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Transitivity of the norm on Banach spaces having a Jordan structure

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Abstract. We study transitivity conditions on the norm of JB^*-triples, C^*-algebras, JB-algebras, and their preduals. We show that, for the predual X of a JBW^*-triple, each one of the following conditions i) and ii) implies that X is a Hilbert space. i) The closed unit ball of X has some extreme point and the norm of X is convex transitive. ii) The set of all extreme points of the closed unit ball of X is non rare in the unit sphere of X. These results are applied to obtain partial affirmative answers to the open problem whether every JB^*-triple with transitive norm is a Hilbert space. We extend to arbitrary C^*-algebras previously known characterizations of transitivity [20] and convex transitivity [36] of the norm on commutative C^*-algebras. Moreover, we prove that the Calkin algebra has convex transitive norm. We also prove that, if X is a JB-algebra, and if either the norm of X is convex transitive or X has a predual with convex transitive norm, then X is associative. As a consequence, a JB-algebra with almost transitive norm is isomorphic to the field of real numbers.

1. Introduction

Throughout this paper X will denote a Banach space, $S = S(X)$ and $B = B(X)$ will be the unit sphere and the closed unit ball of X, respectively, and $\mathcal{G} = \mathcal{G}(X)$ will stand for the group of all surjective linear isometries on X. We recall that the norm of X is said to be transitive if, for every x, y in S there exists F in $\mathcal{G}$ satisfying $F(x) = y$. The famous Banach-Mazur "rotation" problem [1] is the following.

Problem 1.1. If X is separable, and if the norm of X is transitive, is X a Hilbert space?

Examples of non-Hilbert non-separable Banach spaces with transitive norm are known [31]. In fact, it follows from some constructive methods in [27] and [19; Remark, p. 479] (see also [8]) that every Banach space can be isometrically embedded into a Banach space with transitive norm. On the other hand, it is worth to mention that Problem 1.1 has an affirmative answer if the assumption of separability of X is strengthened to the one that X is finite-dimensional [31]. In this case, the

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answer remains affirmative if the requirement of transitivity of the norm of X is relaxed to that of almost transitivity or even convex transitivity (precise definitions of these two concepts will be given in Sections 2 and 3, respectively). The reader is referred to the book of S. Rolewicz [31] and the recent survey paper of F. Cabello [9] for a comprehensive view of known results and fundamental questions related to the Banach–Mazur rotation problem.

A big part of the literature dealing with transitivity conditions of the norm centers its attention in the study of such conditions on the Banach spaces $\mathcal{C}_0^{\mathbb{K}}(L)$ (of all continuous $\mathbb{K}$-valued functions which vanish at infinity on the locally compact Hausdorff topological space L) and $L_1^{\mathbb{K}}(\Gamma, \mu)$ (of all μ-integrable $\mathbb{K}$-valued functions on the localizable measure space (Γ, μ)). Here $\mathbb{K}$ denotes either $\mathbb{R}$ or $\mathbb{C}$. Today such classical Banach spaces have a wider understanding in the setting of C^*-algebras (or even their non associative generalisations, the JB^*-triples) and JB-algebras. Indeed, the $\mathcal{C}_0^{\mathbb{C}}(L)$-spaces are nothing but the commutative C^*-algebras, and the $L_1^{\mathbb{C}}(\Gamma, \mu)$-spaces are precisely the preduals of commutative W^*-algebras. Analogously, the $\mathcal{C}_0^{\mathbb{R}}(L)$-spaces and the $L_1^{\mathbb{R}}(\Gamma, \mu)$-spaces coincide with the associative JB-algebras and the preduals of associative JBW-algebras, respectively.

Motivated by the ideas in the above comment, we study in this paper transitivity conditions on the norm of JB^*-triples, JB-algebras, and their preduals. Sometimes, in the wider setting we are considering, questions and results attain a better formulation. For instance, the Wood conjecture [36] that L is a singleton whenever $\mathcal{C}_0^{\mathbb{C}}(L)$ has transitive norm becomes a particular case of the more ambitious one that complex Hilbert spaces are the unique JB^*-triples with transitive norm (Problem 2.1), and the result in [20] that L is a singleton whenever $\mathcal{C}_0^{\mathbb{R}}(L)$ has transitive norm follows from the more general fact that $\mathbb{R}$ is the unique JB-algebra whose norm is transitive (Corollary 5.4). The remaining part of the paper flows between Problem 2.1 and Corollary 5.4 just mentioned.

Sections 2 and 3 deal with transitivity conditions on the norm of JB^*-triples and preduals of JBW^*-triples. It seems that the first work in this field is the one of S. K. Tarasov [35], where it is shown that the Banach–Mazur rotation problem has an affirmative answer in the class of JB^*-triples. We rediscover this result, and prove that Problem 1.1 also answers affirmatively in the class of preduals of JBW^*-triples (Corollary 2.5). We also prove that, if X is the predual of a JBW^*-triple, and if either the set of all extreme points of B is non rare in S or B has some extreme point and the norm of X is convex transitive, then X is a Hilbert space (Theorems 3.2 and 3.1). These results allow us to improve Tarasov's theorem, by showing that Problem 1.1 has an affirmative answer in the class of non associative generalisations of complex L_1-preduals (namely, the class of Banach spaces whose duals are preduals of JBW^*-triples).

Of course, the results for JB^*-triples reviewed above apply to C^*-algebras, with the added value that $\mathbb{C}$ is the unique C^*-algebra which is also a Hilbert space. Nevertheless, C^*-algebras have their own philosophy (consisting mainly in their order structure), and, with that philosophy, transitivity conditions on the norm get specially nice formulations. We devote Sect. 4 of the paper to this matter. We obtain characterizations of transitivity and convex transitivity of the norm of a C^*-algebra which extend previously known ones in [20] and [36], respectively, for the commutative case. Moreover, we prove that the norm of the Calkin algebra is convex transitive, thus providing the first known example of a non commutative C^*-algebra whose norm is convex transitive.

Finally, in Sect. 5 we show that, if X is a JB-algebra, and if either the norm of X is convex transitive or X has a predual with convex transitive norm, then X is associative (Theorem 5.3 and Proposition 5.2). Then the result pointed out above that $\mathbb{R}$ is the unique JB-algebra with transitive norm follows from [20].

