

Relatively weakly open sets in closed balls of Banach spaces, and real JB^* -triples of finite rank

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Abstract. We prove that, given a real JB^* -triple X , there exists a nonempty relatively weakly open subset of the closed unit ball of X with diameter less than 2 (if and) only if the Banach space of X is isomorphic to a Hilbert space. Moreover we give the structure of real JB^* -triples whose Banach spaces are isomorphic to Hilbert spaces. Such real JB^* -triples are also characterized in two different purely algebraic ways.

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1. Introduction

In [25], O. Nygaard and D. Werner discover, perhaps for the first time, how some of the classical Banach spaces which fail the Radon-Nikodym property actually fail to fulfil much weaker requirements. Indeed, it is proved in [25] that, if X is an infinite-dimensional uniform algebra, then X satisfies Property \mathcal{P} which follows:

(\mathcal{P}) Every nonempty relatively weakly open subset of the closed unit ball of X has diameter equal to 2.

This applies in particular to infinite-dimensional real or complex $C(\Omega)$ -spaces. We remark that Property \mathcal{P} is fulfilled by every Banach space satisfying the so-called Daugavet property [32, Lemma 3]. Other Banach spaces enjoying Property \mathcal{P} are all infinite-dimensional complex C^* -algebras, and all JB -algebras whose Banach spaces are not isomorphic to Hilbert spaces [3]. The proof that infinite-dimensional complex C^* -algebras satisfy Property \mathcal{P} , given in [3], essentially

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relies on the more general fact, also shown in [3], that Property \mathcal{P} is fulfilled by every complex JB^* -triple whose Banach space is not isomorphic to a Hilbert space. For JB -algebras and complex JB^* -triples the reader is referred to [13] and [18], respectively.

In the present paper we prove as the main result that every real JB^* -triple X whose Banach space is not isomorphic to a Hilbert space satisfies Property \mathcal{P} (Theorem 2.3). We note that the class of real JB^* -triples, introduced in [15], contains that of complex JB^* -triples (regarded as real Banach spaces), as well as that of JB -algebras. It is worth mentioning that the proof of Theorem 2.3 provided here is independent of the one given in [3] for complex JB^* -triples, and has the advantage that it works autonomously in all relevant subclasses of the class of real JB^* -triples. In fact the key tool in the proof of Theorem 2.3 is that every nonreflexive Banach space X such that X^* is an L -summand of X^{***} fulfils Property \mathcal{P} (Proposition 2.1). Thus, since infinite-dimensional complex C^* -algebras are nonreflexive Banach spaces, and the dual of every complex C^* -algebra X is an L -summand of X^{***} , we are provided with a JB^* -triple-free proof of the main result in [3] that infinite-dimensional complex C^* -algebras X satisfy Property \mathcal{P} . By the way, as a consequence of Theorem 2.3, every infinite-dimensional real C^* -algebra fulfils Property \mathcal{P} (Corollary 2.6).

Theorem 2.3 also contains a precise description of real JB^* -triples whose Banach spaces are isomorphic to Hilbert spaces. Indeed, such real JB^* -triples are nothing other than those which can be written as finite ℓ_∞ -sums of simple JB^* -triples which are either finite-dimensional, infinite-dimensional generalized real spin factors, or of the form $\mathcal{L}(H, K)$ for some real, complex, or quaternionic Hilbert spaces H, K with $\dim(H) = \infty$ and $\dim(K) < \infty$. (Here, for H and K as above, $\mathcal{L}(H, K)$ means the real Banach space of all bounded linear operators from H to K .) We complete the description just reviewed by showing two different purely algebraic characterizations of real JB^* -triples whose Banach spaces are isomorphic to Hilbert spaces. Indeed, they are precisely the real JB^* -triples of finite rank (Theorem 3.1) as well as those real JB^* -triples such that all single-generated subtriples are finite-dimensional (Theorem 3.8).

To conclude this introduction, let us note that our results include the generalizations to real JB^* -triples of the facts proved by Bunce-Chu [7] and Kaup [19] concerning the Radon-Nikodym Property and finite rank, respectively, on complex JB^* -triples.

2. The main result

Let X be a real or complex Banach space. We denote by S_X , B_X , and X^* the unit sphere, the closed unit ball, and the (topological) dual, respectively, of X . We denote by w the weak topology of X , and by w^* the weak* topology of X^* . We always see X as a subspace of its bidual X^{**} via the canonical injection, and, given

a subspace P of X , we denote by P° the polar of P in X^* . Also, for a bounded linear operator T on X , we denote by T^* the transpose operator on X^* . An **L -projection** (respectively, **M -projection**) on X is a linear projection (say π) on X satisfying $\|x\| = \|\pi(x)\| + \|x - \pi(x)\|$ (respectively, $\|x\| = \max\{\|\pi(x)\|, \|x - \pi(x)\|\}$) for every $x \in X$. A subspace P of X is said to be an **L -summand** (respectively, **M -summand**) of X if it is the range of an L -projection (respectively, M -projection) on X , and an **M -ideal** of X if P° is an L -summand of X^* . The Banach space X is said to be **L -embedded** (respectively, **M -embedded**) whenever X is an L -summand (respectively, an M -ideal) of X^{**} . According to [14, Proposition III.1.2], X is M -embedded if and only if the Dixmier projection on X^{***} is an L -projection. Consequently, if X is M -embedded, then X^* is L -embedded. The converse is not true. Indeed, in the case that X is a complex C^* -algebra, X^* is always L -embedded, whereas X is M -embedded if and only if it is a c_0 -sum of algebras of all compact operators on suitable complex Hilbert spaces (see [14]).

