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The Dunford-Pettis and the Kadec-Klee properties on tensor products of JB*-triples

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1. Introduction

We recall that a Banach space X satisfies the *Dunford-Pettis property* (DPP) if every weakly compact operator T from X to another Banach space is completely continuous, i.e. T maps weakly Cauchy sequences into norm convergent sequences. The question if the projective tensor product of two Banach spaces having the DPP also has this property has focused the attention of several researcher. In 1983, M. Talagrand found a Banach space X such that X^* has the Schur property and $L^1[0,1]\widehat{\otimes}_\pi E^*$ does not satisfy the DPP (see [34]). Four years latter R. Ryan showed that $X\widehat{\otimes}_\pi Y$ satisfies the DPP and contains no copies of ℓ_1 whenever X and Y have both properties [33].

F. Bombal and I. Villanueva prove in [7] (see also [6,20]) that, whenever K_1 and K_2 are two infinite compact Hausdorff spaces, then the projective tensor product of $C(K_1)$ and $C(K_2)$ has the DPP if and only if K_1 and K_2 are scattered, that is $C(K_1)$ and $C(K_2)$ contains no copies of ℓ_1.

The interesting ideas developed by Bombal, Fernández and Villanueva in [7,6] are the inspiration and the starting point for the present paper. In the section 2 we prove that the projective tensor product of two Banach spaces X and Y satisfies the DPP and property (V) of Pelczyński if and only if X and Y have both properties and contain no copies of ℓ_1. When this result is particularized to the class of Banach spaces called JB*-triples (see definition below) we conclude that the projective tensor product of two JB*-triples E and F has the DPP if and only if E and F have the

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DPP and do not contain copies of ℓ_1. As a consequence, in Remark 2.7, we found a wide class of JB*-triples E and F such that $\left(E\widehat{\otimes}_\pi F\right)^*$ has the Schur property and $\left(E\widehat{\otimes}_\pi F\right)^{**}$ does not satisfy the Dunford-Pettis property (compare [20]).

The last section of the paper is devoted to the study of the Kadec-Klee property in the projective tensor product of two real or complex JB*-triples. Our main result shows that the projective tensor product of two real or complex JB*-triples E and F has the Kadec-Klee property if and only if E and F have this property. Among those techniques and results leading us to this result we obtain that the projective tensor product of n Hilbert spaces always has the Kadec-Klee property and a description of every spin factor as the real projective tensor product of $\mathbb{C}$ and a real Hilbert space.

Notation

Let X be a Banach space. Throughout this note we denote by B_X, S_X, and X^* the closed unit ball, the unit sphere, and the dual space of X, respectively. When X is a complex Banach space then $X_{\mathbb{R}}$ will denothe the underlying real Banach space. If Y and Z are also Banach spaces then $L(X, Y)$ (respectively, $L^2(X, Y; Z)$) will stand for the Banach space of all bounded linear operators from X to Y (respectively, the space of all continuous bilinear operators from $X \times Y$ to Z). The projective (respectively, injective) tensor product of X and Y will be denoted by $X\widehat{\otimes}_\pi Y$ (respectively, $X\widehat{\otimes}_\varepsilon Y$).

2. The Dunford-Pettis property and projective tensor products

Let X and Y be Banach spaces. A series $\sum x_n$ in X is called *weakly unconditionally Cauchy (w.u.C.)* if there exists $C > 0$ such that for any finite subset $F \subset \mathbb{N}$ and $\varepsilon_n = \pm 1$ we have $\|\sum_{n\in F} \varepsilon_n x_n\| \leq C$. We say that $\sum x_n$ is *unconditionally convergent* if any subseries is norm converging. We also recall that a bounded linear operator $T : X \to Y$ is said to be *unconditionally converging* if it applies *w.u.C.* series to unconditionally convergent series. A Banach space X is said to have *property (V)* if every unconditional convergent operator from X to another Banach space Y is weakly compact (see [27] for completeness).

The following theorem is due to Emmanuele and Hense [18] and will play an important role in our results.

Theorem 2.1. *[18] Let X and Y be two Banach spaces satisfying property (V) such that $L(X, Y^*) = K(X, Y^*)$. Then $X\widehat{\otimes}_\pi Y$ has property (V).* □

Lemma 2.2. *Let X be a Banach space satisfying DPP and property (V) and Y a Banach space with property (V). Let (x_n) be a weakly null sequence in X and let (y_n) be a bounded sequence in Y. Then $(x_n \otimes y_n)$ is a weakly null sequence in $X\widehat{\otimes}_\pi Y$.*

Proof. We begin by showing that every $T \in L(X, Y^*)$ is weakly compact. Indeed, since Y has property (V), then it cannot contain complemented copies of ℓ_1 and hence Y^* does not contain c_0. It is known that $T \in L(X, Y^*)$ fails to be unconditionally convergent if and only if it fixes a copy of c_0 (compare [15, Exercise V.8]), thus any $T \in L(X, Y^*)$ is unconditionally converging. Since X has property (V), then every unconditionally converging linear operator from X to another Banach space is weakly compact. Therefore every $T \in L(X, Y^*)$ is weakly compact.