2. Transitivity conditions on the norm of JB*-triples: some first observations and comments

We recall that a complex Banach space X is said to be a JB^*-triple if it is equipped with a continuous triple product $\{...\}$ which is conjugate-linear in the middle variable, linear and symmetric in the outer variables, and satisfies the following two conditions.

i) $D(a,b)D(x,y) - D(x,y)D(a,b) = D(D(a,b)(x),y) - D(x,D(b,a)(y))$ for all a,b,x,y in X, where the operator $D(a,b) : X \to X$ is defined by $D(a,b)(x) := \{abx\}$ for all x in X.
ii) For every x in X, $D(x,x)$ is hermitian with non negative spectrum and satisfies $\| D(x,x) \| = \| x \|^2$.

JB^*-triples, introduced by W. Kaup [24], are of capital importance in complex Analysis because their open unit balls are bounded symmetric domains, and every bounded symmetric domain in a complex Banach space is biholomorphically equivalent to the open unit ball of a suitable JB^*-triple [25]. Every complex Hilbert space is a JB^*-triple under the triple product defined by $\{xyz\} := \frac{1}{2}((x \mid y)z + (z \mid y)x)$. Now, it seems reasonable to raise the following problem.

Problem 2.1. If X is a JB^*-triple, and if the norm of X is transitive, is X a Hilbert space?

A JBW^*-triple is a JB^*-triple having a (complete) predual. Such a predual is unique [2] in the strongest sense of the word: two preduals of a JBW^*-triple X coincide when they are canonically regarded as subspaces of the dual X^* of X. JBW^*-triples are very abundant: the bidual of every JB^*-triple X is a JBW^*-triple under a suitable triple product which extends the one of X [14]. The fact that every complex Hilbert space is the predual of a JBW^*-triple could invite us to consider the following question.

Question 2.2. If X is the predual of a JBW^*-triple, and if the norm of X is transitive, is X a Hilbert space?

Contrarily to what happens in relation to Problem 2.1 (which, as far as we know, remains unanswered), it is known that, without additional assumptions, the answer to Question 2.2 can be negative. To explain our assertion by an example, let us recall that every C^*-algebra is a JB^*-triple under the triple product $\{xyz\} :=$

$\frac{1}{2}(xy^*z + zy^*x)$. As a consequence, the classical Banach spaces $\mathcal{C}_0(L)$ (of all continuous complex-valued functions which vanish at infinity on a locally compact Hausdorff topological space L) and $L_\infty(\Gamma, \mu)$ (of all essentially bounded locally μ-measurable complex-valued functions on a localizable measure space (Γ, μ)) are JB^*-triples and JBW^*-triples, respectively, in a natural way. The easiest known counter-example to Question 2.2 is the following (see [31; Proposition 9.6.7] and [19]). Let Γ be the disjoint union of an uncountable family of copies of the closed real interval [0, 1], and let μ be the measure on Γ whose measurable sets are those subsets A of Γ whose intersection with each such copy is measurable relative to the Lebesgue measure, with $\mu(A)$ equal to the sum of the measures of that intersections. Then the Banach space $X := L_1(\Gamma, \mu)$ is the predual of a JBW^*-triple, is not a Hilbert space, and has transitive norm.

The following lemma becomes a common tool to provide partial affirmative answers to Problem 2.1 and Question 2.2.

Lemma 2.3. *Let X be a JB^*-triple such that for all x in X the equality $\{xxx\} = \| x \|^2 x$ holds. Then X is a Hilbert space.*

Proof. It is enough to show that the square of the norm of X is a real-quadratic mapping, a fact that is shown to be true by arguing as in the proof of [30; Lemma 1]. □

Recall that the Banach space X is said to be smooth at a point e of S if there is a unique f in $S(X^*)$ satisfying $f(e) = 1$, and that X is called smooth if it is smooth at every point of S. Note that, if L is a locally compact Hausdorff topological space, and if $\mathcal{C}_0(L)$ is smooth, then L is a singleton (otherwise, by Uryson's lemma, $\mathcal{C}_0(L)$ would contain an isometric copy of the non-smooth complex Banach space ℓ_∞^2). Now let X be a JB^*-triple. Since, for e in S, the smallest closed subtriple of X containing e is isometrically isomorphic to a JB^*-triple of the form $\mathcal{C}_0(L)$ for some L as above [24], it follows that, if X is smooth, then every element e in S is a tripotent (i.e., $\{eee\} = e$), and therefore the equality $\{xxx\} = \| x \|^2 x$ is true for every x in X. In this way, by applying Lemma 2.3, we re-encounter the known result [35] that complex Hilbert spaces are nothing but smooth JB^*-triples. Then, as noticed also in [35], Mazur's theorem on the abundance of smooth points in every separable Banach space (see for instance [31; Proposition 9.4.3]) implies that separable JB^*-triples with transitive norm are Hilbert spaces (a joint partial affirmative answer to Problems 1.1 and 2.1). These results in [35] will be improved in Sect. 3 (see Corollaries 3.5 and 3.6).

Now assume that X is the predual of a JBW^*-triple. Given an element e in S, among the elements f in $S(X^*)$ satisfying $f(e) = 1$ we can find tripotents of X^* (for instance, the so-called support of e [18; p. 75]). It follows that, if X is smooth, then every element in $S(X^*)$ which attains its norm is a tripotent. Since the norm attaining elements of $S(X^*)$ are dense in $S(X^*)$ (by the Bishop-Phelps theorem [4; p. 7]), and the set of all tripotents of X^* is closed in X^*, we actually have that, if X is smooth, then every element in $S(X^*)$ is a tripotent. Now Lemma 2.3 gives us the following geometric characterization of complex Hilbert spaces among the preduals of JBW^*-triples.

Proposition 2.4. *Let X be the predual of a JBW^*-triple. If X is smooth, then X is a Hilbert space.*

Proposition 2.4 provides us with the following joint partial affirmative answer to Problem 1.1 and Question 2.2.

Corollary 2.5. *Let X be the predual of a JBW^*-triple. If X is separable, and if the norm of X is transitive, then X is a Hilbert space.*

Recall that the norm of the Banach space X is called almost transitive if there exists a dense subset D of S such that, for every x, y in D, we can find F in $\mathcal{G}$ satisfying $F(x) = y$. It is well-known and easy to see that the norm of X is almost-transitive if and only if, for every e in S the orbit $\mathcal{G}(e) := \{F(e) \ : \ F \in \mathcal{G}\}$ is dense in S. Now assume that X is a JB^*-triple. Since elements of $\mathcal{G}$ preserve the triple product of X [24; Proposition 5.4], it follows from Lemma 2.3 that, if X has a non-zero tripotent, and if the norm of X is almost transitive, then X is a Hilbert space. As a consequence, we have the following affirmative answer to a natural variant of Problem 2.1.