Proposition 2.1. *Let X be a Banach space such that X^* is L -embedded. If there exists a nonempty relatively w -open subset of B_X with diameter less than 2, then X is reflexive.*

Proof. We have $X^{***} = (X^* \oplus N)_{\ell_1}$ for some subspace N of X^{***} , and hence $X^{****} = ((X^*)^\circ \oplus N^\circ)_{\ell_\infty}$. Assume that there is a nonempty relatively w -open subset U of B_X with $\text{diam}(U) < 2$. Then U contains a set V of the form

$$\{x \in B_X : |f_i(x - x_0)| < 1 \ \forall i = 1, \dots, n\},$$

for suitable $x_0 \in B_X$, $n \in \mathbb{N}$, and $f_1, \dots, f_n \in X^*$. Put

$$V^{**} := \{z \in B_{X^{**}} : |f_i(z - x_0)| < 1 \ \forall i = 1, \dots, n\}.$$

Since V^{**} is relatively w^* -open in $B_{X^{**}}$, and B_X is w^* -dense in $B_{X^{**}}$, the set V ($= V^{**} \cap B_X$) is w^* -dense in V^{**} . Therefore $V - V$ is w^* -dense in $V^{**} - V^{**}$, and consequently, by the lower w^* -semicontinuity of the norm of X^{**} , we have $\text{diam}(V^{**}) = \text{diam}(V) \leq \text{diam}(U) < 2$. In the same way, the set

$$V^{****} := \{\beta \in B_{X^{****}} : |f_i(\beta - x_0)| < 1 \ \forall i = 1, \dots, n\}$$

has diameter less than 2. Write $x_0 = u + v$ with $(u, v) \in (X^*)^\circ \times N^\circ$. We claim that $B_{(X^*)^\circ} + v$ is contained in V^{****} . Indeed, for $\alpha \in B_{(X^*)^\circ}$, $\alpha + v$ belongs to $B_{X^{****}}$ because $X^{****} = ((X^*)^\circ \oplus N^\circ)_{\ell_\infty}$, and, on the other hand, for every $i = 1, \dots, n$ we have $f_i(\alpha + v - x_0) = f_i(\alpha - u) = 0$ because $(\alpha - u, f_i)$ belongs to $(X^*)^\circ \times X^*$. Keeping in mind that $\text{diam}(V^{****}) < 2$, it follows from the claim just shown that $\text{diam}(B_{(X^*)^\circ}) = \text{diam}(B_{(X^*)^\circ} + v) < 2$. Therefore X is reflexive. \square

We recall that a **complex JB^* -triple** is a complex Banach space X with a continuous triple product $\{\dots\} : X \times X \times X \rightarrow X$ which is linear and symmetric in the outer variables, and conjugate-linear in the middle variable, and satisfies:

1. For all x in X , the mapping $y \rightarrow \{xxy\}$ from X to X is a hermitian operator on X and has nonnegative spectrum.
2. The **main identity**

$$\{ab\{xyz\}\} = \{\{abx\}yz\} - \{x\{bay\}z\} + \{xy\{abz\}\}$$

holds for all a, b, x, y, z in X .

3. $\|\{xxx\}\| = \|x\|^3$ for every x in X .

Concerning Condition (1) above, we also recall that a bounded linear operator T on a complex Banach space X is said to be **hermitian** if $\|\exp(irT)\| = 1$ for every r in \mathbb{R} . Following [15], we define **real JB^* -triples** as norm-closed real subtriples of complex JB^* -triples. Here, by a **subtriple** we mean a subspace which is closed under triple products of its elements. A **triple ideal** of a real or complex JB^* -triple X is a subspace M of X such that $\{XXM\} + \{XMX\} \subseteq M$. We say that the JB^* -triple X is **simple** if there are not triple ideals of X others than $\{0\}$ and X . **Real JBW^* -triples** were first introduced as those real JB^* -triples which are dual Banach spaces in such a way that the triple product becomes separately w^* -continuous (see [15, Definition 4.1 and Theorem 4.4]). Later, it was shown in [23] that the requirement of separate w^* -continuity of the triple product is superabundant. We will apply without notice that the bidual of every real JB^* -triple X is a JBW^* -triple under a suitable triple product which extends the one of X [15, Lemma 4.2].

