hence, $(Bu, u) = 1 + u^*(Bu - u) = 0$.

We have shown that A_4^{-1} exists and that $A_3 Z_1 + B_2 = 0$ for $\|Z_1\| < \sqrt{1 - r^2}$, so $A_3 = 0$ and $B_2 = 0$. Now the coefficient matrix of $\hat{g}^{-1}$ is $M^{-1} = \begin{bmatrix} A^* & -C^* \\ B^* & D^* \end{bmatrix}$ and hence the above argument applied to $\hat{g}^{-1}$ shows that $A_2 = 0$ and $C_2 = 0$, as required. □

We deduce Theorem 9 from the following Lemma.

LEMMA 10. *Let $\mathcal{U}$ be a power algebra and let L be a linear isometry of $\mathcal{U}$ onto itself. Then $U = L(I)$ is unitary, $U^* \in \mathcal{U}$, $L(G_I(\mathcal{U}))$ is a component of the set $G(\mathcal{U})$ of invertible operators in $\mathcal{U}$ and*

$$L(Z^{-1}) = UL(Z)^{-1}U \tag{14}$$

for all $Z \in G_I(\mathcal{U})$.

PROOF. By [2, Cor. 2.8], $U^* \in \mathcal{U}$ so by [9, Th. 2], U is unitary and $L(Z) = U\rho(Z)$, where $\rho : \mathcal{U} \to U^*\mathcal{U}$ is an invertible linear isometry satisfying $\rho(I) = I$ and $\rho(Z^2) = \rho(Z)^2$ for all $Z \in \mathcal{U}$. Hence

$$\rho(ZWZ) = \rho(Z)\rho(W)\rho(Z) \tag{15}$$

for all $Z, W \in \mathcal{U}$ since

$$ZWZ = 2(Z \circ W) \circ Z - (Z \circ Z) \circ W, \tag{16}$$

where $Z \circ W = [(Z+W)^2 - Z^2 - W^2]/2$. Let $Z \in G_I(\mathcal{U})$. Then $Z^{-1} \in G_I(\mathcal{U})$ by Example 1. Hence it follows from (15) and the identities $I = Z(Z^{-1})^2 Z$ and $Z = ZZ^{-1}Z$ that $\rho(Z)^{-1}$ exists and $\rho(Z^{-1}) = \rho(Z)^{-1}$. Thus $L(Z)^{-1}$ exists and (14) holds. Also, $L(G_I(\mathcal{U}))$ is a component of $G(\mathcal{U})$ since L is a homeomorphism.□

PROOF OF THEOREM 9. Let $L : \mathcal{U} \to \mathcal{U}$ be an invertible linear isometry with $L(I) \in G_I(\mathcal{U})$. Then by Lemma 10, $L(G_I(\mathcal{U})) = G_I(\mathcal{U})$ so L is a biholomorphic mapping of $\mathcal{E} = \mathcal{U}_0 \cap G_I(\mathcal{U})$. By Example 1, $T(Z) = Z^{-1}$ is a biholomorphic mapping of $\mathcal{D}$ onto $\mathcal{E}$ and hence $g = T \circ L \circ T$ is a biholomorphic mapping of $\mathcal{D}$. Moreover, $L(Z) = Ug(Z)U$ for all $Z \in G_I(\mathcal{U})$ by (14). Therefore, L is a biholomorphic mapping of $\mathcal{D}$.

Now let h be a biholomorphic mapping of $\mathcal{D}$. Then $g = T \circ h \circ T$ is a biholomorphic mapping of $\mathcal{E}$. By hypothesis and the argument given in the proof of Theorem 6, g extends to a biholomorphic mapping $\hat{g}$ of $\mathcal{U}_0$. Put $B = -\hat{g}^{-1}(0)$. Clearly $B \in \mathcal{U}_0$ and it follows from Theorem 2.2(ii) and Theorem 2.6 of [2] that $B^* \in \mathcal{U}$. Hence

$$T_B(Z) = (I - BB^*)^{-\frac{1}{2}}(Z+B)(I+B^*Z)^{-1}(I - B^*B)^{\frac{1}{2}} \tag{17}$$

is a biholomorphic mapping of $\mathcal{U}_0$ by (16), the comments after Proposition 5 of [14] and [10, (12)]. Thus the proof of Theorem 3 of [10] applies to show that $\hat{g} = L \circ T_B$, where $L : \mathcal{U} \to \mathcal{U}$ is an invertible linear isometry. By Lemma 10, L^{-1} takes $G_I(\mathcal{U})$ into $G(\mathcal{U})$ so $Z + B$ is invertible whenever $Z \in G_I(\mathcal{U})$ and $\|Z\| < 1$. We will show that $B = 0$. Then $\hat{g} = L$ and $h = T \circ g \circ T$ so by Lemma 10, $h = L_1$ on $\mathcal{D}$, where $L_1(Z) = VL(Z)V$ and $V = L(I)^*$ is unitary. Hence L_1 is a linear isometry on $\mathcal{U}$ and $L_1(\mathcal{U}) = \mathcal{U}$ since $L_1 = Dh(2I)$. Moreover, $L_1(I) \in G_I(\mathcal{U})$ since $h(2I) \in G_I(\mathcal{U})$.

