Communications in Algebra®, 33: 3293–3310, 2005

ISSN: 0092-7872 print/1532-4125 online
DOI: 10.1081/AGB-200066168

Taylor & Francis
Taylor & Francis Group

ON A CLASS OF PROJECTIONS ON ∗-RINGS#

Maja Fošner
Institute of Mathematics, Physics, and Mechanics, Ljubljana, Slovenia

Dijana Ilišević
Department of Mathematics, University of Zagreb, Zagreb, Croatia

In this article we describe the structure of projections acting on semiprime ∗-rings and satisfying a certain functional identity. The main result is applied to bicircular projections on C*-algebras.

Key Words: ∗-ring, ideal; Bicircular projection; C*-algebra; Derivation; Double centralizer; Essential ideal; Multiplier; Projection.

1991 Mathematics Subject Classification: Primary 16W10, 39B52, 46L05; Secondary 16W25, 16D25, 46L10, 46B04, 46K15.

INTRODUCTION

This work is motivated by the study of bicircular projections on the C*-algebra $B(\mathscr{H})$, the algebra of all bounded linear operators on a Hilbert space $\mathscr{H}$, carried out by Stachó and Zalar. In Stachó and Zalar (2004, Theorem 2.2) they determined the structure of bicircular projections on $B(\mathscr{H})$. According to Stachó and Zalar (2004, Proposition 3.4), every bicircular projection $P : B(\mathscr{H}) \to B(\mathscr{H})$ satisfies the functional identity

$$(FI) \qquad P(xyx) = P(x)yx - xP(y^*)^*x + xyP(x)$$

for all $x, y \in B(\mathscr{H})$. In this article, we investigate the structure of projections satisfying (FI) in the general setting of 2-torsion free semiprime ∗-rings.

The article is organized as follows. In Section 1, we study the functional identity (FI). The main theorem and its corollaries concerning the projections satisfying (FI) are proven in Section 2. In Section 3, the main theorem is applied to obtain a generalization of Stachó and Zalar (2004, Theorem 2.2) on an arbitrary C*-algebra. These sections correspond to the main steps of the proof of Stachó and Zalar (2004, Theorem 2.2).

Received May 30, 2004; Revised January 10, 2005; Accepted January 12, 2005
#Communicated by W. Martindale.
Address correspondence to Dijana Ilišević, Department of Mathematics, University of Zagreb, Bijenička 30, P.O. Box 335, 10002 Zagreb, Croatia; Fax: 385-1-4680-335; E-mail: ilisevic@math.hr

PRELIMINARIES

Throughout this article, all rings and algebras will be associative. A ring R is said to be 2-torsion free if $2a = 0$ (where $a \in R$) implies $a = 0$. A ring R is called a semiprime ring if $aRa = 0$ (where $a \in R$) implies $a = 0$ (this is equivalent to the condition that R has no nonzero nilpotent two-sided ideals) and it is called a prime ring if $aRb = 0$ (where $a, b \in R$) implies $a = 0$ or $b = 0$ (which is equivalent to the condition that the product of two nonzero two-sided ideals of R is not 0). A $*$-ring is a ring R with an involution, that is with an additive mapping $* : R \to R$ such that $(a^*)^* = a$ and $(ab)^* = b^*a^*$ for all $a, b \in R$.

In the sequel the word ideal always stands for a two-sided ideal. An ideal I of a $*$-ring R is called a $*$-ideal if $I^* = I$. Let R be a semiprime ring. If I is an ideal of R, then (for $a \in R$) $Ia = 0$ if and only if $aI = 0$; the set of all such $a \in R$ will be denoted by $I^\perp$. Obviously, $I^\perp$ is also an ideal of R. If I is a $*$-ideal, then $I^\perp$ is a $*$-ideal too. We write $I^{\perp\perp}$ for $(I^\perp)^\perp$. An ideal I of R is said to be an essential ideal if $I \cap J \neq 0$ for every nonzero ideal J of R; this is equivalent to the fact that $I^\perp = 0$. For every ideal I of R, the ideal $I \oplus I^\perp$ is essential.

Let R' be a semiprime ring and let R be a subring of R'. A double centralizer from R to R' is a pair (T, S) of additive mappings $T, S : R \to R'$ satisfying $xT(y) = S(x)y$ for all $x, y \in R$. The set of all double centralizers from R to R' is denoted by $M(R, R')$ and the set of all double centralizers from R to R is denoted by $M(R)$.

If R is an essential ideal of R' and $(T, S) \in M(R, R')$, it is not difficult to verify that $T(xy) = T(x)y$ and $S(xy) = xS(y)$ for all $x, y \in R$. If $(T, S) \in M(R)$, this implies that $T(I^\perp) \subseteq I^\perp$ and $S(I^\perp) \subseteq I^\perp$ for every ideal I of R.

For $a \in R'$, L_a denotes the left multiplication operator (defined by $L_a(x) = ax$ for every $x \in R$) and R_a denotes the right multiplication operator (which is given by $R_a(x) = xa$ for every $x \in R$); (L_a, R_a) is an example of a double centralizer from R to R'.

If R and R' are $*$-rings, for $(T, S) \in M(R, R')$ we define $T^* : R \to R'$ by $T^*(a) = T(a^*)^*$ and $S^* : R \to R'$ by $S^*(a) = S(a^*)^*$. It is easy to see that $(S^*, T^*) \in M(R, R')$.

For $(T_1, S_1), (T_2, S_2) \in M(R)$, we define

$$(T_1, S_1) + (T_2, S_2) = (T_1 + T_2, S_1 + S_2),$$
$$(T_1, S_1)(T_2, S_2) = (T_1T_2, S_2S_1).$$

With respect to these operations, $M(R)$ becomes a semiprime ring with identity (id, id), where $id(x) = x$ for every $x \in R$. The ring $M(R)$ is called the multiplier ring of R.

The ring R is canonically embedded into $M(R)$ by $a \mapsto (L_a, R_a)$. Moreover, R is an essential ideal of $M(R)$. Further, $R = M(R)$ if and only if R is unital. Regarding R as a subring of $M(R)$, every double centralizer (T, S) from R to R is given by the element $a = (T, S)$ in $M(R)$ as a multiplier, that is $(T, S) = (L_a, R_a)$ for some $a \in M(R)$ (more details can be found in Ara and Mathieu, 2003).

1. ON THE FUNCTIONAL IDENTITY (FI)

Let R' be a ring and let R be a subring of R'. Recall that a derivation $\delta : R \to R'$ is an additive mapping satisfying $\delta(xy) = \delta(x)y + x\delta(y)$ for all $x, y \in R$.

Note that, if $D : R \to R$ is an additive mapping satisfying (FI) and the additional assumption $D(x^*)^* = -D(x)$ for every $x \in R$, Brešar (1989, Theorem 4.3) immediately implies that D is a derivation. However, the general situation is more complicated to deal with. It will turn out that the mapping satisfying (FI) is generated by a derivation $\delta : R \to R$ such that $\delta(x^*)^* = -\delta(x)$ for every $x \in R$ and by a double centralizer $(T, S) \in M(R, M(R))$ with the property $T = S^*$.

Let us mention that the methods used in this section are similar to those in Brešar (1989, Theorem 4.3).