Corollary 2.6. *Let X be a JBW^*-triple with almost transitive norm. Then X is a Hilbert space.*

As happens in relation to every mathematical problem which seems to be difficult to answer, it would be convenient to provide us with some non trivial reformulations of Problem 2.1. We will obtain such reformulations as a consequence of the following theorem of F. Cabello. Given a subcategory $\mathcal{J}$ of Banach spaces (see [33; p.161, Definition 9.13]), a $\mathcal{J}$-space will mean an object of $\mathcal{J}$, and a $\mathcal{J}$-subspace of a $\mathcal{J}$-space X will be a closed subspace Y of X which is a $\mathcal{J}$-space such that the inclusion $Y \hookrightarrow X$ is a $\mathcal{J}$-morphism.

Theorem 2.7. *[8] Let $\mathcal{J}$ be a subcategory of Banach spaces satisfying the following two conditions:*

a) *Given a $\mathcal{J}$-space X and a separable subspace Z of X, there is a $\mathcal{J}$-subspace of X which is separable and contains Z.*
b) *Given a $\mathcal{J}$-space X and an increasing sequence $\{Y_n\}$ of $\mathcal{J}$-subspaces of X, the closure of $\cup_{n\in\mathbb{N}} Y_n$ in X is a $\mathcal{J}$-space.*

Then there exists a non-Hilbert separable $\mathcal{J}$-space with almost transitive norm whenever there is some non-Hilbert $\mathcal{J}$-space with transitive norm.

The fact pointed out above that one-generated closed subtriples of a JB^*-triple are JB^*-triples leads easily to see that all closed subtriples of a JB^*-triple also are JB^*-triples. This well-known result is the key tool in verifying that, if $\mathcal{J}$ denotes the category of JB^*-triples, then $\mathcal{J}$ satisfies conditions a) and b) in Theorem 2.7. On the other hand, the class of JB^*-triples is closed under ultraproducts [14], and it is folklore that, if a Banach space X has almost transitive norm, then every non trivial (Banach) ultraproduct of X has transitive norm (see for instance [19; Remark, p. 479]). Therefore we have

Proposition 2.8. *The following assertions are equivalent:*

i) *Every JB^*-triple with transitive norm is a Hilbert space.*
ii) *Every JB^*-triple with almost transitive norm is a Hilbert space.*
iii) *Every separable JB^*-triple with almost transitive norm is a Hilbert space.*

3. Transitivity conditions on the norm of JB*-triples: the main results

In this section we will provide affirmative answers to natural variants of Question 2.2: we will assume that the Banach space X in that question is in fact the predual of an "atomic" JBW^*-triple, but the requirement that the norm of X is transitive will be substantially relaxed. Recall that the norm of the Banach space X is called convex transitive if for every e in S we have $\overline{co}\mathcal{G}(e) = B$, where $\overline{co}$ means closed convex hull.

Theorem 3.1. *Let X be the predual of a JBW^*-triple. Assume that B has some extreme point, and that the norm of X is convex transitive. Then X is a Hilbert space.*

Proof. The convex transitivity of the norm of X and the existence of extreme points of B imply that X is the closed linear hull of the set of all extreme points in B, i.e., the JBW^*-triple X^* is (purely) atomic. Then, by [18; Lemma 2.11] there exists a contractive conjugate-linear mapping $\pi : X \to X^*$ whose value at each extreme point e of B is the support $s(e)$ of e. From the obvious uniqueness of such a mapping π it follows that, for F in $\mathcal{G}$, we have $\pi \circ F = (F^*)^{-1} \circ \pi$. As a consequence, the equivalent norm $|\!|\!|.|\!|\!|$ on X defined by $|\!|\!|x|\!|\!| := \| x \| + \| \pi(x) \|$ satisfies $|\!|\!|F(x)|\!|\!| = |\!|\!|x|\!|\!|$ for all x in X and F in $\mathcal{G}$. Since the norm of X is convex transitive, it follows from [12; Theorem 5] that $|\!|\!|.|\!|\!|$ is a positive multiple of $\| \, . \, \|$ on X, and therefore we have $\| x \| = \| \pi(x) \|$ for all x in X. On the other hand, by [18; Remark 2.8 and the proof of Theorem 1], for every x in X there exist (possibly finite) sequences $\{\lambda_n\}$ of positive numbers and $\{e_n\}$ of extreme points of B such that $\| x \| = \sum_n \lambda_n$, the tripotents $s(e_n)$ are pair-wise orthogonal (i.e., $D(s(e_n), s(e_m)) = 0$ whenever $n \neq m$), and $x = \sum_n \lambda_n e_n$ (conditions which imply $\| \pi(x) \| = Max_n\{\lambda_n\}$). Since we proved that π is an isometry, it follows that every element in S must be an extreme point of B. Therefore we have $\{\pi(x)\pi(x)\pi(x)\} = \| x \|^2 \pi(x)$ for all x in X, so $\pi(X)$ is a JB^*-subtriple of X^* (by polarization law [7; p. 251]), and Lemma 2.3 applies. □

It follows from Theorem 3.1 that, if X is a JB^*-triple, and if the norm of X^* is convex transitive, then X is a Hilbert space. Recall that a subset R of a topological space T is said to be rare in T if the interior of the closure of R in T is empty.

Theorem 3.2. *Let X be the predual of a JBW^*-triple. Assume that there exists some non rare set in S consisting only of extreme points of B. Then X is a Hilbert space.*

Proof. The assumption on X implies that the JBW^*-triple X^* is atomic. Let π be the contractive conjugate-linear mapping from X to X^* introduced in the proof of Theorem 3.1. As we have seen there, given e in S, e is an extreme point of B if (and only if) $\| \pi(e) \|= 1$. Therefore the set U of all extreme points of B is closed in S. Using again the assumption on X, it follows that there exists e in S and $0 < \varepsilon < 1$ such that x lies in U whenever x is in S and $\| x - e \|< \varepsilon$. Let x be in S with $\| x - e \|< \varepsilon$. Then $\pi(x)$ and $\pi(e)$ are non orthogonal (since $\| \pi(x) - \pi(e) \|< 1$) minimal tripotents in X^* (since x and e are extreme points of B and [18; Proposition 4] applies). Therefore, by [18; Corollary 2.5 and Lemma 1.1], there exists G in $\mathcal{G}(X^*)$ satisfying $G(\pi(x)) = \pi(e)$. Since elements in $\mathcal{G}(X^*)$ are w^*-continuous (a consequence of the uniqueness of the predual of X^*), we have $G = F^*$ for some F in $\mathcal{G}$, so