Let $\Phi \in \left(X\widehat{\otimes}_\pi Y\right)^*$. From the identification $\left(X\widehat{\otimes}_\pi Y\right)^* = L(X, Y^*)$, it follows that there exists a unique $T \in L(X, Y^*)$ satisfying

$$\Phi(x \otimes y) = T(x)(y).$$

By the first part of the proof we know that T is weakly compact. Then, since X satisfies DPP, we deduce that T is completely continuous. In particular $(T(x_n)) \to 0$ in the norm-topology of Y^*. Finally, since (y_n) is a bounded sequence in Y, the inequality

$$|\Phi(x_n \otimes y_n)| \leq \|T(x_n)\| \|y_n\|,$$

shows that $\Phi(x_n \otimes y_n) \to 0$. $\square$

A further develop of the arguments given by F. Bombal and I. Villanueva in [7] (see also [20]) allows us to obtain the next result.

Theorem 2.3. *Let X, Y be two infinite-dimensional Banach spaces satisfying DPP and property (V). Then $X\widehat{\otimes}_\pi Y$ fails DPP whenever X or Y contains a copy of ℓ_1.*

Proof. Let us assume that Y contains a copy of ℓ_1. We claim that X contains a copy of c_0. Suppose, contrary to our claim, that X contains no copy of c_0. Let $Id_X : X \to X$ denote the identity mapping on X. By assumption, Id_X can not preserve a copy of c_0 and hence, by [15, Exercise V.8], Id_X is an unconditionally converging linear operator. Moreover, since X satisfies property (V), we deduce that Id_X is a weakly compact operator. Since X has the DPP and Id_X is a weakly compact linear operator then $Id_X^2 = Id_X$ is a compact linear operator, which is impossible since X is infinite-dimensional.

Since $X \supseteq c_0$, there exists a weakly null sequence $(x_n) \subseteq S_X$ and a sequence $(\mu_n) \subseteq B_{X^*}$, satisfying $\mu_n(x_m) = \delta_{n,m}$ for all $n, m \in \mathbb{N}$. Since $Y \supseteq \ell_1$ then, there exists a surjective bounded linear operator $T : Y \to \ell_2$ (see [25, Proposition 3]). For every natural n we take an element $y_n \in Y$ such that $T(y_n) = e_n$, where (e_n) denotes the canonical basis of ℓ_2. Since T is bounded, we can assume that (y_n) is bounded. Define now

$$\Phi : X \times Y \to \ell_2,$$

by $\Phi(x, y) := (\mu_n(x)\ (T(y), e_n))_{n\in\mathbb{N}}$. It is easily seen that Φ is a bounded bilinear mapping, and hence Φ can be extended to a bounded linear operator $\widehat{\Phi} : X\widehat{\otimes}_\pi Y \to \ell_2$. It is evident that $\widehat{\Phi}$ is weakly compact. Lemma 2.2 now shows that $(x_n \otimes y_n) \to 0$ weakly in $X\widehat{\otimes}_\pi Y$. Finally, since for every natural n we have $\widehat{\Phi}(x_n \otimes y_n) = e_n$, we conclude that $\left(\widehat{\Phi}(x_n \otimes y_n)\right)$ does not tend to zero in the norm topology of ℓ_2, which shows that $X\widehat{\otimes}_\pi Y$ fails the DPP. $\square$

As a consequence of the above theorem we get the following result.

Theorem 2.4. *Let X, Y be two infinite dimensional Banach spaces. The following assertions are equivalent:*

(a) *$X\widehat{\otimes}_\pi Y$ satisfies DPP and property (V)*
(b) *X and Y satisfy DPP, property (V) and do not contain copies of ℓ_1.*

Proof. $a) \Rightarrow b)$ Since the DPP and property (V) are inherited by complemented subspaces, it may be concluded that X and Y satisfy DPP and property (V). Theorem 2.3 proves that X and Y do not contain copies of ℓ_1.

$b) \Rightarrow a)$ Since X and Y satisfy DPP and do not contain copies of ℓ_1, it follows by [33] that $X\widehat{\otimes}_\pi Y$ satisfies DPP. Analysis similar to that in the first part of the proof of Lemma 2.2 shows that, since X and Y satisfy property (V), then every bounded linear operator from X to Y^* is weakly compact. Moreover, since X has DPP we deduce that every $T \in L(X, Y^*)$ is in fact a completely continuous linear operator. Finally, Rosenthal's ℓ_1-theorem [15, Chap. XI] together with the observation that X contains no copy of ℓ_1, show that $L(X, Y^*) = K(X, Y^*)$. Theorem 2.1 assures now that $X\widehat{\otimes}_\pi Y$ satisfies property (V). □

We recall that a *JB*-triple* is a complex Banach space E equipped with a continuous triple product

$$\{.,.,.\} : E \otimes E \otimes E \to E$$

$$(x, y, z) \mapsto \{x, y, z\}$$

which is bilinear and symmetric in the outer variables and conjugate linear in the middle one and satisfies:

(a) (*Jordan Identity*)

$$L(x, y)\{a, b, c\} = \{L(x, y)a, b, c\} - \{a, L(y, x)b, c\} + \{a, b, L(x, y)c\},$$

for all $x, y, a, b, c \in E$, where $L(x, y) : E \to E$ is the linear mapping given by $L(x, y)z = \{x, y, z\}$;

(b) The map $L(x, x)$ is an hermitian operator with non-negative spectrum for all $x \in E$;

(c) $\| \{x, x, x\} \| = \|x\|^3$ for all $x \in E$.