Lemma 1.1. *Let R be a 2-torsion free semiprime ring. Suppose that $D : R \to R$ is an additive mapping and $\delta : R \to R$ is a derivation such that*

$$D(xyx) = D(x)yx - xD(y)x + x\delta(y)x + xyD(x) \qquad (x, y \in R). \tag{1}$$

Then

$$D(xyz) = D(x)yz - xD(y)z + x\delta(y)z + xyD(z) \qquad (x, y, z \in R).$$

Proof. Linearizing (1) we obtain

$$\begin{aligned} D(xyz + zxy) = {} & D(x)yz - xD(y)z + x\delta(y)z + xyD(z) \\ & + D(z)yx - zD(y)x + z\delta(y)x + zyD(x) \end{aligned} \tag{2}$$

for all $x, y, z \in R$. Let $a, b, c, x \in R$ and set $W = D(abcxcba + cbaxabc)$. According to (2) we arrive at

$$\begin{aligned} W &= D((abc)x(cba) + (cba)x(abc)) \\ &= D(abc)xcba - abcD(x)cba + abc\delta(x)cba + abcxD(cba) \\ &\quad + D(cba)xabc - cbaD(x)abc + cba\delta(x)abc + cbaxD(abc), \end{aligned}$$

and on the other hand, using (1), we get

$$\begin{aligned} W &= D(a(bcxcb)a + c(baxab)c) = D(a(bcxcb)a) + D(c(baxab)c) \\ &= D(a)bcxcba - aD(bcxcb)a + a\delta(bcxcb)a + abcxcbD(a) \\ &\quad + D(c)baxabc - cD(baxab)c + c\delta(baxab)c + cbaxabD(c) \\ &= D(a)bcxcba - a(D(b)cxcb - bD(cxc)b + b\delta(cxc)b + bcxcD(b))a \\ &\quad + a\delta(bcxcb)a + abcxcbD(a) + D(c)baxabc \\ &\quad - c(D(b)axab - bD(axa)b + b\delta(axa)b + baxaD(b))c \\ &\quad + c\delta(baxab)c + cbaxabD(c) \\ &= D(a)bcxcba - aD(b)cxcba + ab(D(c)xc - cD(x)c + c\delta(x)c + cxD(c))ba \\ &\quad - ab\delta(cxc)ba - abcxcD(b)a + a\delta(bcxcb)a + abcxcbD(a) + D(c)baxabc \\ &\quad - cD(b)axabc + cb(D(a)xa - aD(x)a + a\delta(x)a + axD(a))bc - cb\delta(axa)bc \\ &\quad - cbaxaD(b)c + c\delta(baxab)c + cbaxabD(c) \end{aligned}$$

$$\begin{aligned}
&= D(a)bcxcba - aD(b)cxcba + abD(c)xcba - abcD(x)cba + abc\delta(x)cba \\
&\quad + abcxD(c)ba - ab\delta(c)xcba - abc\delta(x)cba - abcx\delta(c)ba - abcxcD(b)a \\
&\quad + a\delta(b)cxcba + ab\delta(c)xcba + abc\delta(x)cba + abcx\delta(c)ba + abcxc\delta(b)a \\
&\quad + abcxcbD(a) + D(c)baxabc - cD(b)axabc + cbD(a)xabc - cbaD(x)abc \\
&\quad + cba\delta(x)abc + cbaxD(a)bc - cb\delta(a)xabc - cba\delta(x)abc - cbax\delta(a)bc \\
&\quad - cbaxaD(b)c + c\delta(b)axabc + cb\delta(a)xabc + cba\delta(x)abc + cbax\delta(a)bc \\
&\quad + cbaxa\delta(b)c + cbaxabD(c).
\end{aligned}$$

Comparing the identities so obtained and using (2), it follows that

$$\begin{aligned}
&(D(cba) - D(c)ba + cD(b)a - c\delta(b)a - cbD(a))x(abc - cba) \\
&\quad + (abc - cba)x(D(cba) - D(c)ba + cD(b)a - c\delta(b)a - cbD(a)) = 0. \qquad (3)
\end{aligned}$$

Let us write $A(a, b, c) = D(abc) - D(a)bc + aD(b)c - a\delta(b)c - abD(c)$ and $B(a, b, c) = abc - cba$ for brevity. Hence

$$A(c, b, a)xB(a, b, c) + B(a, b, c)xA(c, b, a) = 0$$

for all $x \in R$. According to Brešar (1989, Lemma 1.1) this implies

$$A(c, b, a)xB(a, b, c) = B(a, b, c)xA(c, b, a) = 0$$

for all $a, b, c, x \in R$. By (2) we have $A(a, b, c) = -A(c, b, a)$. Using Brešar (1989, Lemma 1.2) we get

$$A(a_1, a_2, a_3)xB(b_1, b_2, b_3) = B(b_1, b_2, b_3)xA(a_1, a_2, a_3) = 0 \qquad (4)$$

for all $a_1, a_2, a_3, b_1, b_2, b_3, x \in R$. Hence

$$\begin{aligned}
2A(a, b, c)xA(a, b, c) &= (A(a, b, c) - A(c, b, a))xA(a, b, c) \\
&= (D(B(a, b, c)) - B(D(a), b, c) + B(a, D(b), c) \\
&\quad - B(a, \delta(b), c) - B(a, b, D(c)))xA(a, b, c).
\end{aligned}$$

Using (4) it follows that

$$2A(a, b, c)xA(a, b, c) = D(B(a, b, c))xA(a, b, c) \qquad (5)$$

for all $a, b, c, x \in R$. Similarly we can show that

$$2A(a, b, c)xA(a, b, c) = A(a, b, c)xD(B(a, b, c)). \qquad (6)$$

Further we have, using (2),

$$\begin{aligned}0 &= D(A(a,b,c)xB(a,b,c) + B(a,b,c)xA(a,b,c))\\ &= D(A(a,b,c))xB(a,b,c) - A(a,b,c)D(x)B(a,b,c)\\ &\quad + A(a,b,c)\delta(x)B(a,b,c) + A(a,b,c)xD(B(a,b,c))\\ &\quad + D(B(a,b,c))xA(a,b,c) - B(a,b,c)D(x)A(a,b,c)\\ &\quad + B(a,b,c)\delta(x)A(a,b,c) + B(a,b,c)xD(A(a,b,c)).\end{aligned}$$

According to (4), (5) and (6), from above, we get

$$0 = 4A(a,b,c)xA(a,b,c) + D(A(a,b,c))xB(a,b,c) + B(a,b,c)xD(A(a,b,c)).$$

Multiply this relation on the left by $A(a,b,c)xA(a,b,c)y$, $x, y \in R$. Using (4), we obtain

$$4A(a,b,c)xA(a,b,c)yA(a,b,c)xA(a,b,c) = 0$$

for all $x, y \in R$. Since R is a 2-torsion free semiprime ring we arrive at $A(a,b,c) = 0$ for all $a, b, c \in R$. The proof is complete. □