$$\pi(e) = G(\pi(x)) = F^*(\pi(x)) = \pi(F^{-1}(x)),$$

and so $x = F(e)$ (since π is injective). In this way we have shown that $\mathcal{G}(e)$ contains the set $\{x \in S \ : \ \| x - e \|< \varepsilon\}$. Now let x be an arbitrary element in S. We can find a finite sequence $x_0, x_1, ..., x_n$ in S with $x_0 = e, x_n = x$, and $\| x_i - x_{i-1} \|< \varepsilon$ for all $i = 1, ..., n$. Put $k := \max\{i \in \{1, ..., n\} \ : \ x_i \in \mathcal{G}(e)\}$. If x does not belong to $\mathcal{G}(e)$, then we have $k < n$, there exists F in $\mathcal{G}$ with $F(x_k) = e$, so $\| F(x_{k+1}) - e \|< \varepsilon$, and so x_{k+1} belongs to $\mathcal{G}(e)$, a contradiction. Therefore $S = \mathcal{G}(e)$, hence the norm of X is transitive, and Theorem 3.1 applies. □

Either from Theorem 3.1 or Theorem 3.2 it follows that, if X is the predual of a JBW^*-triple, if the norm of X is almost transitive, and if B has extreme points, then X is a Hilbert space. We conclude this section with some corollaries to Theorems 3.1 and 3.2. The first one is a direct consequence of Theorem 3.1 and the following lemma.

Lemma 3.3. *Let X be the predual of a JBW^*-triple. If X^* has convex transitive norm, then X has convex transitive norm too.*

Proof. As observed in [12; Lemma 4], a Banach space E has convex transitive norm if and only if, for every e in $S(E)$ and f in $S(E^*)$, we have

$$\sup\{| f(F(e)) | \ : \ F \in \mathcal{G}(E)\} = 1.$$

Now, the convex transitivity of the norm of X^* implies

$$\sup\{| G(f)(e) |: G \in \mathcal{G}(X^*)\} = 1$$

for every e in $S(X)$ and f in $S(X^*)$. Since elements in $\mathcal{G}(X^*)$ are nothing but those of the form F^* for some F in $\mathcal{G}$, we obtain

$$\sup\{| f(F(e)) |: F \in \mathcal{G}\} = 1$$

for every e in $S(X)$ and f in $S(X^*)$, hence the norm of X is convex transitive. □

Corollary 3.4. *Let X be an atomic JBW^*-triple with convex transitive norm. Then X is a Hilbert space.*

As a consequence, if X is a complex Banach space, and if X^{**} is a JB^*-triple with convex transitive norm, then X is a Hilbert space. The fact that $L_\infty([0,1])$ has convex transitive norm [36] shows that neither the assumption of almost transitivity of the norm of X in Corollary 2.6 can be relaxed to that of convex transitivity nor the assumption that X is atomic in Corollary 3.4 can be removed.

Corollary 3.5. *Assume that the Banach space X is smooth and that X^{**} is a JB^*-triple. Then X is a Hilbert space.*

Proof. The assumption that X is smooth implies that every element in $S(X^*)$ which attains its norm is an extreme point of $B(X^*)$. By the Bishop-Phelps theorem and Theorem 3.2, X^* is a Hilbert space. □

Corollary 3.6. *Assume that the Banach space X is separable, that the norm of X is transitive, and that X^{**} is a JB^*-triple. Then X is a Hilbert space.*

As commented at the beginning of Sect. 2, Corollaries 3.5 and 3.6 above extend the results proved in [35] for JB^*-triples to the more general setting of complex Banach spaces whose biduals are JB^*-triples. In Corollary 3.9 below we will provide further information about the transitivity of the norm on such spaces. For the moment, let X be an arbitrary Banach space. For e in X, we put $\rho(X,e) := \max\{\rho \geq 0 : \rho B \subseteq \overline{co}\mathcal{G}(e)\}$.

Lemma 3.7. *The function $\rho(X,.)$ is continuous on X. More precisely, for u and v in X, we have $|\ \rho(X,u) - \rho(X,v)\ | \leq \|\ u-v\ \|$.*

Proof. Let u, v be in X. For f in $S(X^*)$, we have

$$\rho(X,u) \leq \sup\{Re[f(F(u))] : F \in \mathcal{G}\} \leq \|\ u-v\ \| + \sup\{Re[f(F(v))] : F \in \mathcal{G}\},$$

and hence

$$\rho(X,u) \leq \|\ u-v\ \| + \inf\{\ \sup\{Re[f(F(v))] : F \in \mathcal{G}\}\ :\ f \in S(X^*)\ \}.$$

But, by the Hahn–Banach separation theorem, the equality

$$\rho(X,v) = \inf\{\sup\{Re[f(F(v))] : F \in \mathcal{G}\} : f \in S(X^*)\}$$

holds. It follows $|\ \rho(X,u) - \rho(X,v)\ | \leq \|\ u-v\ \|$. □

Proposition 3.8. *Assume that the norm of X is transitive, and that every element in $B(X^{**})$ is the w^*-limit of a sequence of elements of B. Then the norm of X^* is convex transitive.*

Proof. Let f be an element in $S(X^*)$ which attains its norm. By the transitivity of the norm of X, for every x in S, there exists g in the set $\{F^*(f) : F \in \mathcal{G}\}$ such that g attains its norm at x. Now, the remaining assumption on X, together with [13; Lemma I.5.10], leads to $\overline{co}\{F^*(f) : F \in \mathcal{G}\} = B(X^*)$, so $\overline{co}\mathcal{G}(X^*)(f) = B(X^*)$, and so $\rho(X^*, f) = 1$. By the Bishop-Phelps theorem and Lemma 3.7, we actually have $\rho(X^*, g) = 1$ for every g in $S(X^*)$, i.e., the norm of X^* is convex transitive.
□

Corollary 3.9. *Assume that the norm of X is transitive, that X^{**} is a JB^*-triple, and that every element in $B(X^{**})$ is the w^*-limit of a sequence of elements of B. Then X is a Hilbert space.*

Proof. Apply Proposition 3.8 and Theorem 3.1. □

Banach spaces whose biduals are JB^*-triples have been systematically studied in [17] (see also [11]). However, we have not found in the literature any example showing that the enlargement of the class of JB^*-triples provided by such spaces is strict. In what follows we give such an example.