JB*-triples were introduced by W. Kaup in the study of bounded symmetric domains in infinite dimensional complex Banach spaces [22]. Every C*-algebra is a JB*-triple with respect to $\{x, y, z\} = 2^{-1}(xy^*z + zy^*x)$, every JB*-algebra is a JB*-triple with triple product $\{a, b, c\} = (a \circ b^*) \circ c + (c \circ b^*) \circ a - (a \circ c) \circ b^*$, and the Banach space $B(H, K)$ of all bounded linear operators between two complex Hilbert spaces H, K is also an example of a JB*-triple with respect to $\{R, S, T\} = 2^{-1}(RS^*T + TS^*R)$. We refer to [31,32] as excellent surveys on JB*-triples.

In [12] the authors show that every JB*-triple satisfies property (V). The following corollary follows now from this fact and the above Theorem 2.4.

Corollary 2.5. *Let E and F be two JB*-triples. The following statements are equivalent:*

(a) *$E\widehat{\otimes}_\pi F$ satisfies DPP*

(b) *E and F satisfy DPP and do not contain copies of ℓ_1.* □

Corollary 2.6. *Let E and F be two infinite dimensional JB*-triples. Then $\left(E\widehat{\otimes}_\pi F\right)^{**}$ does not satisfy the DPP.*

Proof. We may assume that E and F satisfy DPP and do not contain copies of ℓ_1, otherwise, by Corollary 2.5, $E\widehat{\otimes}_\pi F$ (and hence $\left(E\widehat{\otimes}_\pi F\right)^{**}$) does not satisfy the DPP. Since E^{**} and F^{**} are complemented subspaces of $\left(E\widehat{\otimes}_\pi F\right)^{**}$, we may also assume that E^{**} and F^{**} satisfy DPP.

Since E is infinite dimensional then E^{**} dose not satisfy the Schur property. We claim that F^{**} contains a copy of ℓ_1. Indeed, if F^{**} does not contain a copy of ℓ_1, then F^* neither does ([15, §XI, Exercise 2.(iv)]) and thus F is reflexive by [5], which is impossible. Since E^{**} has the DPP and every bounded linear operator from E^{**} into F^{***} is weakly compact, we deduce that every bounded linear operator from E^{**} into F^{***} is completely continuous. From [20, Theorem 10] it follows that $\left(E^*\widehat{\otimes}_\varepsilon F^*\right)^*$ does not satisfy the DPP.

Since E does not contain ℓ_1, then E^* has the approximation property by [8, Lemma 3.3 and Theorem 3.4]. Therefore, it follows from [14, Corollary 5.3] and the arguments given in the proof of Theorem 2.4 ($b) \Rightarrow a)$) that

$$\left(E\widehat{\otimes}_\pi F\right)^* = L(E, F^*) = K(E, F^*) = E^*\widehat{\otimes}_\varepsilon F^*.$$

In particular $\left(E\widehat{\otimes}_\pi F\right)^{**} = \left(E^*\widehat{\otimes}_\varepsilon F^*\right)^*$ does not satisfy the DPP. □

Remark 2.7. Let E and F be two JB*-triples satisfying the DPP and not containing ℓ_1. By Corollary 2.5 it follows that $E\widehat{\otimes}_\pi F$ has the DPP. Similar arguments to those given in the proof of Theorem 2.4 ($b) \Rightarrow a)$) show that $L(E, F^*) = K(E, F^*)$ and hence, by [17], $E\widehat{\otimes}_\pi F$ does not contain ℓ_1. Therefore, by [16, Theorem 3], we conclude that $\left(E\widehat{\otimes}_\pi F\right)^*$ has the Schur property. However, by Corollary 2.6 $\left(E\widehat{\otimes}_\pi F\right)^{**}$ does not satisfy the DPP.

3. The Kadec-Klee property and projective tensor products

We recall that a Banach space X is said to have the *Kadec-Klee property (KKP)* if whenever x_n tends to x weakly in X with x and x_n in the unit sphere of X we have $x_n \to x$ in norm. By [10, Proposiiton 2.13], we know that a JB*-triple E satisfy the KKP if and only if E is finite dimensional or a spin factor (see definition below) or a Hilbert space, in particular E is isomorphic to a Hilbert space.

It is natural to ask whether the projective tensor product of two JB*-triples satisfying the Kadec-Klee property satisfies the Kadec-Klee property. This is the main goal of the present section.

Let T_1, T_2 be bounded linear operator on a Banach space X. We will denote by $(T_1, T_2) : X \times X \to X \times X$, the bounded bilinear operator given by $(T_1, T_2)(x, y) := (T_1(x), T_2(y))$. The following inequality generalices [2, Proposition in Appendix].