Proposition 1.2. *Let R be a 2-torsion free semiprime $*$-ring. Suppose that $D : R \to R$ is an additive mapping satisfying (FI). If $\delta(x) = D(x) - D(x^*)^*$, then δ is a derivation on R with the property $\delta(x) = -\delta(x^*)^*$. Further, there exists $(T, S) \in M(R, M(R))$ such that $T = S^*$ and*

$$2D(x) = \delta(x) + T(x) + S(x) \qquad (x \in R).$$

Proof. A linearization of (FI) implies

$$\begin{aligned}D(xyz + zyx) &= D(x)yz - xD(y^*)^*z + xyD(z)\\ &\quad + D(z)yx - zD(y^*)^*x + zyD(x)\end{aligned} \tag{7}$$

for all $x, y, z \in R$. According to the definition of δ, we have

$$\delta(xyz + zyx) = D(xyz + zyx) - D(x^*y^*z^* + z^*y^*x^*)^*$$

for all $x, y, z \in R$. Using (7), we get

$$\delta(xyz + zyx) = \delta(x)yz + x\delta(y)z + xy\delta(z) + zy\delta(x) + \delta(z)yx + z\delta(y)x \tag{8}$$

for all $x, y, z \in R$. In particular,

$$\delta(xyx) = \delta(x)yx + x\delta(y)x + xy\delta(x).$$

According to Brešar (1989, Theorem 4.3) δ is a derivation on R. It is obvious that for all $x \in R$ we have $\delta(x) = -\delta(x^*)^*$.

From (7), we get

$$\begin{aligned} D(xyz + zyx) = {} & D(x)yz - xD(y)z + x\delta(y)z + xyD(z) \\ & + D(z)yx - zD(y)x + z\delta(y)x + zyD(x) \end{aligned}$$

for all $x, y, z \in R$. In particular,

$$D(xyx) = D(x)yx - xD(y)x + x\delta(y)x + xyD(x)$$

for all $x, y \in R$. Lemma 1.1 implies

$$D(xyz) = D(x)yz - xD(y)z + x\delta(y)z + xyD(z)$$

for all $x, y, z \in R$. Consider the expression $W = D(vxyz)$, $v, x, y, z \in R$. On the one hand, we have

$$W = D((vx)yz) = D(vx)yz - vxD(y)z + vx\delta(y)z + vxyD(z),$$

and on the other hand, we obtain

$$\begin{aligned} W &= D(v(xy)z) \\ &= D(v)xyz - vD(xy)z + v\delta(xy)z + vxyD(z) \\ &= D(v)xyz - vD(xy)z + v\delta(x)yz + vx\delta(y)z + vxyD(z). \end{aligned}$$

Comparing these two relations we arrive at

$$(D(vx) - D(v)x - v\delta(x))yz + v(D(xy) - xD(y))z = 0$$

for all $v, x, y, z \in R$. Using the semiprimeness of R it follows that

$$(D(vx) - D(v)x - v\delta(x))y + v(D(xy) - xD(y)) = 0.$$

For a fixed $x \in R$ let us write

$$T'(y) = -D(xy) + xD(y) \quad \text{and} \quad T''(v) = D(vx) - D(v)x - v\delta(x). \tag{9}$$

Hence we have $vT'(y) = T''(v)y$ for all $v, y \in R$. Therefore (T', T'') is a double centralizer from R to R, which yields $(T', T'') = (L_{\Delta(x)}, R_{\Delta(x)})$ for some $\Delta(x) \in M(R)$. Consequently,

$$T'(y) = \Delta(x)y \quad \text{and} \quad T''(v) = v\Delta(x),$$

which in turn implies

$$D(vx) = D(v)x + v\delta(x) + v\Delta(x), \tag{10}$$

$$D(xy) = xD(y) - \Delta(x)y \tag{11}$$

for all $v, x, y \in R$, by (9).

Finally, from (10) and (11), we get

$$x(D(y) - \delta(y) - \Delta(y)) = (D(x) + \Delta(x))y$$

for all $x, y \in R$. If we define $T : R \to M(R)$ by $T(x) = D(x) - \delta(x) - \Delta(x)$ and $S : R \to M(R)$ by $S(x) = D(x) + \Delta(x)$, then $(T, S) \in M(R, M(R))$ and $2D(x) = \delta(x) + T(x) + S(x)$ holds.

Since $\delta(x) = -\delta(x^*)^*$ for all $x \in R$, we have

$$\begin{aligned}
2\delta(x) &= 2D(x) - 2D(x^*)^* \\
&= (\delta(x) + T(x) + S(x)) - (\delta(x^*) + T(x^*) + S(x^*))^* \\
&= \delta(x) + T(x) + S(x) - \delta(x^*)^* - T^*(x) - S^*(x) \\
&= 2\delta(x) + (T - S^*)(x) + (S - T^*)(x).
\end{aligned}$$

Therefore, $T - S^* = T^* - S$. If we define $V = T - S^* = T^* - S$, then

$$xV(y) = xT(y) - xS^*(y) = S(x)y - T^*(x)y = -V(x)y \tag{12}$$

for all $x, y \in R$, so $(V, -V) \in M(R, M(R))$. This implies

$$V(xy) = V(x)y,$$
$$V(xy) = xV(y)$$

and thus we have

$$xV(y) = V(x)y \tag{13}$$

for all $x, y \in R$. From (12) and (13), using that R is 2-torsion free, we conclude that

$$V(x)y = 0 \qquad (x, y \in R).$$

Since R is an essential ideal of $M(R)$, we get $V(x) = 0$ for every $x \in R$. Hence, $T = S^*$. □

Corollary 1.3. *Let R be a 2-torsion free semiprime $*$-ring with the property that $R^2 = R$ (resp. $\overline{R^2} = R$ if R is a semiprime Banach $*$-algebra). Suppose that $D : R \to R$ is an additive mapping satisfying (FI). Then there exists a selfadjoint $a \in M(R)$ such that*

$$2D(x) = \delta(x) + ax + xa \qquad (x \in R),$$

where δ is a derivation on R defined by $\delta(x) = D(x) - D(x^*)^*$.

Proof. According to Proposition 1.2, there exists $(T, S) \in M(R, M(R))$ such that $T = S^*$ and

$$2D(x) = \delta(x) + T(x) + S(x) \qquad (x \in R),$$

where $\delta : R \to R$ is a derivation on R given by $\delta(x) = D(x) - D(x^*)^*$, so it is sufficient to prove the existence of $a \in M(R)$ such that $(T, S) = (L_a, R_a)$.

For all $x, y \in R$,

$$T(xy) = T(x)y \in R,$$
$$S(xy) = xS(y) \in R.$$

If $R^2 = R$, we have $T(x) \in R$ and $S(x) \in R$ for all $x \in R$. If R is a semiprime Banach $*$-algebra, the condition $R^2 = R$ can be replaced by $\overline{R^2} = R$ and the same is true since T and S are continuous, which can be easily verified via the closed graph theorem.

Therefore, (T, S) is a double centralizer from R to R, so there exists $a \in M(R)$ such that $(T, S) = (L_a, R_a)$. Since $T = S^*$, we have $L_a = R_a{}^*$ and thus

$$ax = L_a(x) = R_a{}^*(x) = R_a(x^*)^* = (x^*a)^* = a^*x$$

for every $x \in R$. This implies $a = a^*$ because R is an essential ideal of $M(R)$. □

2. MAIN THEOREM

A projection P on a ring R is an additive mapping $P : R \to R$ such that $P^2 = P$. If R is a $*$-ring, an element p in $M(R)$ is called a selfadjoint projection if $p^* = p = p^2$.