Example 3.10. Let Y be the C^*-algebra of all compact operators on an infinite-dimensional complex Hilbert space, so that Y is an M-embedded Banach space [22; Example III.1.4.(f)] in the sense of [22; Definition III.1.1.(a)]. By [22; Proposition III.2.10.(b)], there exists a complex Banach space X and a surjective linear isometry $F : X^* \to Y^*$ which is not the transpose of a linear isometry from Y onto X. We claim that X cannot be linearly isometric to Y. Indeed, if X is linearly isometric to Y, then X is an M-embedded Banach space, and we can argue as in the proof of [22; Proposition III.2.2] to obtain that $F^* = G^{**}$ for some linear isometry G from Y onto X, and hence $F = G^*$, a contradiction. Now, the situation is that Y is a JB^*-triple, Y^{**} is a Cartan factor, X^{**} is linearly isometric to Y^{**}, but X is not linearly isometric to Y. It follows from [6; Lemma 3.2] that X cannot be a JB^*-triple.

The argument in the above example actually shows that, for every non reflexive Cartan factor Z, there exists a complex Banach space X which is not a JB^*-triple and satisfies $X^{**} = Z$. Given a non negative integer number n, we could consider the class $\mathcal{J}_n$ of complex Banach spaces whose n-th dual is a JB^*-triple, obtaining in such a way increasing sequences $\{\mathcal{J}_{2p-2}\}_{p\geq 1}$ and $\{\mathcal{J}_{2p-1}\}_{p\geq 1}$ of classes of Banach spaces whose first terms are the one of JB^*-triples and that of preduals of JBW^*-triples, respectively. However, since for every Banach space X, X^* is the range of a contractive projection on X^{***}, and the class of JB^*-triples is closed by passing to ranges of contractive projections ([26], [34]), it follows from Example 3.7 that the actual situation is the following:

$$\mathcal{J}_0 \subset \mathcal{J}_2 = \mathcal{J}_4 = ... = \mathcal{J}_{2p} = ... \text{ and } \mathcal{J}_1 = \mathcal{J}_3 = \ldots = \mathcal{J}_{2p-1} = ... $$

Therefore, as we have done along this paper, among the classes $\mathcal{J}_n$, only $\mathcal{J}_0$, $\mathcal{J}_1$, and $\mathcal{J}_2$ deserve to be considered.

4. Transitivity conditions on the norm of C*-algebras

The results obtained in Sections 2 and 3 for JB^*-triples automatically get a stronger form when they are applied to C^*-algebras. The reason lies in the folklore fact that $\mathbb{C}$ is the unique C^*-algebra whose C^*-norm derives from an inner product. (Indeed, from the continuous functional calculus for a single self-adjoint element of a C^*-algebra, it follows that, if X is a smooth C^*-algebra, then every norm-one element e in the self-adjoint part X_{sa} of X satisfies either $e^2 = e$ or $e^2 = -e$, which implies that $S(X_{sa})$ is disconnected, and hence the real Banach space X_{sa} is one-dimensional.) By the folklore result just mentioned, an affirmative answer to Problem 2.1 would imply the verification of Wood's conjecture [36] that, if L is a locally compact Hausdorff topological space such that $\mathcal{C}_0(L)$ has transitive norm, then L is a singleton. Actually, if Problem 2.1 had an affirmative answer, then the natural conjecture that $\mathbb{C}$ is the unique (non necessarily commutative) C^*-algebra with transitive norm would be right. We note also that the category $\mathcal{J}$ of C^*-algebras is closed under ultraproducts and satisfies conditions a) and b) in Theorem 2.7.

Let us say that a C^*-algebra is proper whenever it is different from $\mathbb{C}$. It follows from the above comments that the existence of a proper C^*-algebra with transitive norm is equivalent to the existence of a proper C^*-algebra with almost transitive norm, and implies the existence of a separable proper C^*-algebra with almost transitive norm. Accordingly to previous comments in Sect. 2, a proper C^*-algebra with transitive norm must be non separable, and a proper C^*-algebra with almost transitive norm cannot have non-zero self-adjoint idempotents. In the next proposition we characterize the transitivity of the norm of a C^*-algebra in purely algebraic terms. Such a characterization will follow from the Kadison-Paterson-Sinclair determination of surjective linear isometries on C^*-algebras [28], and becomes the non-commutative generalization of [20; Proposition 4.2]. The reader is referred to the books [15], [29], and [32] for basic results in the theory of C^*-algebras.

Let X be a C^*-algebra, and let $M(X)$ denote the C^*-algebra of multipliers of X. The so called Jordan $*$-automorphisms of X, as well as the operators of left multiplication on X by unitary elements in $M(X)$, become distinguished examples of surjective linear isometries on X. Jordan $*$-automorphisms of X are nothing but linear bijections from X to X preserving the C^*-involution and the squares. Consequently, if $Pos(X)$ denotes the set of all positive elements in X, and if F is a Jordan $*$-automorphism of X, then we have $F(S \cap Pos(X)) = S \cap Pos(X)$. Let us denote by U the set of all unitary elements of $M(X)$, and by $\mathcal{G}^+$ the group of all Jordan $*$-automorphisms of X. The Kadison-Paterson-Sinclair theorem asserts that every surjective linear isometry on X is the composition of an element of $\mathcal{G}^+$ with the operator of left multiplication by an element of U. The modulus $\mid x \mid$ of an element x of X is defined as the unique positive square root of x^*x.

Proposition 4.1. *Let X be a C^*-algebra. Then the following assertions are equivalent:*

i) *The norm of X is transitive.*
ii) *$\mathcal{G}^+$ acts transitively on $S \cap Pos(X)$, and every element x in X has a "polar decomposition" $x = u \mid x \mid$, where u is in U.*

Proof. Assume that the norm of X is transitive. Then, for p, q in the set $S \cap Pos(X)$ there exist F in $\mathcal{G}^+$ and v in U such that $q^{1/2} = vF(p^{1/2})$, and hence we have $q = F(p^{1/2})v^*vF(p^{1/2}) = (F(p^{1/2}))^2 = F(p)$. Therefore $\mathcal{G}^+$ acts transitively on $S \cap Pos(X)$. On the other hand, for every x in S we can find G in $\mathcal{G}^+$ and u in U such that $x = uG(x^*x)$, which implies $x^*x = (G(x^*x))^2$, and hence $G(x^*x) = \mid x \mid$. Now assume that assertion ii) holds. For x, y in S, we can write

$x = u \mid x \mid$ and $y = v \mid y \mid$ for suitable elements u, v in U, and there exists F in $\mathcal{G}^+$ such that $F(\mid x \mid) = \mid y \mid$. Then the mapping $G : z \to vF(u^*z)$ from X to X is a surjective linear isometry satisfying $G(x) = y$. $\square$

Now, we pass to provide a characterization of C^*-algebras with convex transitive norm, which extends the one in [36; Theorem 3.3] for the commutative case. Let X be a W^*-algebra. It is well-known that the predual X_* of X is an X-bimodule in a natural way. Indeed, if v belongs to X, and if g is in X_* , then it is enough to define vg and gv as the (automatically w^*-continuous) linear functionals on X given by $(vg)(x) := g(xv)$ and $(gv)(x) := g(vx)$, respectively, for all x in X.