Lemma 3.1. *Let H be a real or complex Hilbert space. Let $T : H \times H \to H$ be a bounded bilinear operator, let p be a projection in $L(H)$ and let $q = 1 - p$. Then*

$$\|T\|^2 \leq \|pT(p,p)\|^2 + \|qT(p,p)\|^2 + \|pT(p,q)\|^2 + \|qT(p,q)\|^2 \\ +\|pT(q,p)\|^2 + \|qT(q,p)\|^2 + \|pT(q,q)\|^2 + \|qT(q,q)\|^2.$$

In particular, for each $\varphi \in \left(L^2(H,H;H)\right)_*$ we have:

$$\|\varphi\|^2 \geq \|p\varphi(p,p)\|^2 + \|q\varphi(p,p)\|^2 + \|p\varphi(p,q)\|^2 + \|q\varphi(p,q)\|^2 \\ +\|p\varphi(q,p)\|^2 + \|q\varphi(q,p)\|^2 + \|p\varphi(q,q)\|^2 + \|q\varphi(q,q)\|^2.$$

Proof. Let $x \in L(H)$. By [2, Proposition in Appendix] it follows that

$$\|x\|^2 \leq \|pxp\|^2 + \|qxp\|^2 + \|pxq\|^2 + \|qxq\|^2.$$

Fix $\xi_1 \in B_H$. From the above inequality we have

$$\|T(\xi_1,.)\|^2 \leq \|pT(\xi_1,.)p\|^2 + \|qT(\xi_1,.)p\|^2 + \|pT(\xi_1,.)q\|^2 + \|qT(\xi_1,.)q\|^2 \\ \leq \|pT(1,p)\|^2 + \|qT(1,p)\|^2 + \|pT(1,q)\|^2 + \|qT(1,q)\|^2,$$

and hence

$$\|T\|^2 \leq \|pT(1,p)\|^2 + \|qT(1,p)\|^2 + \|pT(1,q)\|^2 + \|qT(1,q)\|^2. \tag{1}$$

Similarly,

$$\|T\|^2 \leq \|pT(p,1)\|^2 + \|qT(p,1)\|^2 + \|pT(q,1)\|^2 + \|qT(q,1)\|^2. \tag{2}$$

Finally, applying inequality (2) on each summand in (1) we obtain the first statement of the lemma. The last statement follows by standard duality arguments. □

The inequality obtained in the above Lemma is the main tool to prove the following result.

Proposition 3.2. *Let H be a real or complex Hilbert space. Then the projective tensor product $H\widehat{\otimes}_\pi H\widehat{\otimes}_\pi H$ satisfies the KKP.*

Proof. It is well known that $\left(H\widehat{\otimes}_\pi H\widehat{\otimes}_\pi H\right)^* = L^2(H,H;H)$. Let $(\varphi_n) \to \varphi$ weakly in $\left(L^2(H,H;H)\right)_*$ with $\|\varphi_n\| = \|\varphi\| = 1$. Let $\varepsilon > 0$. There exists a finite rank projection $p \in L(H)$ such that

$$\|p\varphi(p,p) - \varphi\| < \varepsilon \tag{3}$$

Since $(p\varphi_n(p,p))$ converges weakly to $p\varphi(p,p)$ and $p(H)$ is finite dimensional, we conclude that $(p\varphi_n(p,p))$ converges to $p\varphi(p,p)$ in norm. Therefore, there exists $m_0 \in \mathbb{N}$ such that for each $n \geq m_0$ we have

$$\|p\varphi_n(p,p) - p\varphi(p,p)\| < \varepsilon \quad \text{and} \quad \|p\varphi_n(p,p)\| \geq 1 - \varepsilon \tag{4}$$

By Lemma 3.1 we conclude that

$$\begin{aligned}\|\varphi_n\|^2 - \|p\varphi_n(p,p)\|^2 &\geq \|q\varphi(p,p)\|^2 + \|p\varphi(p,q)\|^2 + \|q\varphi(p,q)\|^2 \\ &\quad + \|p\varphi(q,p)\|^2 + \|q\varphi(q,p)\|^2 + \|p\varphi(q,q)\|^2 + \|q\varphi(q,q)\|^2 \\ &\geq \frac{1}{7^2}\big(\|q\varphi(p,p)\| + \|p\varphi(p,q)\| + \|q\varphi(p,q)\| + \|p\varphi(q,p)\| + \|q\varphi(q,p)\| \\ &\quad + \|p\varphi(q,q)\| + \|q\varphi(q,q)\|\big)^2.\end{aligned}$$

Thus for each $n \geq m_0$ we have

$$\begin{aligned}&\|q\varphi(p,p)\| + \|p\varphi(p,q)\| + \|q\varphi(p,q)\| + \|p\varphi(q,p)\| + \|q\varphi(q,p)\| \\ &+ \|p\varphi(q,q)\| + \|q\varphi(q,q)\| \leq 7\sqrt{\|\varphi_n\|^2 - \|p\varphi_n(p,p)\|^2} \leq 7\sqrt{2\varepsilon}.\end{aligned}$$

Finally, for each $n \geq m_0$ it follows from (3) and (4) that

$$\begin{aligned}\|\varphi_n - \varphi\| &\leq \|p\varphi_n(p,p) - p\varphi(p,p)\| + \|p\varphi(p,p) - \varphi\| + \|q\varphi(p,p)\| \\ &\quad + \|p\varphi(p,q)\| + \|q\varphi(p,q)\| + \|p\varphi(q,p)\| + \|q\varphi(q,p)\| + \|p\varphi(q,q)\| \\ &\quad + \|q\varphi(q,q)\| \\ &\leq 2\varepsilon + 7\sqrt{2\varepsilon}.\end{aligned}$$