In this section, we are going to study projections satisfying the functional identity (FI). Let us begin with an example of such a projection.

Example 2.1. Let R be a semiprime $*$-ring and let I, J be $*$-ideals of R such that $I \oplus J = R$. Suppose that $p \in M(R)$ is a selfadjoint projection. Define a mapping $P : R \to R$ by $P(x) = pu + vp$, where $x = u + v$, $u \in I$, $v \in J$.

Note that $pI \subseteq I$ and $Jp \subseteq J$. Namely, for $a \in I$ we have $pa \in R$. Therefore, there exist $b \in I$ and $c \in J$ such that $pa = b + c$. Multiplying this by rc from the right, where r is an arbitrary element in R, we arrive at $0 = crc$ for all $r \in R$. The semiprimeness of R implies $c = 0$, which in turn yields $pa = b \in I$. In the same way it can be proven that $Jp \subseteq J$.

The mapping P is a projection since

$$P^2(x) = P(pu + vp) = p(pu) + (vp)p = pu + vp = P(x)$$

for all $x \in R$.

Let $x, y \in R$. There exist $u_1, u_2 \in I$, $v_1, v_2 \in J$ such that $x = u_1 + v_1$ and $y = u_2 + v_2$. We have

$$P(xyx) = P(u_1u_2u_1 + v_1v_2v_1) = p(u_1u_2u_1) + (v_1v_2v_1)p$$

and, on the other hand,

$$\begin{aligned} &P(x)yx - xP(y^*)^*x + xyP(x) \\ &\quad = (pu_1 + v_1p)(u_2 + v_2)(u_1 + v_1) - (u_1 + v_1)(pu_2^* + v_2^*p)^*(u_1 + v_1) \\ &\qquad + (u_1 + v_1)(u_2 + v_2)(pu_1 + v_1p) \end{aligned}$$

$$= (pu_1)u_2u_1 + (v_1p)v_2v_1 - u_1(u_2p)u_1 - v_1(pv_2)v_1 + u_1u_2(pu_1) + v_1v_2(v_1p)$$
$$= p(u_1u_2u_1) + (v_1v_2v_1)p.$$

Hence, P satisfies the identity (FI).

We are going to prove that for every projection $P : R \to R$ satisfying the functional identity (FI), there exist a $*$-ideal I of R and a selfadjoint projection $p \in M(R)$ such that P on $I^{\perp} \oplus I^{\perp\perp}$ is of the form described in Example 2.1.

In the main theorem, we have to deal with derivations $\delta_1, \delta_2 : R \to M(R)$ such that $\delta_1(R)R\delta_2(R) = 0$. This implies $(R\delta_1(R))R(\delta_2(R)R) = 0$. When R is a prime ring, it immediately follows that $R\delta_1(R) = 0$ or $\delta_2(R)R = 0$. Since R is an essential ideal of $M(R)$, we conclude $\delta_1 = 0$ or $\delta_2 = 0$. In the following proposition, we are going to generalize this fact to semiprime rings.

Proposition 2.2. *Let R be a semiprime ring. If $\delta_1, \delta_2 : R \to M(R)$ are derivations satisfying $\delta_1(x)R\delta_2(y) = 0$ for all $x, y \in R$, then $\delta_1(I^{\perp\perp}) = 0$ and $\delta_2(I^{\perp}) = 0$, where $I = R\delta_2(R)R$.*

Proof. If $x \in I^{\perp}$, then for all $y \in I$ and $z \in R$,

$$y\delta_2(x)z = \delta_2(yx)z - \delta_2(y)xz = -\delta_2(y)xz \in I^{\perp}.$$

On the other hand, $y\delta_2(x)z \in I$. Hence, $y\delta_2(x)z = 0$. Since y is an arbitrary element in I, we conclude that $\delta_2(x)z \in I^{\perp}$ for every $z \in R$. This implies $R\delta_2(x)R \subseteq I^{\perp}$. Because of $R\delta_2(x)R \subseteq I$, we obtain $R\delta_2(x)R = 0$. This implies $R\delta_2(x) = 0$ since R is semiprime. Using that R is an essential ideal of $M(R)$, we finally get $\delta_2(x) = 0$ for all $x \in I^{\perp}$.

If $x \in I^{\perp\perp}$, $y \in I^{\perp}$ and $z \in R$, then

$$y\delta_1(x)z = \delta_1(yx)z - \delta_1(y)xz = -\delta_1(y)xz \in I^{\perp\perp}.$$

Since $y\delta_1(x)z \in I^{\perp}$, we have $y\delta_1(x)z = 0$. Because $y \in I^{\perp}$ is arbitrary, we have $\delta_1(x)z \in I^{\perp\perp}$, that is $\delta_1(x)R \subseteq I^{\perp\perp}$. On the other hand,

$$\delta_1(x)RI = \delta_1(x)R^2\delta_2(R)R \subseteq \delta_1(R)R\delta_2(R)R = 0,$$

so $\delta_1(x)R \subseteq I^{\perp}$. Finally, $\delta_1(x)R = 0$, so we get $\delta_1(x) = 0$ for every $x \in I^{\perp\perp}$. □

Now we are in the position to prove our main result.

Theorem 2.3. *Let R be a 2-torsion free semiprime $*$-ring. Let $P : R \to R$ be a projection satisfying the functional identity (FI). Then there exist a $*$-ideal I of R and a selfadjoint projection $p \in M(I^{\perp} \oplus I^{\perp\perp})$ such that $P(x) = px$ for every $x \in I^{\perp}$ and $P(x) = xp$ for every $x \in I^{\perp\perp}$.*

Proof. By Proposition 1.2 we have

$$2P(x) = \delta(x) + T(x) + S(x) \qquad (x \in R), \tag{14}$$

where $\delta : R \to R$ is a derivation defined by $\delta(x) = P(x) - P(x^*)^*$ and $(T, S) \in M(R, M(R))$ satisfies $T = S^*$. Using $P^2 = P$, it follows that $(2P)^2 = 2 \cdot 2P$. Taking into account (14), we get

$$\delta^2(x) + T^2(x) + S^2(x) + \delta T(x) + \delta S(x) + T\delta(x) \\ + S\delta(x) + TS(x) + ST(x) = 2\delta(x) + 2T(x) + 2S(x). \tag{15}$$

In the sequel, we are going to use $xT(y) = S(x)y$, $T(xy) = T(x)y$ and $S(xy) = xS(y)$ for all $x, y \in R$, as well as the fact that δ is a derivation. Inserting yx instead of x in (15), we obtain

$$\begin{aligned}&\delta(\delta(y)x + y\delta(x)) + T(T(y)x) + S(yS(x)) + \delta(T(y)x) + \delta(yS(x))\\ &\quad + T(\delta(y)x + y\delta(x)) + S(\delta(y)x + y\delta(x)) + T(yS(x)) + S(T(y)x)\\ &\quad = 2\delta(y)x + 2y\delta(x) + 2T(y)x + 2yS(x),\end{aligned}$$

that is

$$\begin{aligned}&\delta^2(y)x + 2\delta(y)\delta(x) + y\delta^2(x) + T^2(y)x + yS^2(x)\\ &\quad + \delta T(y)x + T(y)\delta(x) + \delta(y)S(x) + y\delta S(x)\\ &\quad + T\delta(y)x + T(y)\delta(x) + \delta(y)S(x) + yS\delta(x) + 2T(y)S(x)\\ &\quad = 2\delta(y)x + 2y\delta(x) + 2T(y)x + 2yS(x).\end{aligned} \tag{16}$$