To show that $B = 0$, let U be any unitary operator in $G_I(\mathcal{U})$ and note that $U^* \in G_I(\mathcal{U})$ since $U^* = T(U)$. By what we have shown above, $(-\lambda U^* + B)^{-1}$ exists for all $|\lambda| < 1$ so $|UB|_\sigma = 0$, where $|\ |_\sigma$ denotes the spectral radius. The straightforward extension of [10, Prop. 2a] to $\mathcal{U}$ shows that $T_{B^*}(\lambda I)$ is a unitary operator in $G_I(\mathcal{U})$ for all $|\lambda| = 1$. Thus $f(\lambda) = T_{B^*}(\lambda I)B$ is an operator-valued function which is holomorphic on a neighborhood of the closed unit disc with $|f(\lambda)|_\sigma = 0$ for all $|\lambda| = 1$ so $|f(0)|_\sigma = 0$ by a maximum principle of Vesentini [26]. Therefore, $|B^*B|_\sigma = 0$ so $B = 0$, as required. □

PROOF OF PROPOSITION 8. Put $\mathcal{E} = \mathcal{U}_0 \cap G_I(\mathcal{U})$ and let $h : \mathcal{E} \to \mathfrak{X}$ be a holomorphic function with $\|h(Z)\| \le 1$ for all $Z \in \mathcal{E}$. Given $0 < t < 1/2$, put $r = 1 - t$. Let $W \in \mathcal{U}_0$ and suppose $\sigma(W)$ is a finite set. Then $f(\lambda) = h(tI + \lambda W)$ is a vector-valued function with unit bound which is holomorphic at all but a finite number of points in the disc $|\lambda| < r$ so f has a holomorphic extension to this disc by [17, Th. 3.13.3]. By the Cauchy estimates [17, (3.11.3)], $|f^{(n)}(0)| \leq n! r^{-n}$ so

$$\|\hat{D}^n h(tI)(W)\| \leq n! r^{-n}, \tag{18}$$

where $\hat{D}^n h(Z)$ denotes the homogeneous polynomial associated with the nth order Fréchet derivative at Z.

By the spectral theorem, each normal operator in $\mathcal{U}$ is a limit in the operator norm of a linear combination W of orthogonal projections in $\mathcal{U}$ which sum to I. Hence (18) holds for all normal operators $W \in \mathcal{U}_0$. By the maximum principle for unitary operators [10, Th. 9], inequality (18) holds for all $W \in \mathcal{U}_0$.

Define

$$h_t(Z) = \sum_{n=0}^{\infty} \frac{1}{n!} \hat{D}^n h(tI)(Z - tI)$$

and put $\mathcal{D}_t = \{Z \in \mathcal{U} : \|Z - tI\| < r\}$. Then h_t is holomorphic in $\mathcal{D}_t$ since by (18) the above series converges uniformly on each ball about tI with radius less than r. Also, $\mathcal{E}$ is connected and $h_t(Z) = h(Z)$ for all $Z \in \mathcal{U}$ with $\|Z - tI\| < t$ by Taylor's theorem [17, Th. 3.17.1]. Since $\{\mathcal{D}_t : 0 < t < 1/2\}$ is a monotone family of domains with union $\mathcal{U}_0$, it follows from the identity theorem [17, Th. 3.16.4] that there exists a holomorphic extension $\hat{h}$ of h to $\mathcal{U}_0$. □

In the remainder of this section, we give some types of circular domains which are homogeneous under the group of linear automorphisms. Our first result reduces to [14, Example 4] when $H = \mathbb{C}$.

PROPOSITION 11. *Let $\mathcal{U} = \mathcal{L}(H, K \times H)$ and put*

$$\mathcal{D} = \{Z \in \mathcal{U} : Z_1^* Z_1 < Z_2^* Z_2, Z_2^{-1} \text{ exists}\}.$$

Then $\mathcal{D}$ is symmetric and the group of linear biholomorphic mappings of $\mathcal{D}$ acts transitively on $\mathcal{D}$. Moreover, $\mathcal{D}$ is holomorphically equivalent to the product domain $\mathcal{D}' = \mathcal{L}(H, K)_0 \times G\mathcal{L}(H)$, where $G\mathcal{L}(H)$ is the set of all invertible operators in $\mathcal{L}(H)$.