The next proposition becomes the real variant of [2, Proposition 3.4].

Proposition 2.2. *The predual of every real JBW^* -triple is L -embedded.*

Proof. Let X be a real JBW^* -triple, and let X_* stand for the predual of X . For $x, y \in X$, let $L(x, y)$ and $Q(x, y)$ denote the operators on X defined by $L_{x,y}(z) := \{xyz\}$ and $Q_{x,y}(z) := \{xzy\}$, respectively. By standard theory of duality, the separate w^* -continuity of the triple product of X is equivalent to the inclusions $(L_{x,y})^*(X_*) \subseteq X_*$ and $(Q_{x,y})^*(X_*) \subseteq X_*$ for all $x, y \in X$. Therefore we have $(L_{x,y})^{**}((X_*)^\circ) \subseteq (X_*)^\circ$ and $(Q_{x,y})^{**}((X_*)^\circ) \subseteq (X_*)^\circ$ for all $x, y \in X$. Keeping in mind the separate w^* -continuity of the triple product of X^{**} and the w^* -density of X in X^{**} , the above inclusions read as $\{XX(X_*)^\circ\} \subseteq (X_*)^\circ$ and $\{X(X_*)^\circ X\} \subseteq (X_*)^\circ$, respectively. Applying again the separate w^* -continuity of the triple product of X^{**} and the w^* -density of X in X^{**} , we deduce

$$\{X^{**}X^{**}(X_*)^\circ\} + \{X^{**}(X_*)^\circ X^{**}\} \subseteq (X_*)^\circ.$$

Therefore $(X_*)^\circ$ is a w^* -closed triple ideal of X^{**} , and hence we have $X^{**} = (X_*)^\circ \oplus P$ for a suitable closed triple ideal P of X^{**} [15, Lemma 4.3]. Now, the abstract ℓ_∞ -product $((X_*)^\circ \times P)_{\ell_\infty}$ is a real JB^* -triple in a natural way, and the mapping $\Phi : (u, v) \rightarrow u + v$ from $((X_*)^\circ \times P)_{\ell_\infty}$ to X^{**} becomes a linear bijection preserving triple products. By [15, Theorem 4.8], Φ is an isometry, that is $X^{**} = ((X_*)^\circ \oplus P)_{\ell_\infty}$. Thus $(X_*)^\circ$ is an M -summand of X^{**} , and hence, by [14, Theorem I.1.9], X_* is an L -summand of X^* . □

Examples of real JB^* -triples are the spaces $\mathcal{L}(H, K)$, for arbitrary real, complex, or quaternionic Hilbert spaces H and K , under the triple product $\{xyz\} := \frac{1}{2}(xy^*z + zy^*x)$. The above examples become particular cases of those arising by considering either the so-called complex Cartan factors (regarded as real JB^* -triples) or real forms of complex Cartan factors [20]. We recall that **real forms** of a complex Banach space X are defined as the real closed subspaces of X of the form $X^\tau := \{x \in X : \tau(x) = x\}$, for some conjugation (i.e., conjugate-linear isometry of period two) on X . We note that, if X is a complex JB^* -triple, then every real form of X is a real JB^* -triple (since conjugations on X preserve triple products [18]). Among complex Cartan factors, the so-called **complex spin factors** become specially relevant for our present approach. They are built from an arbitrary complex Hilbert space $(H, (\cdot|\cdot))$ of hilbertian dimension ≥ 3 , by taking a conjugation σ on H , and then by defining the triple product and the norm by

$$\{xyz\} := (x|y)z + (z|y)x - (x|\sigma(z))\sigma(y)$$

and

$$\|x\|^2 := (x|x) + \sqrt{(x|x)^2 - |(x|\sigma(x))|^2},$$

respectively, for all x, y, z in H . Following [24], we say that a real JB^* -triple is a **generalized real spin factor** if it is either a complex spin factor (regarded as a real JB^* -triple) or a real form of a complex spin factor.

Theorem 2.3. *Let X be a real JB^* -triple. Then the following assertions are equivalent:*

1. *There exists a relatively w -open subset of B_X with diameter less than two.*
2. *The Banach space of X is reflexive.*
3. *X is a finite ℓ_∞ -sum of closed simple triple ideals which are either finite-dimensional, infinite-dimensional generalized real spin factors, or of the form $\mathcal{L}(H, K)$ for some real, complex, or quaternionic Hilbert spaces H, K with $\dim(H) = \infty$ and $\dim(K) < \infty$.*
4. *The Banach space of X is isomorphic to a Hilbert space.*
5. *X has the Radon-Nikodym property.*

Proof. (1) \Rightarrow (2).- By Proposition 2.2, X^* is L -embedded. Then Assertion (2) follows from the assumption (1) and Proposition 2.1.