Multiplying (15) from the left by y we get

$$y\delta^2(x) + S(y)T(x) + yS^2(x) + y\delta T(x) + y\delta S(x) + S(y)\delta(x) \\ + yS\delta(x) + S(y)S(x) + yST(x) = 2y\delta(x) + 2S(y)x + 2yS(x). \tag{17}$$

Subtracting (17) from (16), we arrive at

$$\begin{aligned}&\delta^2(y)x + 2\delta(y)\delta(x) + T^2(y)x + \delta T(y)x + T(y)\delta(x) + \delta(y)S(x)\\ &\quad + T\delta(y)x + T(y)\delta(x) + \delta(y)S(x) + 2T(y)S(x) - S(y)T(x)\\ &\quad - y\delta T(x) - S(y)\delta(x) - S(y)S(x) - yST(x)\\ &\quad = 2\delta(y)x + 2T(y)x - 2S(y)x.\end{aligned} \tag{18}$$

Let us write xz instead of x in (18):

$$\begin{aligned}&\delta^2(y)xz + 2\delta(y)\delta(x)z + 2\delta(y)x\delta(z) + T^2(y)xz + \delta T(y)xz\\ &\quad + T(y)\delta(x)z + T(y)x\delta(z) + \delta(y)xS(z) + T\delta(y)xz + T(y)\delta(x)z\\ &\quad + T(y)x\delta(z) + \delta(y)xS(z) + 2T(y)xS(z) - S(y)T(x)z - y\delta T(x)z\\ &\quad - yT(x)\delta(z) - S(y)\delta(x)z - S(y)x\delta(z) - S(y)xS(z) - yT(x)S(z)\\ &\quad = 2\delta(y)xz + 2T(y)xz - 2S(y)xz.\end{aligned} \tag{19}$$

Multiplying (18) from the right by z, we get

$$\begin{aligned}&\delta^2(y)xz + 2\delta(y)\delta(x)z + T^2(y)xz + \delta T(y)xz + T(y)\delta(x)z\\&\quad + \delta(y)S(x)z + T\delta(y)xz + T(y)\delta(x)z + \delta(y)S(x)z + 2T(y)S(x)z\\&\quad - S(y)T(x)z - y\delta T(x)z - S(y)\delta(x)z - S(y)S(x)z - yST(x)z\\&\quad = 2\delta(y)xz + 2T(y)xz - 2S(y)xz.\end{aligned}\tag{20}$$

Subtracting (20) from (19) yields

$$\begin{aligned}&2\delta(y)x\delta(z) + 2T(y)x\delta(z) + 2\delta(y)xS(z) + 2T(y)xS(z)\\&\quad - yT(x)\delta(z) - S(y)x\delta(z) - S(y)xS(z) - yT(x)S(z)\\&\quad - 2\delta(y)xT(z) - 2T(y)xT(z) + S(y)S(x)z + yST(x)z = 0,\end{aligned}$$

which implies

$$\begin{aligned}&\delta(y)x\delta(z) + T(y)x\delta(z) + \delta(y)xS(z) + T(y)xS(z) - S(y)x\delta(z)\\&\quad - S(y)xS(z) - \delta(y)xT(z) - T(y)xT(z) + S(y)xT(z) = 0.\end{aligned}$$

Finally,

$$(\delta(y) + T(y) - S(y))x(\delta(z) + S(z) - T(z)) = 0 \qquad (x, y, z \in R).$$

It is not difficult to verify that $T - S$ is a derivation from R to $M(R)$ since $(T, S) \in M(R, M(R))$. Let us define

$$\begin{aligned}\delta_1 &= \delta + T - S,\\ \delta_2 &= \delta + S - T.\end{aligned}$$

The mappings δ_1 and δ_2 are derivations from R to $M(R)$ satisfying $\delta_1(x)R\delta_2(y) = 0$ for all $x, y \in R$. Proposition 2.2 implies that for the ideal $I = R\delta_2(R)R$ of R we have $\delta_1(I^{\perp\perp}) = 0$ and $\delta_2(I^{\perp}) = 0$. Note that I is a $*$-ideal since

$$\begin{aligned}\delta_2(x)^* &= \delta(x)^* + S(x)^* - T(x)^* = -\delta(x^*) + S^*(x^*) - T^*(x^*)\\&= -\delta(x^*) + T(x^*) - S(x^*) = -(\delta(x^*) + S(x^*) - T(x^*)) = -\delta_2(x^*).\end{aligned}$$

From $\delta_1(I^{\perp\perp}) = 0$ and $\delta_2(I^{\perp}) = 0$ we conclude

$$\delta(x) = S(x) - T(x) \qquad (x \in I^{\perp\perp}),\tag{21}$$

$$\delta(x) = T(x) - S(x) \qquad (x \in I^{\perp}).\tag{22}$$

By (14) this implies

$$P(x) = S(x) \quad (x \in I^{\perp\perp}), \qquad P(x) = T(x) \quad (x \in I^{\perp}).\tag{23}$$

Since $P(x) \in R$, it follows that $S : I^{\perp\perp} \to R$ and $T : I^{\perp} \to R$. More precisely, $S : I^{\perp\perp} \to I^{\perp\perp}$ and $T : I^{\perp} \to I^{\perp}$. Furthermore,

$$T(x) = 2P(x) - \delta(x) - S(x) \in R \qquad (x \in I^{\perp\perp}),$$

so $T : I^{\perp\perp} \to R$, which implies $T : I^{\perp\perp} \to I^{\perp\perp}$. Analogously, $S : I^{\perp} \to I^{\perp}$. Consequently, (T, S) is a double centralizer from $I^{\perp} \oplus I^{\perp\perp}$ to $I^{\perp} \oplus I^{\perp\perp}$. Therefore, there exists $p \in M(I^{\perp} \oplus I^{\perp\perp})$ such that $(T, S) = (L_p, R_p)$. In other words,

$$T(x) = px, \qquad S(x) = xp \quad (x \in I^{\perp} \oplus I^{\perp\perp}). \tag{24}$$

Comparing (23) and (24), we arrive at

$$P(x) = xp \ (x \in I^{\perp\perp}),$$
$$P(x) = px \ (x \in I^{\perp}).$$

Since $T = S^*$, for every $x \in I^{\perp} \oplus I^{\perp\perp}$ we have

$$(p - p^*)x = px - (x^*p)^* = T(x) - S(x^*)^* = T(x) - S^*(x) = 0.$$

This implies $p = p^*$.