□

Remark 3.3. Let $H_1, \ldots, H_n$ be Hilbert spaces. The same arguments given in the proof of Proposition 3.2 show that $H_1 \widehat{\otimes}_\pi H_2 \widehat{\otimes}_\pi \ldots \widehat{\otimes}_\pi H_n$ satisfies the KKP, since the latter is a 1-complemented subspace of $H \widehat{\otimes}_\pi H \widehat{\otimes}_\pi \ldots \widehat{\otimes}_\pi H$ for a suitable Hilbert space H.

Let $\mathbb{K}$ denote $\mathbb{R}$ or $\mathbb{C}$. We will denote by $\widehat{\otimes}_\pi^{\mathbb{K}}$ (respectively, $\otimes_\pi^{\mathbb{K}}$) the complete projective tensor product over $\mathbb{K}$ (respectively, the projective tensor product over $\mathbb{K}$).

We focus now our attention to the projective tensor product of two spin factors. We recall that a *complex spin factor* is a JB*-triple built from a complex Hilbert space $(H, (\cdot,\cdot))$, and a conjugation $\bar{\cdot}$ on H, by defining the triple product and the norm by

$$\{xyz\} := (x, y)z + (z, y)x - (x, \overline{z})\overline{y}$$

and

$$\|x\|^2 := (x, x) + \sqrt{(x, x)^2 - |(x, \overline{x})|^2}, \tag{5}$$

respectively, for all x, y, z in H.

We will need some technical lemmas in order to prove that the projective tensor product of two spin factors has the KKP.

Lemma 3.4. *Let X and Y be real Banach spaces. Then $\left(\mathbb{C}\widehat{\otimes}_\pi^{\mathbb{R}} X\right) \widehat{\otimes}_\pi^{\mathbb{C}} \left(\mathbb{C}\widehat{\otimes}_\pi^{\mathbb{R}} Y\right)$ is isometrically isomorphic to $\mathbb{C}\widehat{\otimes}_\pi^{\mathbb{R}} X \widehat{\otimes}_\pi^{\mathbb{R}} Y$.*

Proof. The mapping $(\alpha, x, y) \mapsto (\alpha \otimes x) \otimes (1 \otimes y) = (1 \otimes x) \otimes (\alpha \otimes y)$ is a trilinear contractive operator from $\mathbb{C} \times X \times Y$ to $\left(\mathbb{C}\widehat{\otimes}_\pi^{\mathbb{R}} X\right) \widehat{\otimes}_\pi^{\mathbb{C}} \left(\mathbb{C}\widehat{\otimes}_\pi^{\mathbb{R}} Y\right)$. Then there exists a contractive linear operator

$$\Phi : \mathbb{C}\widehat{\otimes}_\pi^{\mathbb{R}} X \widehat{\otimes}_\pi^{\mathbb{R}} Y \to \left(\mathbb{C}\widehat{\otimes}_\pi^{\mathbb{R}} X\right) \widehat{\otimes}_\pi^{\mathbb{C}} \left(\mathbb{C}\widehat{\otimes}_\pi^{\mathbb{R}} Y\right).$$

Moreover, Φ is a linear bijection from $\mathbb{C} \otimes_\pi^{\mathbb{R}} X \otimes_\pi^{\mathbb{R}} Y$ to $\left(\mathbb{C} \otimes_\pi^{\mathbb{R}} X\right) \otimes_\pi^{\mathbb{C}} \left(\mathbb{C} \otimes_\pi^{\mathbb{R}} Y\right)$. We claim that for each (u, v) in $\left(\mathbb{C} \otimes_\pi^{\mathbb{R}} X\right) \times \left(\mathbb{C} \otimes_\pi^{\mathbb{R}} Y\right)$ we have

$$\|\Phi^{-1}(u \overset{\mathbb{C}}{\otimes} v)\| \leq \|u\| \, \|v\|.$$

Indeed, for each (u, v) in $\left(\mathbb{C} \otimes_\pi^{\mathbb{R}} X\right) \times \left(\mathbb{C} \otimes_\pi^{\mathbb{R}} Y\right)$ let us write

$$u = \sum_i \alpha_i \otimes x_i, \quad v = \sum_j \beta_j \otimes y_j,$$

where $\alpha_i, \beta_j \in \mathbb{C}$, $x_i \in X$, and $y_j \in Y$ for all i, j. Then

$$\begin{aligned}\|\Phi^{-1}(u \otimes v)\| &= \left\| \sum_{i,j} \alpha_i \beta_j \otimes x_i \otimes y_j \right\| \leq \sum_{i,j} |\alpha_i| \, |\beta_j| \, \|x_i\| \, \|y_j\| \\ &= \left(\sum_i |\alpha_i| \, \|x_i\| \right) \left(\sum_j |\beta_j| \, \|y_j\| \right),\end{aligned}$$

which implies that $\|\Phi^{-1}(u \overset{\mathbb{C}}{\otimes} v)\| \leq \|u\| \, \|v\|$. Therefore Φ extends to a surjective linear isometry from $\mathbb{C}\widehat{\otimes}_\pi^{\mathbb{R}} X \widehat{\otimes}_\pi^{\mathbb{R}} Y$ to $\left(\mathbb{C}\widehat{\otimes}_\pi^{\mathbb{R}} X\right) \widehat{\otimes}_\pi^{\mathbb{C}} \left(\mathbb{C}\widehat{\otimes}_\pi^{\mathbb{R}} Y\right)$. □

The result contained in the following lemma was implicitly proved in [26, Theorem 3.4]. We include here an explicit and shorter proof of this fact for completeness reasons.