(2) \Rightarrow (3).- By [15, Proposition 2.2], there exists a complex JB^* -triple Y , and a conjugation τ on Y such that $X = Y^\tau$. Since the Banach space of X is reflexive (by the assumption (2)), and $Y = X \oplus iX$, the Banach space of Y is reflexive. Then, by the concluding part of the proof of [7, Proposition 4.5], we have $Y = (\bigoplus_{i=1}^n Y_i)_{\ell_\infty}$ where $\{Y_i\}_{i=1, \dots, n}$ is the family of all minimal triple ideals of Y , and moreover, for $i = 1, \dots, n$, Y_i is either finite-dimensional, an infinite-dimensional complex spin factor, or of the form $\mathcal{L}(H, K)$ for suitable complex Hilbert spaces H, K with $\dim(H) = \infty$ and $\dim(K) < \infty$. Since τ preserves

the triple product of Y , and is of period two, the set $\{1, \dots, n\}$ must be the disjoint union of three subsets A, B, C such that Y_i is τ -invariant whenever i belongs to A , and for each $i \in B$ there is a unique $j \in C$ with $\tau(Y_i) = Y_j$. It follows that, putting $Z_j := (Y_j \oplus \tau(Y_j))_{\ell_\infty}$ whenever j lies in B , each Z_j is τ -invariant, and we have

$$X = ((\oplus_{i \in A} Y_i^\tau)_{\ell_\infty} \oplus (\oplus_{j \in B} Z_j^\tau)_{\ell_\infty})_{\ell_\infty}.$$

Since, for $j \in B$, the mapping $y_j \rightarrow y_j + \tau(y_j)$ from Y_j (regarded as a real JB^* -triple) to Z_j^τ is a surjective linear isometry preserving triple products, we can write

$$X = ((\oplus_{i \in A} Y_i^\tau)_{\ell_\infty} \oplus (\oplus_{j \in B} Y_j)_{\ell_\infty})_{\ell_\infty}.$$

Now, to conclude the proof it is enough to show that, if there exists $i \in A$ such that Y_i is of the form $\mathcal{L}(H, K)$ for suitable complex Hilbert spaces H, K with $\dim(H) = \infty$ and $\dim(K) < \infty$, then Y_i^τ is of the form $\mathcal{L}(H', K')$ for suitable real or quaternionic Hilbert spaces H', K' with $\dim(H') = \infty$ and $\dim(K') < \infty$. But this follows from [20, Theorem 4.1].

(3) \Rightarrow (4).- This implication is clear.

(4) \Rightarrow (5) and (5) \Rightarrow (1).- These implications are true even if X is only assumed to be an arbitrary Banach space. Indeed, Banach spaces isomorphic to Hilbert spaces are reflexive, reflexive Banach spaces have the Radon-Nikodym property, the Radon-Nikodym property implies the existence of “slices” of the unit closed ball with arbitrarily small diameter, and such slices are w -open relative to the closed unit ball. □

For the determination of finite-dimensional simple real JB^* -triples (including finite-dimensional simple complex JB^* -triples) and of real forms of complex spin factors, the reader is referred to [22] and [20], respectively.

The next corollary follows straightforwardly from Propositions 2.2 and 2.1, and Theorem 2.3. The Banach space appearing in it could be considered as an arbitrary “nonassociative Lindenstrauss space”.

Corollary 2.4. *Let X be a real Banach space such that X^{**} is a real JB^* -triple (for some triple product). Then there exists a nonempty relatively weakly open subset of the closed unit ball of X with diameter less than 2 (if and) only if X is isomorphic to a Hilbert space.*

Let X be a Banach space. For u in S_X , we define the **roughness of X at u** , $\eta(X, u)$, by the equality

$$\eta(X, u) := \limsup_{\|h\| \rightarrow 0} \frac{\|u + h\| + \|u - h\| - 2}{\|h\|}.$$

We remark that the absence of roughness of X at u (i.e., $\eta(X, u) = 0$) is nothing other than the Fréchet differentiability of the norm of X at u [10, Lemma

I.1.3]. Given $\epsilon > 0$, the Banach space X is said to be ϵ -**rough** if, for every u in S_X , we have $\eta(X, u) \geq \epsilon$. We say that X is **rough** whenever it is ϵ -rough for some $\epsilon > 0$, and **extremely rough** whenever it is 2-rough.

Invoking the proof of [10, Proposition I.1.11], as is done in the proof of [3, Corollary 2.7], the following corollary follows from Theorem 2.3.

Corollary 2.5. *Let X be the predual of a real JBW^* -triple. Then X is extremely rough if (and only if) it is not isomorphic to a Hilbert space.*

Despite **real C^* -algebras** can be defined by different systems of intrinsic axioms (see [16] for a summary), we prefer to introduce them as the norm-closed self-adjoint real subalgebras of complex C^* -algebras. Since complex C^* -algebras are complex JB^* -triples under the triple product

$$\{xyz\} := \frac{1}{2}(xy^*z + zy^*x),$$

certainly real C^* -algebras are real JB^* -triples.