For every $x \in I^{\perp\perp}$,

$$0 = P^2(x) - P(x) = P(xp) - xp = (xp)p - xp = x(p^2 - p).$$

Similarly, for every $x \in I^{\perp}$ we have

$$0 = P^2(x) - P(x) = P(px) - px = p(px) - px = (p^2 - p)x.$$

Since $p = p^*$ and $I^{\perp}$ is a $*$-ideal, we get $x(p^2 - p) = 0$ for every $x \in I^{\perp}$. Therefore,

$$(I^{\perp} \oplus I^{\perp\perp})(p^2 - p) = 0,$$

which implies $p^2 = p$. □

According to Theorem 2.3, for every projection P on R satisfying (FI) there exists an essential ideal of R on which the structure of P is completely determined. The question when the obtained essential ideal is equal to the whole ring R appears naturally. The following corollaries are related to this question.

Corollary 2.4. *Let R be a 2-torsion free commutative semiprime $*$-ring. If $P : R \to R$ is a projection satisfying (FI), then there exists a selfadjoint projection $p \in M(R)$ such that $P(x) = px = xp$ for every $x \in R$.*

Proof. First note that, for $(T, S) \in M(R, M(R))$, since R is commutative, we have

$$\begin{aligned} xT(y)z &= S(x)yz = S(x)zy = yS(x)z \\ &= S(yx)z = S(xy)z = xS(y)z \qquad (x, y, z \in R). \end{aligned}$$

The semiprimeness of R implies $xT(y) = xS(y)$ for all $x, y \in R$. Since R is an essential ideal of $M(R)$, $T = S$ follows.

From now on, we use the same notations as in the proof of Theorem 2.3. After inspection of the proof of that theorem, we observe that, using $T = S$, (21) and (22) imply $\delta(x) = 0$ for every $x \in I^\perp \oplus I^{\perp\perp}$. Therefore, for $x \in R$ and $y \in I^\perp \oplus I^{\perp\perp}$ we have

$$\delta(x)y = \delta(xy) - x\delta(y) = 0.$$

Using that $I^\perp \oplus I^{\perp\perp}$ is an essential ideal of R, we conclude $\delta(x) = 0$ for every $x \in R$.

Since the mapping $\delta_2 : R \to M(R)$ is defined by $\delta_2 = \delta + S - T$, we have $\delta_2(x) = 0$ for every $x \in R$. Then $I = R\delta_2(R)R = 0$. Hence, $p \in M(R)$ and $P(x) = px$ for every $x \in R$.

For all $x, y \in R$,

$$(px - xp)y = p(xy) - x(py) = p(yx) - (py)x = 0,$$

so $px = xp$. □

Corollary 2.5. *Let R be a 2-torsion free prime $*$-ring. If $P : R \to R$ is a projection satisfying (FI), then there exists a selfadjoint projection $p \in M(R)$ such that $P(x) = px$ for every $x \in R$ or $P(x) = xp$ for every $x \in R$.*

Proof. For every ideal I of R, in particular for the one obtained in Theorem 2.3, $I^\perp = 0$ or $I^{\perp\perp} = 0$ since R is prime.

Suppose that $I^\perp = 0$. This implies $I^{\perp\perp} = R$. According to Theorem 2.3, there exists a selfadjoint projection $p \in M(R)$ such that $P(x) = xp$ for every $x \in R$.

In the case when $I^{\perp\perp} = 0$, then $I^\perp = R$ and Theorem 2.3 implies the existence of a selfadjoint projection $p \in M(R)$ such that $P(x) = px$ for every $x \in R$. □

Corollary 2.6. *Let $\{R_k : k \in K\}$ be a family of 2-torsion free prime $*$-rings and let $R = \bigoplus_{k\in K} R_k$. Then for every ideal I of R we have $I^\perp \oplus I^{\perp\perp} = R$. Further, if $P : R \to R$ is a projection satisfying (FI), then there exist a $*$-ideal I of R and a selfadjoint projection $p \in M(R)$ such that $P(x) = py + zp$, where $y \in I^\perp$ and $z \in I^{\perp\perp}$ are such that $y + z = x$.*

Proof. Let us define $K_0 = \{k \in K : I \cap R_k = 0\}$. We are going to prove that $I^\perp = \bigoplus_{k\in K_0} R_k$. Obviously, $R_k \subseteq I^\perp$ for every $k \in K_0$ and thus $\bigoplus_{k\in K_0} R_k \subseteq I^\perp$.

Take $x = x_1 + x_2 + \cdots + x_n \in I^\perp$, where $x_i \in R_{k_i}$ are nonzero and $k_i \neq k_j$ if $i \neq j$. Suppose that, for example, $k_1 \notin K_0$. Hence, $I \cap R_{k_1} \neq 0$. From $x(I \cap R_{k_1}) = 0$ it follows that $x_1(I \cap R_{k_1}) = 0$. Since R_{k_1} is prime and $I \cap R_{k_1} \neq 0$ we arrive at $x_1 = 0$, which contradicts our assumption. Consequently, $I^\perp \subseteq \bigoplus_{k\in K_0} R_k$ and therefore $I^\perp = \bigoplus_{k\in K_0} R_k$.

Analogously, it can be proven that $I^{\perp\perp} = \bigoplus_{k\in K\setminus K_0} R_k$. Finally, $I^\perp \oplus I^{\perp\perp} = R$. □

Let us recall that a von Neumann algebra is a strongly closed $*$-subalgebra of $B(\mathscr{H})$.

Corollary 2.7. *Let A be a von Neumann algebra. Then $I^{\perp} \oplus I^{\perp\perp} = A$ for every ideal I of A. Further, if $P : A \to A$ is a projection satisfying (FI), then there exist a $*$-ideal I of A and a selfadjoint projection $p \in M(A)$ such that $P(x) = py + zp$, where $y \in I^{\perp}$ and $z \in I^{\perp\perp}$ are such that $y + z = x$.*

Proof. According to e.g. (Murphy, 1990, Theorem 4.1.8(2)), the annihilator of every ideal of A is of the form qA for some central selfadjoint projection $q \in A$. In particular, there exist central selfadjoint projections $e, f \in A$ such that $I^{\perp} = eA$, $I^{\perp\perp} = fA$. If $x \in I^{\perp} \oplus I^{\perp\perp}$, there are $y \in I^{\perp}$ and $z \in I^{\perp\perp}$ such that $x = y + z$, so

$$(e + f - 1)x = (e + f - 1)(y + z) = ey + fz - y - z = y + z - y - z = 0.$$

Since $I^{\perp} \oplus I^{\perp\perp}$ is an essential ideal of A, we conclude that $e + f = 1$. Hence, $I^{\perp} \oplus I^{\perp\perp} = eA \oplus fA = A$. □

The most important role in theory of von Neumann algebras is played by the so-called factors. A factor is a von Neumann algebra whose centre only contains the scalar operators. Thus for every ideal I of a factor A we have $I^{\perp} = 0$ and $I^{\perp\perp} = A$, or $I^{\perp} = A$ and $I^{\perp\perp} = 0$, so the previous result directly implies

Corollary 2.8. *Let A be a factor and let $P : A \to A$ be a projection satisfying (FI). Then there exists a selfadjoint projection $p \in M(A)$ such that $P(x) = px$ for every $x \in A$ or $P(x) = xp$ for every $x \in A$.*

Let us also recall that a proper H*-algebra is a complex Banach $*$-algebra A which is a Hilbert space with respect to the inner product $\langle ., . \rangle$ such that $\langle ab, c \rangle = \langle b, a^*c \rangle$ and $\langle ba, c \rangle = \langle b, ca^* \rangle$ for all $a, b, c \in A$. We have:

Corollary 2.9. *Let A be a proper H*-algebra and let $P : A \to A$ be a projection satisfying (FI). Then there exist a $*$-ideal I of A and a selfadjoint projection $p \in M(A)$ with the property $P(x) = py + zp$, where $y \in I^{\perp}$ and $z \in I^{\perp\perp}$ are such that $y + z = x$.*

Howewer, there exist a $*$-ring R and a projection $P : R \to R$ satisfying (FI) such that, for the ideal I obtained in Theorem 2.3, $I^{\perp} \oplus I^{\perp\perp} \neq R$. We are indebted to Matej Brešar for suggesting us the following example.