Lemma 3.5. *Let S be a complex spin factor. Then there exists a real Hilbert space H such that S is isometrically isomorphic to $\mathbb{C}\widehat{\otimes}_\pi^{\mathbb{R}} H$.*

Proof. Let H be a real Hilbert space with inner product denoted by $< ., . >$. We first claim that for every $1 \otimes x + i \otimes y$ in $\mathbb{C}\widehat{\otimes}_\pi^{\mathbb{R}} H$, its norm can be computed as follows:

$$\|1 \otimes x + i \otimes y\|^2 = \|x\|_2^2 + \|y\|_2^2 + 2\sqrt{\|x\|_2^2 \, \|y\|_2^2 - < x, y >^2}, \tag{6}$$

where $\|.\|_2$ denotes the Hilbertian norm in H. Indeed, since the dual of $\mathbb{C}\widehat{\otimes}_\pi^{\mathbb{R}} H$ can be identified with $L(H, \mathbb{C}_{\mathbb{R}})$ then, every $1 \otimes x + i \otimes y \in \mathbb{C}\widehat{\otimes}_\pi^{\mathbb{R}} H$ is a trace-class operator in $L(H, \mathbb{C}_{\mathbb{R}})$. Let T denote the operator $1 \otimes x + i \otimes y$ and let $U = TT^*$. Since U satisfies the identity

$$U^2 - (\|x\|_2^2 + \|y\|_2^2)U + \left(\|x\|_2^2 \, \|y\|_2^2 - < x, y >^2\right) Id = 0,$$

we conclude that the eigenvalues of U are

$$\lambda_1 = \frac{1}{2}\left(\|x\|_2^2 + \|y\|_2^2 + \sqrt{\left(\|x\|_2^2 + \|y\|_2^2\right)^2 + 4 < x, y >^2}\right)$$

and

$$\lambda_2 = \frac{1}{2}\left(\|x\|_2^2 + \|y\|_2^2 - \sqrt{\left(\|x\|_2^2 + \|y\|_2^2\right)^2 + 4 < x, y >^2}\right).$$

As a consequence, we get that

$$\begin{aligned}\|1 \otimes x + i \otimes y\|^2 = \|T\|^2 = \left(\sqrt{\lambda_1} + \sqrt{\lambda_2}\right)^2 &= \lambda_1 + \lambda_2 + 2\sqrt{\lambda_1\, \lambda_2}\\ &= \|x\|_2^2 + \|y\|_2^2 + 2\sqrt{\|x\|_2^2\, \|y\|_2^2 - < x, y >^2}.\end{aligned}$$

Let now S be a spin factor with inner product and conjugation denoted by $(.,.)$ and $\bar{\cdot}$, respectively. Let $H := \{z \in S : \bar{z} = z\}$. It is not hard to see that $(H, \Re\mathrm{e}(.,.))$ is a real Hilbert space. Since for every $z \in S$ there are unique $x, y \in H$ such that $z = x + iy$, we conclude that the law

$$\Phi : z = x + iy \mapsto 1 \otimes x + i \otimes y$$

defines a linear bijection from S to $\mathbb{C}\widehat{\otimes}_\pi^{\mathbb{R}} H$. Finally, from the formulae of the norms in the tensor product and in the spin factor (see (6) and (5)) we conclude that Φ is an isometry. $\square$

In [1, Proposition 5], M. D. Acosta and the second author of the present note prove that every spin factor satisfies the KKP. The same fact can be now derived from Lemma 3.5 and Remark 3.3 in a shorter way.

Corollary 3.6. *Let S be a spin factor. Then S satisfies the KKP.* $\square$

Corollary 3.7. *Let S_1 and S_2 be two (complex) spin factors. Then $S_1\widehat{\otimes}_\pi S_2$ satisfies the KKP.*

Proof. By Lemma 3.5 there are real Hilbert spaces H_1 and H_2 such that $S_1 = \mathbb{C}\widehat{\otimes}_\pi^{\mathbb{R}} H_1$ and $S_2 = \mathbb{C}\widehat{\otimes}_\pi^{\mathbb{R}} H_2$. From Lemma 3.4 we conclude that $S_1\widehat{\otimes}_\pi^{\mathbb{C}} S_2$ is isometrically isomorphic to $\mathbb{C}\widehat{\otimes}_\pi^{\mathbb{R}} H_1\widehat{\otimes}_\pi^{\mathbb{R}} H_2$. Since, by Remark 3.3, the realification of $\mathbb{C}\widehat{\otimes}_\pi^{\mathbb{R}} X\widehat{\otimes}_\pi^{\mathbb{R}} Y$ satisfies the KKP, it follows that $S_1\widehat{\otimes}_\pi S_2$ satisfies the KKP. $\square$

Our last goal is to show that the projective tensor product of a spin factor and a Hilbert space always satisfies the KPP.