Corollary 2.6. *Let X be a real C^* -algebra. If there exists a non-empty relatively w -open subset of B_X with diameter less than two, then X is finite dimensional.*

Proof. By [28, 4.1.13] and [12, 15.4], there exists a complex C^* -algebra Y with a conjugate-linear $*$ -automorphism τ of period two such that $X = Y^\tau$. Assume that there is a non-empty relatively w -open subset of B_X with diameter less than two. Then, by Theorem 2.3, the Banach space of X (and hence that of Y) is reflexive. Finally, apply that complex C^* -algebras whose Banach spaces are reflexive actually are finite-dimensional [31]. \square

Real W^* -algebras are usually defined as those real C^* -algebras which are dual Banach spaces in such a way that the product becomes separately w^* -continuous (see for instance [8]). Nevertheless, the requirement of separate w^* -continuity of the product is superabundant [16].

Corollary 2.7. *Let X be the predual of a real W^* -algebra. If X is not extremely rough, then X is finite-dimensional.*

3. Algebraic characterizations of real JB^* -triples whose Banach spaces are isomorphic to Hilbert spaces

In this section we are going to prove two purely algebraic characterizations of real JB^* -triples whose Banach spaces are isomorphic to Hilbert spaces. To this end, it seems to us convenient to introduce the appropriate concepts and basic results at a suitable level of generality. The relevant references in such a level are [17] and [21].

From now on, let \mathbb{F} be an arbitrary field of characteristic different from 2 and 3. A **Jordan algebra** over \mathbb{F} is a (possibly non associative) commutative algebra over \mathbb{F} satisfying the identity $(xy)x^2 = x(yx^2)$. An element x in a Jordan algebra X with a unit $\mathbf{1}$ is said to be **invertible** if there exists $y \in X$ such that $xy = \mathbf{1}$ and $x^2y = x$. A **division Jordan algebra** is a nonzero unital Jordan algebra whose nonzero elements are invertible. A **Jordan triple** over \mathbb{F} is a vector space (say X) over \mathbb{F} endowed with a TRILINEAR triple product $\{\dots\} : X \times X \times X \rightarrow X$ satisfying the same main identity required in the definition of complex JB^* -triples. Now, let X be a Jordan triple over \mathbb{F} . A **tripotent** of X is an element $u \in X$ such that $\{uuu\} = \sigma u$ where $\sigma = \sigma(u) = \pm 1$. Those tripotents u in X satisfying $\sigma(u) = 1$ are called **positive**. Given a tripotent u in X , we have $X = X_0(u) \oplus X_1(u) \oplus X_2(u)$, where, for $j \in \{0, 1, 2\}$, $X_j(u)$ denotes the eigenspace of the operator $x \rightarrow \sigma\{u, u, x\}$ corresponding to the eigenvalue $\frac{1}{2}j$. The space $X_2(u)$ becomes a Jordan algebra over \mathbb{F} with unit u under the product $xy := \sigma\{xuy\}$, and moreover the operator $x \rightarrow \sigma\{uxu\}$ on $X_2(u)$ is a linear algebra involution. The tripotent u is said to be a **division tripotent** whenever $X_2(u)$ is a division Jordan algebra. Two tripotents u, v of X are called **orthogonal** if $u \in X_0(v)$, or equivalently $v \in X_0(u)$. By a **frame** in X we mean a family \mathcal{E} of pairwise orthogonal division tripotents of X such that $\bigcap_{u \in \mathcal{E}} X_0(u) = 0$. We say that X is of **finite rank** if there exists a finite frame in X .

Now, the first main result of this section reads as follows.

Theorem 3.1. *Let X be a real JB^* -triple. Then the Banach space of X is isomorphic to a Hilbert space if and only if X is of finite rank.*

The proof of Theorem 3.1 above needs further auxiliary notions and results. **Complex JB^* -algebras** are defined as those complete normed Jordan complex algebras X endowed with a conjugate-linear algebra involution $*$ satisfying $\|U_x(x^*)\| = \|x\|^3$ for every x in X , where, for x in X , the operator $U_x : X \rightarrow X$ is defined by $U_x(y) = 2x(xy) - x^2y$ [34, 6, 33]. We define **real JB^* -algebras** as the norm-closed self-adjoint real subalgebras of complex JB^* -algebras. Real JB^* -algebras were introduced by K. Alvermann [1] (under the name of J^*B -algebras), who provided a system of intrinsic axioms for them. **JB -algebras** are defined as those complete normed Jordan real algebras X satisfying $\|x\|^2 \leq \|x^2 + y^2\|$ for all $x, y \in X$ [13]. Note that, although complex JB^* -triples are only real Jordan triples, if u is an (automatically positive) tripotent in a complex JB^* -triple X , then the Jordan algebra $X_2(u)$ is a complex algebra in a natural way, and the canonical involution of $X_2(u)$ becomes conjugate-linear, so that the norm of X converts $X_2(u)$ into a complex JB^* -algebra.