Example 2.10. Let A be a nonunital prime $*$-algebra (over the field $\mathbb{F}$ with the identity 1) with zero-centre, in which the involution is linear. Let $\mathcal{A} = A \times A \oplus \mathbb{F}$ be the unitization of the algebra $A \times A$ and let us define an involution in $\mathcal{A}$ by $((a, b) + \lambda)^* = (a^*, b^*) + \lambda$. Therefore $\mathcal{A}$ becomes a semiprime $*$-algebra over $\mathbb{F}$. If there exist such ideals U and V of $\mathcal{A}$ that $\mathcal{A} = U \oplus V$, then $U = e\mathcal{A}$ and $V = (1 - e)\mathcal{A}$, where e is a central idempotent in $\mathcal{A}$. But the only central idempotents in $\mathcal{A}$ are 0 and 1, so $U = \mathcal{A}$ or $V = \mathcal{A}$.

Let $q \in A$ be a nonzero selfadjoint projection. Let $P : \mathcal{A} \to \mathcal{A}$ be the mapping defined by

$$P((a, b) + \lambda) = (qa + \lambda q, bq + \lambda q) \qquad (a, b \in A, \lambda \in \mathbb{F}).$$

Then $P^2 = P$ and P satisfies (FI). Theorem 2.3 implies the existence of an ideal I of $\mathscr{A}$ and a projection $p \in M(I^{\perp} \oplus I^{\perp\perp})$ such that $P(x) = px$ for every $x \in I^{\perp}$ and $P(x) = xp$ for every $x \in I^{\perp\perp}$.

Assume that $I^{\perp} \oplus I^{\perp\perp} = \mathscr{A}$ for the obtained I. Then $I^{\perp} = \mathscr{A}$ or $I^{\perp\perp} = \mathscr{A}$, that is $p \in M(\mathscr{A}) = \mathscr{A}$ and $P(x) = px$ for every $x \in \mathscr{A}$ or $P(x) = xp$ for every $x \in \mathscr{A}$. Since $P(x) \in A \times A$ for every $x \in \mathscr{A}$, we conclude that $p = (p_1, p_2)$ for some projections $p_1, p_2 \in A$.

If $P(x) = px$ for every $x \in \mathscr{A}$, then in particular

$$(0, bq) = P(0, b) = (p_1, p_2)(0, b) = (0, p_2 b),$$

so $bq = p_2 b$ for every $b \in A$. Then

$$b(q + p_2) = bq + bp_2 = p_2 b + (p_2 b)^* = p_2 b + (b^* q)^* = (p_2 + q)b,$$

which implies that $q + p_2$ is in the centre of A. Hence, $q + p_2 = 0$. Then

$$q = q^2 = (-p_2)^2 = p_2^2 = p_2 = -q,$$

so $q = 0$, which is a contradiction.

In the case when $P(x) = xp$ for every $x \in \mathscr{A}$, we have

$$(qa, 0) = P(a, 0) = (a, 0)(p_1, p_2) = (ap_1, 0).$$

Hence, $qa = ap_1$ for every $a \in A$. Analogously, as in the previous case, we arrive at a contradiction.

3. BICIRCULAR PROJECTIONS ON C*-ALGEBRAS

Let $(X, \|\cdot\|)$ be a complex Banach space and let $P : X \to X$ be a linear projection. We denote by $\overline{P}$ its complementary projection, that is the projection $1_X - P$. A projection P is called bicircular if the mapping $e^{i\alpha}P + e^{i\beta}\overline{P}$ is an isometry for all $\alpha, \beta \in \mathbb{R}$. Obviously, this is equivalent to the fact that the mapping $P + e^{i\varphi}\overline{P}$ is an isometry for every $\varphi \in \mathbb{R}$. In particular, $P - \overline{P}$ is an isometry. It follows that P is bounded. More precisely, $\|P\| \le 1$ and $\|\overline{P}\| \le 1$, so P is a bicontractive projection.

In this section, we investigate bicircular projections on C*-algebras. A C*-algebra is a complex Banach $*$-algebra $(A, \|\cdot\|)$ such that $\|a^* a\| = \|a\|^2$ for every $a \in A$.

First we shall give an example of a bicircular projection on a C*-algebra.

Example 3.1. Let A be a C*-algebra and let I, J be $*$-ideals of A such that $I \oplus J = A$. Let p be a selfadjoint projection in $M(A)$. Define $P : A \to A$ by $P(x) = py + zp$, where $y \in I$ and $z \in J$ are such that $x = y + z$. Using $pI \subseteq I$ and $Jp \subseteq J$ (as proved in Example 2.1),

$$\begin{aligned}\|P(x) + e^{i\varphi}\overline{P}(x)\|^2 &= \|py + zp + e^{i\varphi}(y + z - py - zp)\|^2 \\ &= \big\|\big(py + e^{i\varphi}(y - py)\big) + \big(zp + e^{i\varphi}(z - zp)\big)\big\|^2\end{aligned}$$

$$
\begin{aligned}
&= \big\| \big(\big(py + e^{i\varphi}(y - py)\big) + \big(zp + e^{i\varphi}(z - zp)\big) \big)^* \\
&\quad \times \big(\big(py + e^{i\varphi}(y - py)\big) + \big(zp + e^{i\varphi}(z - zp)\big) \big) \big\| \\
&= \big\| \big(py + e^{i\varphi}(y - py)\big)^* \big(py + e^{i\varphi}(y - py)\big) \\
&\quad + \big(zp + e^{i\varphi}(z - zp)\big)^* \big(zp + e^{i\varphi}(z - zp)\big) \big\| \\
&= \big\| (y^* y) + \big(zp + e^{i\varphi}(z - zp)\big)^* \big(zp + e^{i\varphi}(z - zp)\big) \big\| \\
&= \big\| \big(y + zp + e^{i\varphi}(z - zp)\big)^* \big(y + zp + e^{i\varphi}(z - zp)\big) \big\| \\
&= \big\| \big(y + zp + e^{i\varphi}(z - zp)\big) \big(y + zp + e^{i\varphi}(z - zp)\big)^* \big\| \\
&= \big\| \big(yy^* + (zp + e^{i\varphi}(z - zp))(zp + e^{i\varphi}(z - zp))^* \big\| \\
&= \|yy^* + zz^*\| = \|(y + z)(y + z)^*\| = \|y + z\|^2 = \|x\|^2.
\end{aligned}
$$

Therefore, P is a bicircular projection on A.