Lemma 4.2. *Let X be a C^*-algebra. Then the set*

$$\{uf \ : \ f \in Pos(X^*) \cap S(X^*), \ u \in U\}$$

is norm-dense in $S(X^*)$.

Proof. Let h be in $S(X^*)$, and let $0 < \varepsilon < 2$. Since $B(M(X))$ is the closed convex hull of U (by the Russo-Dye theorem [4; Theorem 30.2]), there exists v in U such that $\mid 1 - h(v) \mid < \frac{\varepsilon^2}{16}$. By the Bishop-Phelps-Bollobás theorem [4; Theorem 16.1], there are elements x and g in $S(X^{**})$ and $S(X^*)$, respectively, satisfying $\| x - v \| < \frac{\varepsilon}{2}$, $\| g - h \| < \frac{\varepsilon}{2}$, and $g(x) = 1$. Put $u := v^*$ and $f := xg$. Then u belongs to U, f belongs to $Pos(X^*) \cap S(X^*)$ (because, if $\mathbf{1}$ denotes the unit of X^{**}, then $1 = g(x) = (xg)(\mathbf{1}) = f(\mathbf{1}) \leq \| f \| = = \| xg \| \leq \| x \| \| g \| = 1)$, and

$$\| h - uf \| \leq \| h - x^*f \| + \| (x^* - u)f \| = \| h - g \| + \| (x^* - u)f \|$$

$$\leq \| h - g \| + \| x^* - u \| = \| h - g \| + \| x - v \| < \varepsilon. \qquad \square$$

Let X be a C^*-algebra. The extreme points of the w^*-compact convex set $Pos(X^*) \cap B(X^*)$ are zero and the so called (normalized) pure states of X. It is well-known that pure states of X are extreme points of $B(X^*)$.

Theorem 4.3. *Let X be a C^*-algebra. Then X has convex transitive norm if and only if, for every pure state g of X and every norm-one positive linear functional f on X, g belongs to the w^*-closure in X^* of the set $\{F^*(f) : F \in \mathcal{G}^+\}$.*

Proof. Assume that X has convex transitive norm. Then, by the Hahn-Banach theorem, for every φ in $S(X^*)$ the equality

$$B(X^*) = {}^{w^*}\overline{co}\{G^*(\varphi) \ : \ G \in \mathcal{G}\}$$

holds. Let g be a pure state of X, and f a norm-one positive linear functional on X. It follows from [3; Theorem 36.10] that g belongs to the w^*-closure of $\{G^*(f) \ : \ G \in \mathcal{G}\}$. Therefore there exist nets $\{u_\alpha\}$ and $\{F_\alpha\}$ in U and $\mathcal{G}^+$, respectively, such that $\{f(u_\alpha F_\alpha(x))\} \to g(x)$ for all x in X. Let h be a w^*-cluster point in $\mathcal{B}(X^*)$ of the net $\{F_\alpha^*(f)\}$. To prove that g belongs to the w^*-closure of the set $\{F^*(f) \ : \ F \in \mathcal{G}^+\}$ it is enough to show that $h = g$. Let $x = x^*$ be in X. Then, by the Cauchy-Schwarz inequality, we have

$$\mid f(u_\alpha F_\alpha(x)) \mid^2 \leq f(u_\alpha^* u_\alpha) f(F_\alpha(x)^2) = f(\mathbf{1}) f(F_\alpha(x^2)) = F_\alpha^*(f)(x^2) \ ,$$

and hence $g(x)^2 \leq h(x^2)$. Note that this inequality implies $\| h \| = 1$. Let $\{x_\lambda\}$ be an increasing approximate unit for X bounded by one. Then $\{x_\lambda\}$ converges to $\mathbf{1}$ in the w^*-topology of X^{**}, so $0 \leq h((\mathbf{1}-x_\lambda)^2) \leq h(\mathbf{1}-x_\lambda) \to 0$, and so $h(x_\lambda^2) \to 1$. Now, for ρ in $\mathbb{R}$, we have

$$\rho^2 + 2\rho g(x) + g(x)^2 = \lim\{g(\rho x_\lambda + x)^2\}$$

$$\leq \lim\{h((\rho x_\lambda + x)^2)\} = \rho^2 + 2\rho h(x) + h(x^2) \ ,$$

and therefore $2\rho(g(x)-h(x)) \leq h(x^2)-g(x)^2$. Since ρ is arbitrary in $\mathbb{R}$, it follows $g(x) - h(x) = 0$, and hence $h = g$, as required.

Now assume that, for every pure state g of X and every f in the set $Pos(X^*) \cap S(X^*)$, g belongs to the w^*-closure of $\{F^*(f) : F \in \mathcal{G}^+\}$. Let $|\!|\!| . |\!|\!|$ be an equivalent norm on (the Banach space of) X such that $\mathcal{G} \subseteq \mathcal{G}(X, |\!|\!| . |\!|\!|)$. Then, for g and f as above, the dual norm $|\!|\!| . |\!|\!|$ is constant on $\{F^*(f) : F \in \mathcal{G}^+\}$, so that our assumption implies $|\!|\!| g |\!|\!| \leq |\!|\!| f |\!|\!|$. As a consequence, $|\!|\!| . |\!|\!|$ is constant (say equal to M) on the set of pure states of X. Now, for every f in $Pos(X^*) \cap S(X^*)$, the inequality $M \leq |\!|\!| f |\!|\!|$ holds. But the converse inequality is also true because the set $\{h \in X^* : |\!|\!| h |\!|\!| \leq M\}$ is w^*-closed and convex and contains all extreme points of $B(X^*) \cap Pos(X^*)$, and hence, by the Krein-Milman theorem, it also contains $B(X^*) \cap Pos(X^*)$. Since, for u in U, the mapping $G : x \to xu$ from X to X is an element of $\mathcal{G}$, for every f in $Pos(X^*) \cap S(X^*)$ we have $|\!|\!| uf |\!|\!| = |\!|\!| G^*(f) |\!|\!| = |\!|\!| f |\!|\!| = M$. It follows from Lemma 4.2 that the dual norm $|\!|\!| . |\!|\!|$ is constant on $S(X^*)$. Therefore the norm $|\!|\!| . |\!|\!|$ on X is a positive multiple of the original norm. Finally, the convex transitivity of the norm of X follows from the already applied result in [12; Theorem 5]. □

Our concluding goal in this section is to prove that the Calkin algebra [10] has convex transitive norm. As far as we know, this becomes the first known example of a non commutative C^*-algebra whose norm is convex transitive. We recall that the Calkin algebra is defined as the quotient $L(H)/K(H)$, where H is an infinite-dimensional separable complex Hilbert space, $L(H)$ denotes the C^*-algebra of all bounded linear operators on H, and $K(H)$ stands for the closed ideal of $L(H)$ consisting of all compact operators on H.