Proposition 3.2. *If K is a nonzero real Hilbert space, and if u is any element in S_K , then the Banach space of K , with product and involution defined by*

$$xy := (x|u)y + (y|u)x - (x|y)u$$

and

$$x^* := 2(x|u)u - x,$$

respectively, becomes a division real JB^* -algebra (denoted by $J(K, u)$), whose unit is precisely u . Moreover, there are no division real JB^* -algebras others than those constructed above. More precisely, for a nonzero unital real JB^* -algebra X , the following condition are equivalent:

1. X is a division Jordan algebra.
2. The self-adjoint part X_{sa} of X reduces to $\mathbb{R}\mathbf{1}$.
3. $X = J(K, u)$ for some nonzero real Hilbert space K and some $u \in S_K$.

Proof. Let K be a nonzero real Hilbert space. If the dimension of K is 1 or 2, then $J(K, u)$ is \mathbb{R} or \mathbb{C} , so that certainly $J(K, u)$ is a division real JB^* -algebra. Assume that $\dim(K) > 2$. Let H denote the hilbertian complexification of K , let σ stand for the natural conjugation

$$k_1 + ik_2 \rightarrow k_1 - ik_2$$

on H , and let Y be the complex spin factor built from (H, σ) . Since u is a tri-potent in Y with $Y_2(u) = Y$, Y becomes naturally a complex JB^* -algebra with unit u . Since σ is a conjugate linear $*$ -automorphism of the complex JB^* -algebra Y , and $J(K, u) = Y^\sigma$, we deduce that $J(K, u)$ is a real JB^* -algebra. Moreover, $J(K, u)$ is a division Jordan algebra because, for $x \in J(K, u) \setminus \mathbb{R}u$, the subalgebra generated by x is a copy of \mathbb{C} .

Now, let X be a unital real JB^* -algebra. We have just proved that Condition (3) in the statement implies Condition (1).

(1) \Rightarrow (2).- As a consequence of [13, Proposition 3.8.2], the self-adjoint part X_{sa} of X is a JB -algebra. Assume that (2) does not holds, so that there exists $x \in X_{sa} \setminus \mathbb{R}\mathbf{1}$. Then, since Jordan algebras are power associative [13, Lemma 2.4.5], the closed subalgebra of X_{sa} generated by x and $\mathbf{1}$ is a unital associative JB -algebra, and hence it is of the form $C^{\mathbb{R}}(\Omega)$ for some compact Hausdorff topological space Ω [13, Theorem 3.2.2], which must have at least two points. Take $y, z \in C^{\mathbb{R}}(\Omega) \setminus \{0\}$ such that $yz = 0$. Then we have $U_y(z) = 0$, which, in view of [17, Theorem I.13.(2)], implies that y is not invertible in X , and hence that X is not a Jordan division algebra.

(2) \Rightarrow (3).- Let X_{sk} denote the skew part of X . Since the product of two skew elements of X is self-adjoint, the assumption (2) provides us with a symmetric bilinear form $(\cdot|\cdot)$ on X_{sk} satisfying $zt = -(z|t)\mathbf{1}$ for all $z, t \in X_{sk}$. Since $X = \mathbb{R}\mathbf{1} \oplus X_{sk}$, we can extend $(\cdot|\cdot)$ to a symmetric bilinear form on X by defining $(\mathbf{1}|\mathbf{1}) := 1$ and $(\mathbf{1}|z) := 0$ for every $z \in X_{sk}$. Then we have

$$xy := (x|\mathbf{1})y + (y|\mathbf{1})x - (x|y)\mathbf{1}$$

and

$$x^* := 2(x|\mathbf{1})\mathbf{1} - x$$

for all $x, y \in X$. Moreover, since, for $x \in X$, the (automatically self-adjoint) subalgebra of X generated by x is a real C^* -algebra, and the equality $x^*x = (x|x)\mathbf{1}$ holds, we deduce that $(\cdot|\cdot)$ is an inner product on X satisfying $(x|x) = \|x\|^2$. Thus the Banach space of X is a Hilbert space (say K), and we have $X = J(K, \mathbf{1})$. \square

We note that, if K is a real Hilbert space, and if u, v are in S_K , then we have $J(K, u) = J(K, v)$ structurally. Indeed, taking a surjective linear isometry T on K with $T(u) = v$, T becomes an isometric $*$ -isomorphism from $J(K, u)$ onto $J(K, v)$. It is also worth mentioning that the algebras $J(K, u)$ have appeared in the literature as the solutions to other problems different from the characterization of division real JB^* -algebras given by Proposition 3.2. Indeed, the algebras $J(K, u)$ are the unique norm-unital complete normed commutative real algebras whose Banach spaces are smooth at their unit [29]. They are also the unique unital complete normed Jordan real algebras satisfying $\|x\|\|x^{-1}\| = 1$ for every invertible element x [5], as well as the unique norm-unital complete normed commutative real algebras X such that the group of all surjective linear isometries on X acts transitively on S_X [4]. Moreover, such algebras become the natural parameters in the construction of all one-sided division complete absolute-valued real algebras [30].