In order to determine the structure of a bicircular projection on a C*-algebra, first we have to relate bicircular projections with the functional identity (FI).

Lemma 3.2. *Let A be a C*-algebra and let $P : A \to A$ be a bicircular projection. Then P satisfies (FI).*

Proof. Let us fix an arbitrary $\varphi \in \mathbb{R}$ and define the mapping $T_\varphi = P + e^{i\varphi}\overline{P}$. According to the definition of a bicircular projection, T_φ is a linear isometry. Since for every $y \in A$ there exists $x = T_{-\varphi}(y) \in A$ with the property $T_\varphi(x) = y$, the mapping T_φ is a surjection. From Dang et al. (1990, Theorem E) we get

$$
T_\varphi(xy^*z + zy^*x) = T_\varphi(x)T_\varphi(y)^*T_\varphi(z) + T_\varphi(z)T_\varphi(y)^*T_\varphi(x)
$$

for all $x, y, z \in A$. Since this holds for every $\varphi \in \mathbb{R}$, the same calculation as in Stachó and Zalar (2004, Proposition 3.4) leads to the functional identity (FI). □

The above lemma enables us to transfer the results from Section 2 to bicircular projections on C*-algebras.

Theorem 3.3. *Let A be a C*-algebra. Let $P : A \to A$ be a bicircular projection. Then there exist a $*$-ideal I of A and a selfadjoint projection $p \in M(I^\perp \oplus I^{\perp\perp})$ such that $P(x) = px$ for every $x \in I^\perp$ and $P(x) = xp$ for every $x \in I^{\perp\perp}$.*

Corollary 3.4. *Let A be a commutative C*-algebra and let $P : A \to A$ be a bicircular projection. Then there exists a selfadjoint projection $p \in M(A)$ such that $P(x) = px = xp$ for every $x \in A$.*

Corollary 3.5. *Let A be a prime C*-algebra and let $P : A \to A$ be a bicircular projection. Then there exists a selfadjoint projection $p \in M(A)$ such that $P(x) = px$ for every $x \in A$ or $P(x) = xp$ for every $x \in A$.*

If A is $K(\mathscr{H})$ or $B(\mathscr{H})$ (the algebra of all compact linear operators on a Hilbert space $\mathscr{H}$ and the algebra of all bounded linear operators on $\mathscr{H}$), then $M(A) = B(\mathscr{H})$. For $A = K(\mathscr{H})$ this follows from Busby (1968, Theorem 3.9) and for $A = B(\mathscr{H})$, we use the fact that $M(A) = A$ since A is unital. Thus, from the above corollary, we get

Corollary 3.6. *Let A be $K(\mathscr{H})$ or $B(\mathscr{H})$ and let $P : A \to A$ be a bicircular projection. Then there exists a selfadjoint projection $p \in B(\mathscr{H})$ such that $P(x) = px$ for every $x \in A$ or $P(x) = xp$ for every $x \in A$.*

Let us emphasize that the structure of bicircular projections in the case when $A = B(\mathscr{H})$ was already determined in Stachó and Zalar (2004, Theorem 2.2).

Corollary 3.7. *Let $\{A_k : k \in K\}$ be a family of prime C^*-algebras and let $A = \overline{\bigoplus_{k\in K} A_k}$ (the closure with respect to the norm given by $\|(x_k)\| = \sup_{k\in K} \|x_k\|$). Then, for every ideal I of A we have $I^{\perp} \oplus I^{\perp\perp} = A$. Further, if $P : A \to A$ is a bicircular projection, then there exist a $*$-ideal I of A and a selfadjoint projection $p \in M(A)$ such that $P(x) = py + zp$, where $y \in I^{\perp}$ and $z \in I^{\perp\perp}$ are such that $y + z = x$.*

Proof. As in the proof of Corollary 2.6, we conclude that

$$I^{\perp} = \overline{\bigoplus_{k\in K_0} A_k}, \qquad I^{\perp\perp} = \overline{\bigoplus_{k\in K\setminus K_0} A_k},$$

where $K_0 = \{k \in K : I \cap R_k = 0\}$. It is easy to verify that

$$\overline{\bigoplus_{k\in K} A_k} = \overline{\bigoplus_{k\in K_0} A_k} \oplus \overline{\bigoplus_{k\in K\setminus K_0} A_k},$$

that is $A = I^{\perp} \oplus I^{\perp\perp}$. □

Corollary 3.8. *Let A be a C^*-subalgebra of $K(\mathscr{H})$. Then for every ideal I of A we have $I^{\perp} \oplus I^{\perp\perp} = A$. Furthermore, if $P : A \to A$ is a bicircular projection, then there exist a $*$-ideal I of A and a selfadjoint projection $p \in M(A)$ such that $P(x) = py + zp$, where $y \in I^{\perp}$ and $z \in I^{\perp\perp}$ are such that $y + z = x$.*

Proof. According to Arveson (1976, Theorem 1.4.5), A is isometrically $*$-isomorphic to the closure (with respect to the norm given by $\|(x_j)\| = \sup_{j\in J} \|x_j\|$) of the direct sum of prime C*-algebras $K(\mathscr{H}_j)$, $j \in J$, so we can apply the previous corollary. □

Corollary 3.9. *Let A be a von Neumann algebra. Then $I^{\perp} \oplus I^{\perp\perp} = A$ for every ideal I of A. Further, if $P : A \to A$ is a bicircular projection, then there exist a $*$-ideal I of A and a selfadjoint projection $p \in M(A)$ such that $P(x) = py + zp$, where $y \in I^{\perp}$ and $z \in I^{\perp\perp}$ are such that $y + z = x$.*

Corollary 3.10. *Let A be a factor and let $P : A \to A$ be a bicircular projection. Then there exists a selfadjoint projection $p \in M(A)$ such that $P(x) = px$ for every $x \in A$ or $P(x) = xp$ for every $x \in A$.*

ACKNOWLEDGMENT

This article was written during the second author's stay at the University of Maribor, supported by the Alps-Adria Research scholarship. The authors would like to thank Matej Brešar for his help and encouragement, Borut Zalar, who acquainted us with the recent results on bicircular projections, and the referee, whose comments helped us improve the article.

REFERENCES

Ara, P., Mathieu, M. (2003). Local multipliers of C*-algebras. London: Springer-Verlag.

Arveson, W. B. (1976). An invitation to C*-algebras. Graduate Texts in Mathematics No. 39. New York-Heidelberg: Springer-Verlag.

Brešar, M. (1989). Jordan mappings of semiprime rings. *J. Algebra* 127:218–228.

Busby, R. C. (1968). Double centralizers and extensions of C*-algebras. *Trans. Amer. Math. Soc.* 132:79–99.

Dang, T., Friedman, Y., Russo, B. (1990). Affine geometric proof of the Banach Stone theorems of Kadison and Kaup. *Rocky Mountain J. Math.* 20:409–428.

Murphy, G. J. (1990). C*-algebras and operator theory. Boston: Academic Press, Inc.

Stachó, L. L., Zalar, B. (2004). Bicircular projections on some matrix and operator spaces. *Linear Algebra Appl.* 384:9–20.