Lemma 4.4. *Let X be a C^*-algebra, let x be in X, and let y, z be in $B(M(X))$. Then yxz belongs to the closed convex hull of $\mathcal{G}(x)$.*

Proof. The set $\{t \in M(X) : tx \in \overline{co}\mathcal{G}(x)\}$ is closed and convex in $M(X)$, and contains U. By the Russo-Dye theorem, it also contains y. Now the closed convex set $\{t \in M(X) : yxt \in \overline{co}\mathcal{G}(x)\}$ contains U, hence it contains z. □

In what follows H will denote an infinite-dimensional separable complex Hilbert space. For x in $L(H)$, we put $\| x \|_{ess} := \| x + K(H) \|$.

Theorem 4.5. *Let X denote the C^*-algebra $L(H)$, and let x be in S. Then $\overline{co}\mathcal{G}(x) = B$ if and only if $\| x \|_{ess} = 1$.*

Proof. Let π be a self-adjoint idempotent in X whose range is an infinite-dimensional subspace of H. Then there exists u in X satisfying $u^*u = \mathbf{1}$ and $uu^* = \pi$, and hence $\mathbf{1} = u^*\pi u$. By Lemma 4.4, $\mathbf{1}$ belongs to $\overline{co}\mathcal{G}(\pi)$, and, since $B = \overline{co}\mathcal{G}(\mathbf{1})$ (by the Russo-Dye theorem), we actually have $B = \overline{co}\mathcal{G}(\pi)$.

Now, let x be in $S \cap Pos(X)$ such that $\| x \|_{ess} = 1$. Let $0 < \varepsilon < 1$. By the spectral decomposition for x (see for instance [21; Proposition 4.2.3]), there are pairwise orthogonal self-adjoint idempotents $\pi_1, ..., \pi_n$ in X and real numbers $\lambda_1, ..., \lambda_n$ such that $\| x - \sum_{i=1}^n \lambda_i \pi_i \| \leq \varepsilon$. We may assume that there exists a positive integer $k \leq n$ such that $\pi_1, ..., \pi_k$ have infinite-dimensional range, $\pi_{k+1}, ..., \pi_n$ have finite-dimensional range, and $| \lambda_1 | \geq | \lambda_j |$ for all $j = 2, ..., k$. Then we have

$$| \lambda_1 | = \| \sum_{i=1}^{k} \lambda_i \pi_i \| \geq \| x - \sum_{i=k+1}^{n} \lambda_i \pi_i \| - \| x - \sum_{i=1}^{n} \lambda_i \pi_i \|$$

$$\geq \| x \|_{ess} - \varepsilon = 1 - \varepsilon.$$

On the other hand, if we put $v := \sum_{i=1}^n \lambda_i \pi_i$, then we have $\pi_1 v = \lambda_1 \pi_1$, hence, by Lemma 4.4, $\lambda_1 \pi_1$ belongs to $\overline{co}\mathcal{G}(v)$. By the first paragraph in the proof, the inclusion $\lambda_1 B \subseteq \overline{co}\mathcal{G}(v)$ holds. With the notation before Lemma 3.7, this means $\rho(X, v) \geq | \lambda_1 |$. It follows from Lemma 3.7 that

$$\rho(X, x) \geq \rho(X, v) - \| x - v \| \geq | \lambda_1 | - \varepsilon \geq 1 - 2\varepsilon.$$

By letting $\varepsilon \to 0$, we obtain $\rho(X, x) = 1$, i.e., $B = \overline{co}\mathcal{G}(x)$.

Now, let x be in S such that $\| x \|_{ess} = 1$. Then x^*x lies in $S \cap Pos(X)$ and $\| x^*x \|_{ess} = 1$. By the second paragraph in the proof, we have $B = \overline{co}\mathcal{G}(x^*x)$, and, by Lemma 4.4, also $B = \overline{co}\mathcal{G}(x)$. This concludes the proof of the "if" part in the theorem.

To prove the "only if" part, first note that $K(H)$ is a $\mathcal{G}$-invariant subspace of X (see for instance [22; Proposition III.2.2]), so that, if Y denotes the Calkin algebra, and if $P : X \to Y$ is the quotient mapping, then every F in $\mathcal{G}$ gives rise to an element $\hat{F}$ in $\mathcal{G}(Y)$ satisfying $P \circ F = \hat{F} \circ P$. Then it follows easily that, if x is in S, and if $B = \overline{co}\mathcal{G}(x)$, then $B(Y) \subseteq \overline{co}\mathcal{G}(Y)(P(x))$, and therefore $\| x \|_{ess} = \| P(x) \| = 1$. □

Corollary 4.6. *The Calkin algebra has convex transitive norm.*

Proof. Let us take the notation in Theorem 4.5 and its proof. Let y be in $S(Y)$. Since $K(H)$ is proximinal in X [22; Proposition II.1.1], there exists x in S such that $P(x) = y$, and therefore $\| x \|_{ess} = 1$. By Theorem 4.5, for such an x we have $B = \overline{co}\mathcal{G}(x)$. Finally, since $K(H)$ is $\mathcal{G}$-invariant, the equality $B(Y) = \overline{co}\mathcal{G}(Y)(y)$ holds. □

5. Transitivity conditions on the norm of JB-algebras

JB-algebras are defined as those Jordan-Banach real algebras X satisfying $\| x \|^2 \leq \| x^2 + y^2 \|$ for all x, y in X. A natural example of a JB-algebra is the Banach space X of all self-adjoint operators on a complex Hilbert space, when we define the Jordan product $x.y$ of elements x, y in X as $x.y := \frac{1}{2}(xy + yx)$. Other examples are provided by the real Banach algebras $\mathcal{C}_0^{\mathbb{R}}(L)$, with L a locally compact Hausdorff topological space. Actually these last Banach algebras are the unique associative JB-algebras [21; 3.2.2]. JB-algebras are closely related to JB^*-triples. Indeed, if X is a JB-algebra, and if we define a triple product on X by $\{xyz\} := (x.y).z + (y.z).x - (x.z).y$, then $(X, \{...\})$ can be regarded as a closed real subtriple of a suitable JB^*-triple (cf. [21; 3.3.9], [37], and [5]).