Lemma 3.8. *Let M be a 1-complemented subspace of a Banach space X. Then M^* is isometrically isomorphic to a 1-complemented subspace of X^*.*

Proof. Let P be contractive projection on X with $P(X) = M$, and let π be the natural projection of X^* onto X^*/M°. Since $X = N \oplus M$, where $N = \ker(P)$ it follows that $X^* = M^\circ \oplus N^\circ$. It is clear that P^* is a contractive projection from X^* onto N°.

Let $u \in N^\circ$. The inequality

$$\|\pi(u)\| = \inf\{\|u + v\| : v \in M^\circ\} \geq \inf\{\|P^*(u + v)\| : v \in M^\circ\} = \|u\|,$$

shows that $\|\pi(u)\| = \|u\|$. Thus $\pi|_{N^\circ}$ is a surjective isometry from N° to M^*. $\square$

Lemma 3.9. *Let X and Y be complex Banach spaces. Then $K(X, Y)_\mathbb{R}$ is a 1-complemented subspace of $K(X_\mathbb{R}, Y_\mathbb{R})$.*

Proof. For each $T \in K(X_\mathbb{R}, Y_\mathbb{R})$ let $\sigma(T) \in K(X_\mathbb{R}, Y_\mathbb{R})$ defined by

$$\sigma(T)(x) := -iT(ix).$$

The law $T \mapsto \sigma(T)$ is a linear surjective isometry of period 2 on $K(X_\mathbb{R}, Y_\mathbb{R})$. Let $P : K(X_\mathbb{R}, Y_\mathbb{R}) \to K(X_\mathbb{R}, Y_\mathbb{R})$ be the projection defined by $P(T) := \frac{1}{2}(T + \sigma(T))$. It is easily seen that P is a linear contractive projection whose image is $K(X, Y)_\mathbb{R}$. $\square$

Corollary 3.10. *Let S be a (complex) spin factors and let H be a complex Hilbert space. Then $S\widehat{\otimes}_\pi H$ satisfies the KKP.*

Proof. By Lemma 3.5 there is a real Hilbert spaces H_1 such that $S = \mathbb{C}\widehat{\otimes}_\pi^\mathbb{R} H_1$. Lemma 3.9 implies that $K(S, H^*)$ is 1-complemented in $K(S_\mathbb{R}, H^*_\mathbb{R})$. From Lemma 3.8 we deduce that $K(S, H^*)^*$ is 1-complemented in $K(S_\mathbb{R}, H^*_\mathbb{R})^*$. Since H satisfies the approximation property and the Radon-Nikodym property it follows that $K(S, H^*)^* = S\widehat{\otimes}_\pi^\mathbb{C} H$ and $K(S_\mathbb{R}, H^*_\mathbb{R})^* = S_\mathbb{R}\widehat{\otimes}_\pi^\mathbb{R} H_\mathbb{R}$ (compare [14, Corollary 5.3 and Theorem 16.5]). Therefore, $S\widehat{\otimes}_\pi^\mathbb{C} H$ is 1-complemented in $S_\mathbb{R}\widehat{\otimes}_\pi^\mathbb{R} H_\mathbb{R} = \mathbb{C}\widehat{\otimes}_\pi^\mathbb{R} H_1\widehat{\otimes}_\pi^\mathbb{R} H_\mathbb{R}$. Since by Remark 3.3 the space $\mathbb{C}\widehat{\otimes}_\pi^\mathbb{R} H_1\widehat{\otimes}_\pi^\mathbb{R} H_\mathbb{R}$ has the KKP we conclude that $S\widehat{\otimes}_\pi^\mathbb{C} H$ satisfies the KKP. $\square$

Remark 3.11. Let F be a finite dimensional Banach space and let H be a Hilbert space. It is not hard to see that for each $x \in L(H, F^*)$ and for every projection $p \in L(H)$ we have

$$\|x\|^2 \leq \|xp\|^2 + \|xq\|^2,$$

where $q = 1 - p$ (compare [2, Proposition in Appendix]). Analysis similar to that in the proof of Proposition 3.2 or [2, Appendix] shows that $L(H, F^*)_* = H\widehat{\otimes}_\pi F$ has the KKP.

Let S be a spin factor. Similar arguments to those given in the proof of Corollary 3.10 show that $S\widehat{\otimes}_\pi F$ has the KKP.