Let u be a tripotent in a real JB^* -triple X . Then the Jordan algebra $X_2(u)$, endowed with its natural involution, is a real JB^* -algebra. Indeed, taking a complex JB^* -triple Y containing X as a closed real subtriple, $X_2(u)$ becomes a closed self-adjoint real subalgebra of the complex JB^* -algebra $Y_2(u)$. We denote by $X^1(u)$ the JB -algebra of all self-adjoint elements of the real JB^* -algebra $X_2(u)$. With the convention of symbols just made, the following corollary follows from Proposition 3.2.

Corollary 3.3. *Let u be a tripotent in a real JB^* -triple X . Then u is a division tripotent if and only if $X^1(u) = \mathbb{R}u$.*

Corollary 3.4. *Let u be a tripotent in a complex JB^* -triple X . Then u is a division tripotent if and only if $X_2(u) = \mathbb{C}u$.*

Proof. We have $X_2(u) = X^1(u) \oplus iX^1(u)$, and Corollary 3.3 applies. \square

Thus, in view of Corollary 3.3 (respectively, 3.4), division tripotents in a real (respectively, complex) JB^* -triple X coincide with the so-called in the literature **minimal tripotents** of X (see for instance [11] and [19]), which are defined as those tripotents u of X satisfying $X^1(u) = \mathbb{R}u$ (respectively, $X_2(u) = \mathbb{C}u$). The name “minimal” is in agreement with the fact that, in the case that X is actually a

real or complex JBW^* -triple, minimal tripotents of X are precisely those tripotents of X which are minimal relative to the order defined in the set of all tripotents of X by $u \leq v$ if and only if $v = u + w$ for some tripotent w orthogonal to u [27, Proposition 2.2]. Now, by arguing as in the proof of [27, Lemma 3.2], we obtain the following corollary.

Corollary 3.5. *Let Y be a complex JB^* -triple with conjugation τ , and let u be a division tripotent of Y^τ . Then u is the sum of at most two orthogonal division tripotents of Y .*

Proof of Theorem 3.1. Assume that the real JB^* -triple X is of finite rank, so that there exists a finite frame $\{u_1, \dots, u_n\}$ in X . Then, taking a complex JB^* -triple Y with conjugation τ such that $X = Y^\tau$, Corollary 3.5 gives (up to a rearrangement, if necessary) the existence of $1 \leq m \leq n$ and division tripotents $v_1, \dots, v_m, w_1, \dots, w_m$ of Y such that $u_i = v_i + w_i$ and v_i is orthogonal to w_i for $i = 1, \dots, m$, whereas u_j is a division tripotent of Y whenever $j = m + 1, \dots, n$. Now it is easily seen that $\{v_1, \dots, v_m, w_1, \dots, w_m, u_{m+1}, \dots, u_n\}$ is a frame in Y , and hence the complex JB^* -triple Y is of finite rank. By [19, Theorem 4.10], Y is a finite ℓ_∞ -sum of simple triple ideals which are either finite-dimensional, infinite-dimensional complex spin factors, or of the form $\mathcal{L}(H, K)$ for complex Hilbert spaces H, K with $\dim(H) = \infty$ and $\dim(K) < \infty$. It follows that the Banach space of Y (and hence that of X) is isomorphic to a Hilbert space.

To conclude the proof it is enough to show that X is of finite rank whenever the Banach space of X is isomorphic to a Hilbert space. But, in view of the implication (4) \Rightarrow (3) in Theorem 2.3, we are reduced to the case that X is either finite-dimensional, an infinite-dimensional generalized real spin factor, or of the form $\mathcal{L}(H, K)$ for some real, complex, or quaternionic Hilbert spaces H, K with $\dim(H) = \infty$ and $\dim(K) < \infty$. In all these cases, we realize that X is of finite rank by a direct inspection (see [20, Table 1 and Proposition 5.8.(ii)]). \square

Let X be a Jordan triple over a field \mathbb{F} of characteristic different from 2 and 3. We say that X is **algebraic** if all single-generated subtriples of X are finite-dimensional over \mathbb{F} . If in fact there exists $m \in \mathbb{N}$ such that all single-generated subtriples of X have dimension $\leq m$, then we say that X is of **bounded degree**, and the minimum such an m will be called the **degree** of X . We define inductively the odd powers of an element $x \in X$ by $x^1 := x$ and $x^{2n+1} := \{x^{2n-1}xx\}$.