Let X be a JB-algebra with a unit $\mathbf{1}$. If u is an element in X satisfying $u^2 = \mathbf{1}$, then we say that u is a symmetry in X. Central symmetries in X are characterized as the isolated points of the set of all extreme points of B [23; Proposition 1.3]. It follows that the orbit $\mathcal{G}(\mathbf{1})$ is contained in the centre of X. Therefore we have

Proposition 5.1. *Let X be a JB-algebra with a unit $\mathbf{1}$. If the linear hull of $\mathcal{G}(\mathbf{1})$ is dense in X (for instance, if the norm of X is convex transitive), then X is associative.*

JBW-algebras (see [21; 4.1.1] for a definition) can actually be characterized as those JB-algebras which are Banach dual spaces [21; 4.4.16]. If X is a JBW-algebra, then X has a unit [21; 4.1.7], and the product of X is separately w^*-continuous [21; 4.4.16 and 4.1.6].

Proposition 5.2. *Let X be the predual of a JBW-algebra. If X has no non trivial $\mathcal{G}$-invariant closed subspaces (for instance, if the norm of X is convex transitive), then X^* is associative.*

Proof. Assume that X^* is not associative. Then, denoting by $\mathbf{1}$ the unit of X^*, the linear hull of $\mathcal{G}(X^*)(\mathbf{1})$ is not w^*-dense in X^*. Therefore there exists a non-zero element x in X such that $(\mathcal{G}(X^*)(\mathbf{1})(x) = 0$. As a consequence, for every F in $\mathcal{G}$ we have $\mathbf{1}(F(x)) = 0$, and hence the closed linear hull of $\mathcal{G}(x)$ is a non trivial $\mathcal{G}$-invariant closed subspace of X. □

Let X be a JB-algebra. Then the bidual X^{**} of X is a JBW-algebra containing X as a subalgebra [21; 4.4.3], and the set

$$M(X) := \{z \in X^{**} \ : \ z.X \subseteq X\}$$

is a subalgebra of X^{**} [16] called the multiplier algebra of X. According to the Kadison type theorem in [23], every surjective linear isometry on X is the composition of an algebra automorphism of X with the operator of multiplication by a central symmetry in $M(X)$. Recall that an element x in X is said to be positive if there exists y in X such that $y^2 = x$.

Theorem 5.3. *Let X be a JB-algebra. If there exists a norm-one positive element p in X such that the convex hull of $\mathcal{G}(p)$ is dense in B (for instance, if the norm of X is convex transitive), then X is associative.*

Proof. Recall that a JBW-factor is a JBW-algebra with no non trivial central idempotents, and that a factor representation of X is an algebra homomorphism from X to some JBW-factor Y, whose range is w^*-dense in Y. Assume that there exists some norm-one positive element p in X such that $\overline{co}\mathcal{G}(p) = B$. Since the family of all factor representations of X is faithful [21; 4.6.4], to prove that X is associative it is enough to show that every factor representation of X has 1-dimensinal range. Let $\Phi : X \to Y$ be such a factor representation. By [21; 4.6.2], we may assume that Y is equal to $e.X^{**}$ for some minimal central idempotent in X^{**}, and that Φ is nothing but the mapping $x \to e.x$. Let x be in $\mathcal{G}(p)$. Then there exists an algebra automorphism F of X and a central symmetry u in $M(X)$ such that $x = u.F(p)$, and hence $x = u.q$ for some norm-one positive element q in X. Now $\frac{1}{2}(\mathbf{1} + u)$ is a central idempotent in X^{**}, so that, since e is a minimal central idempotent in X^{**}, we have either $(\mathbf{1}+u).e = 0$ or $(\mathbf{1}+u).e = 2e$, and hence either $\Phi(x) = -e.q$ or $\Phi(x) = e.q$. Since x is arbitrary in $\mathcal{G}(p)$, the above shows that, if P denotes the set of all positive elements in $B(Y)$, then $\Phi(\mathcal{G}(p))$ is contained in $P \cup (-P)$. Since P is convex and w^*-compact, $co(P \cup (-P))$ is w^*-compact and hence norm-closed in Y. Since $\overline{co}\mathcal{G}(p) = B$, it follows that $\Phi(B)$ is contained in $co(P \cup (-P))$. By [21; 3.4.2 and 3.4.3], for y in $\Phi(X)$ with $\| y \| < 1$ there exists x in X satisfying $\| x \| < 1$ and $\Phi(x) = y$, and therefore the closed unit ball of $\Phi(X)$ is contained in $co(P \cup (-P))$. Since $\Phi(X)$ is w^*-dense in Y, we may apply the Kaplansky density theorem [21; 4.5.12] to obtain that $co(P \cup (-P)) = B(Y)$. As a consequence, if z is an extreme point of $B(Y)$, then z lies in $P \cup (-P)$. Since such a z is a symmetry in Y $(= e.X^{**})$ [23; Lemma 1.2], we have that either $z = e$ or $z = -e$. It follows from the Krein-Milman theorem that $Y = \mathbb{R}e$. $\square$

It has been proved recently that, if L is a locally compact Hausdorff topological space, and if $\mathcal{C}_0^{\mathbb{R}}(L)$ has almost transitive norm, then L is a singleton [20; Theorem 3.1]. Therefore we have

Corollary 5.4. $\mathbb{R}$ *is the unique* JB*-algebra with almost transitive norm.*

It follows from Theorem 5.3 and Proposition 5.2 that the question of convex transitivity of the norm on JB-algebras and preduals of JBW-algebras reduces to the consideration of a similar question on the classical Banach spaces $\mathcal{C}_0^{\mathbb{R}}(L)$ (for locally compact Hausdorff topological spaces L) and $L_1^{\mathbb{R}}(\Gamma, \mu)$ (for localizable measure spaces (Γ, μ)), respectively. The reader is referred to [36] for the $\mathcal{C}_0^{\mathbb{R}}(L)$ case. As far as we know, the convex transitivity of the norm for $L_1^{\mathbb{R}}(\Gamma, \mu)$ spaces has not been systematically studied. For the particular case of the almost transitivity of the norm on such spaces, the reader is referred to [19].

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