Let E be a JB*-triple. We have already commented that E has the KKP if and only if F is finite dimensional or a Hilbert space or a spin factor (compare [10, Proposiiton 2.13]). From Corollaries 3.7 and 3.10 and Remark 3.11 we obtain now the following:

Theorem 3.12. *Let E and F be JB*-triples. Then $E\widehat{\otimes}_\pi F$ satisfies the KKP if and only if E and F have the KKP.* □

In the last twenty years a theory of real Jordan triples has been developed by several authors (see for example [3,11,13,21,23,24,28–30]). According to [21] a *real JB*-triple* is a real norm-closed subtriple of a (complex) JB*-triple. The class of real JB*-triples includes all (complex) JB*-triples, all real C*-algebras and real von Neumann algebras and the real Banach space of all bounded linear operators between two real, complex or quaternionic Hilbert spaces. Another interesting example of a real JB*-triple is the so-called *real spin factor* defined as follows: let X be a real Hilbert space, of dimension greater or equal to three, and let X_1 and X_2 be closed linear subspaces of X so that $X_2 = X_1{}^\perp$, a real spin factor is the real JB*-triple $E = X_1 \oplus^1 X_2$ with triple product given by

$$\{x, y, z\} = \langle x, y\rangle\, z + \langle z, y\rangle\, x - \langle x, \bar{z}\rangle\, \bar{y},$$

where $\langle ., .\rangle$ is the inner product in X and the involution $x \to \bar{x}$ on E is defined by $\bar{x} = (x_1, -x_2)$ for every $x = (x_1, x_2)$ (compare [23]).

Our next goal consists in describing those real JB*-triples having the KKP in order to generalize [10, Proposiiton 2.13].

Let x be a element in a real JB*-triple E and let $E(x)$ denote the real closed subtriple of E generated by x. It is known that there exists a locally compact Hausdorff space $S_x \subseteq (0, +\infty)$ with $S_x \cup \{0\}$ compact and a conjugation (conjugate linear isometry of period 2) τ on $C_0(S_x, \mathbb{C})$ such that $E(x)$ is isometrically isomorphic to $C_0(S_x, \mathbb{C})^\tau := \{f \in C_0(S_x, \mathbb{C}) : \tau(f) = f\}$ (compare [4, Proof of Theorem 2.9]). S_x is called the complex triple spectrum of x.

Let E be a real JB*-triple satisfying the KKP. Since the KKP property is inherited by closed subspaces it follows that, for each x in E, the closed subtriple of E generated by x, $E(x)$, also has the KKP. Since $E(x)$ is isometrically isomorphic to $C_0(S_x, \mathbb{C})^\tau$ it is not hard to see that S_x must be finite. Therefore, if E is a real JB*-triple having the KKP then S_x must me finite for all $x \in E$ and by [3, Theorems 3.1, 3.8 and 2.3] we conclude that E is isometric to a finite ℓ_∞-sum of closed simple triple ideals which are either finite-dimensional, infinite-dimensional real or complex spin factors, or of the form $\mathcal{L}(H, K)$ for some real, complex, or quaternionic Hilbert spaces H, K, with $\dim(H) = \infty$ and $\dim(K) < \infty$. Now applying [1, Remark 1 (3)], which is also valid for the KKP instead of DP1, we get that E is finite dimensional, or a real, complex, or quaternionic Hilbert space or a real or complex spin factor.

Proposition 3.13. *Let E be a real JB*-triple. Then E has the KKP if and only if E is finite dimensional or a real, complex, or quaternionic Hilbert space or a real or complex spin factor.* □

When in the arguments and tools applied in the proof of Theorem 3.12 we replace [10, Proposiiton 2.13] with Proposition 3.13 above we get:

Theorem 3.14. *Let E and F be real JB*-triples. Then $E\widehat{\otimes}_\pi F$ satisfies the KKP if and only if E and F have the KKP.* □

From [27] we know that a Banach space X has property (V) if and only if a set $K \subset X^*$ is relatively weakly compact whenever $\lim_{n\to\infty} \sup\{|\varphi(x_n)| : \varphi \in K\} = 0$, for every *w.u.C.* series $\sum x_n$ in X.

Given a real JB*-triple E, it is know that there exists a (complex) JB*-triple $\widehat{E}$ and a conjugation τ on $\widehat{E}$ such that

$$E = \widehat{E}^{\tau} := \{z \in \widehat{E} : \tau(z) = z\} \quad \text{(see [21]).}$$

Since every (complex) JB*-triples has property (V) and this property inherited by complemented subspaces it follows that every real JB*-triple also has this property. The following results follows now from Theorem 2.4.

Corollary 3.15. *Let E and F be two real JB*-triples. The following statements are equivalent:*

(a) $E\widehat{\otimes}_\pi F$ *satisfies DPP*
(b) *E and F satisfy DPP and do not contain copies of* ℓ_1. □

Problem: In [19], W. Freedman introduces an strictly weaker version of the DPP called *the alternative Dunford-Pettis property* (DP1). A Banach space X has the DP1 if and only if whenever $x_n \to x$ and $\rho_n \to 0$ weakly in X and X^*, respectively, with $x_n, x \in S_X$, then we have $\rho_n(x_n) \to 0$. In the same paper Freedman proved that the DP1 and the DPP are equivalent whenever X is a von Neumann algebra. In [9] L. J. Bunce and the second author of the present note show that DPP and DP1 coincide for every C*-algebra.

It is known that the DPP and the KKP both imply the DP1. In the present paper we have studied the DPP and the KKP in the projective tensor product of two JB*-triples, it seems natural to study the DP1 on the projective tensor product of two JB*-triples. Concerning this question, we would like to note that this problem is even open for $C(K)$ spaces.

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