PROOF. Note that $\mathcal{D} = \{Z \in \mathcal{E} : Z^* J Z < 0\}$, where $\mathcal{E} = \{Z \in \mathcal{U} : Z_2^{-1} \text{ exists}\}$ and $J = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}$. Given $W \in \mathcal{D}$, put $B = W_1 W_2^{-1}$. Since $\|B\| < 1$, we may define $L(Z) = MZR$, where

$$M = \begin{bmatrix} (I - BB^*)^{-\frac{1}{2}} & B(I - B^*B)^{-\frac{1}{2}} \\ (I - B^*B)^{-\frac{1}{2}} B^* & (I - B^*B)^{-\frac{1}{2}} \end{bmatrix}$$

and $R = (I - B^*B)^{1/2} W_2$. (Here M is the coefficient matrix of (17).) Clearly $L \begin{bmatrix} 0 \\ I \end{bmatrix} = W$. Moreover, $M^* J M = J$ so $L(\mathcal{D}) \subseteq \mathcal{D}$ since $L(Z)^* J L(Z) = R^* Z^* J Z R$ and $(I - B^*B)^{-1/2}(B^* Z_1 Z_2^{-1} + I) Z_2 R$ is invertible for all $Z \in \mathcal{D}$. Since M^{-1} is obtained from M by replacement of B by $-B$ in the expression for M, a similar argument shows that $L^{-1}(\mathcal{D}) \subseteq \mathcal{D}$. Therefore, L is a biholomorphic mapping of $\mathcal{D}$.

It is easy to verify that $h(Z) = (Z_1 Z_2^{-1}, Z_2^{-1})$ is a biholomorphic mapping of $\mathcal{D}$ onto $\mathcal{D}'$ and hence $\mathcal{D}$ is symmetric since each of the factors of $\mathcal{D}'$ is symmetric. $\square$

PROPOSITION 12. *Let $J \in \mathcal{L}(H)$ with $J^* = J$ and suppose* 1 *and* -1 *are eigenvalues for J. Then the group G of linear biholomorphic mappings of $\mathcal{D} = \{z \in H : (Jz, z) < 0\}$ acts transitively on $\mathcal{D}$.*

PROOF. By hypothesis, $H = \mathbb{C}e \bigoplus K \bigoplus \mathbb{C}f$, where e and f are unit eigenvectors for 1 and -1, respectively, and $K = \{e, f\}^{\perp}$. Thus if $z \in H$, we may write

$$z = \begin{bmatrix} \alpha \\ w \\ \beta \end{bmatrix}, \quad J = \begin{bmatrix} 1 & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & -1 \end{bmatrix},$$

$$\mathcal{D} = \{z \in H : |\alpha|^2 + (Bw, w) < |\beta|^2\}.$$

Suppose $z_1 \in \mathcal{D}$ and let α_1, w_1 and β_1 be the coordinates of z_1. We will obtain an $L \in G$ with $Lf = z_1$.

First suppose $A \equiv |\alpha_1| + |\beta_1| > 0$ and put

$$a = |\alpha_1| + \frac{1}{2A}(Bw_1, w_1), \quad b = |\beta_1| - \frac{1}{2A}(Bw_1, w_1).$$

Clearly $a + b = A$ and $A(b - a) > 0$ so $b^2 - a^2 > 0$. Given $w \in K$, let $L_w : H \to H$ be the linear map defined by the operator matrix

$$L_w = \begin{bmatrix} 1 - \frac{1}{2}(Bw, w) & -w^*B & -\frac{1}{2}(Bw, w) \\ w & I & w \\ \frac{1}{2}(Bw, w) & w^*B & 1 + \frac{1}{2}(Bw, w) \end{bmatrix}.$$

Then $L_w^{-1} = L_{-w}$ and $L_w^* J L_w = J$ so $L_w \in G$. Take $w = w_1/A$. Then by a short computation,

$$L_w \begin{bmatrix} a \\ 0 \\ b \end{bmatrix} = \begin{bmatrix} |\alpha_1| \\ w_1 \\ |\beta_1| \end{bmatrix}.$$

Note that $L_1 = \begin{bmatrix} b & 0 & a \\ 0 & \sqrt{b^2 - a^2} & 0 \\ a & 0 & b \end{bmatrix}$ satisfies $L_1 f = \begin{bmatrix} a \\ 0 \\ b \end{bmatrix}$ and L_1 is in G since $L_1^* J L_1 = (b^2 - a^2)J$ and L_1^{-1} exists. Also, there is an $L_2 \in G$ (given by a diagonal matrix) such that $L_2 \begin{bmatrix} |\alpha_1| \\ w_1 \\ |\beta_1| \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ w_1 \\ \beta_1 \end{bmatrix}$. Hence $L_2 L_w L_1 f = z_1$.

Now suppose $\alpha_1 = \beta_1 = 0$. Then $L_{w_1}z_1 = \begin{bmatrix} (Bw_1, w_1) \\ w_1 \\ (Bw_1, w_1) \end{bmatrix}$ and $(Bw_1, w_1) \neq 0$ since $z_1 \in \mathcal{D}$. By what we have shown, there is an $L \in G$ with $Lf = L_{w_1}z_1$ so $z_1 = L_{w_1}^{-1}Lf$. $\square$

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