Lemma 3.6. *Let X be a Jordan triple of degree m over a field \mathbb{F} of characteristic different from 2 and 3 containing at least $2m + 3$ elements. Then every family of nonzero pairwise orthogonal positive tripotents of X has at most m elements.*

Proof. Assume on the contrary that we can find nonzero pairwise orthogonal positive tripotents u_1, \dots, u_{m+1} in X . Since \mathbb{F} has at least $2m + 3$ elements, we can also find nonzero elements a_1, \dots, a_{m+1} in \mathbb{F} such that a_1^2, \dots, a_{m+1}^2 are pairwise

different. Put $x := \sum_{i=1}^{m+1} a_i u_i$. Since for $j = 1, \dots, m+1$ we have $x^{2j-1} = \sum_{i=1}^{m+1} a_i^{2j-1} u_i$, and the matrix $(a_i^{2j-1})_{i,j=1,\dots,m+1}$ is invertible, the subtriple of X generated by x contains $\{u_1, \dots, u_{m+1}\}$, and hence is of dimension $\geq m+1$. This contradicts that X is of degree m . \square

We note that Jordan triples are “power-associative”. Indeed, for an element x in a Jordan triple X , we have $\{x^{2i-1} x^{2j-1} x^{2k-1}\} = x^{2(i+j+k)-3}$ for all $i, j, k \in \mathbb{N}$, and therefore the subtriple of X generated by x is equal to the linear hull of the set of all odd powers of x . Keeping in mind this fact, the proof of [9, Theorem 1], originally made in the setting of complete normed power-associative algebras, works almost verbatim in the setting of complete normed Jordan triples. Thus we have the following lemma.

Lemma 3.7. *Let X be an algebraic Jordan triple over \mathbb{R} endowed with a complete norm making the triple product continuous. Then X is of bounded degree.*

Now we can prove the concluding main result of the paper.

Theorem 3.8. *Let X be a real JB^* -triple. Then the Banach space of X is isomorphic to a Hilbert space if and only if X is algebraic.*

Proof. Assume that X is algebraic. By Lemma 3.7, X is of bounded degree. Let m denote the degree of X . We claim that X^{**} is also of bounded degree (equal to m). To prove the claim we invoke the so-called strong* (in short s^*) topology of X^{**} [26, Section 4]. The strong* topology of X^{**} is a locally convex topology on X^{**} compatible with the duality (X^{**}, X^*) [26, Corollary 9], and makes the triple product of X^{**} jointly continuous on bounded subsets of X^{**} [26, Theorem 9]. Let x be in $B_{X^{**}}$. Since B_X is w^* -dense in $B_{X^{**}}$, the compatibility of s^* with the duality (X^{**}, X^*) gives that B_X is also s^* -dense in $B_{X^{**}}$, and hence there is a net $\{x_\lambda\}_{\lambda \in \Lambda}$ in B_X s^* -convergent to x . Since X is of degree m , for $\lambda \in \Lambda$ there exist $a_{0\lambda}, a_{1\lambda}, \dots, a_{m\lambda} \in \mathbb{R}$ such that $|a_{0\lambda}| + |a_{1\lambda}| + \dots + |a_{m\lambda}| = 1$ and $a_{0\lambda} x_\lambda + a_{1\lambda} x_\lambda^3 + \dots + a_{m\lambda} x_\lambda^{2m+1} = 0$. Taking a cluster point (a_0, a_1, \dots, a_m) of the net $\{(a_{0\lambda}, a_{1\lambda}, \dots, a_{m\lambda})\}_{\lambda \in \Lambda}$ in \mathbb{R}^{m+1} , and applying the joint s^* -continuity of the triple product of X^{**} on bounded sets, we obtain $|a_0| + |a_1| + \dots + |a_m| = 1$ and $a_0 x + a_1 x^3 + \dots + a_m x^{2m+1} = 0$. Therefore, the subtriple of X^{**} generated by x has dimension $\leq m$. Since x is arbitrary in $B_{X^{**}}$, and X^{**} contains X , and X is of degree m , we deduce that X^{**} is also of degree m . Now that the claim is proved, to show that the Banach space of X is isomorphic to a Hilbert space we can assume that X is in fact a JBW^* -triple of degree m . Then, invoking the Krein-Milman theorem, and applying [15, Lemma 3.3], we are provided with a tripotent u in X such that $X_0(u) = 0$. Let u_1, \dots, u_k be nonzero pairwise orthogonal tripotents of X such that $u = u_1 + \dots + u_k$. According to Lemma 3.6, we must have $k \leq m$. Therefore, we may choose the family $\{u_1, \dots, u_k\}$ above of maximum length, and then each u_i becomes a minimal (equivalently, division) tripotent of X (see the

comments after Corollary 3.4). Now, since $u = u_1 + \dots + u_k$ and $X_0(u) = 0$, we have $\bigcap_{i=1}^k X_0(u_i) = 0$, so $\{u_1, \dots, u_k\}$ is a frame in X , and so X is of finite rank. By Theorem 3.1, the Banach space of X is isomorphic to a Hilbert space.

Now assume that the Banach space of X is isomorphic to a Hilbert space. In view of the implication (4) \Rightarrow (3) of Theorem 2.3, to show that X is algebraic we can additionally assume that X is either a generalized real spin factor or of the form $\mathcal{L}(H, K)$ for some real, complex, or quaternionic Hilbert spaces H, K with $\dim(K) < \infty$. Then we realize that X is algebraic by a direct inspection. \square

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