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Fixed Point and Equilibrium Theorems in Pseudoconvex and Related Spaces

FERENC FORGO and ISTVÁN JOÓ
Budapest University of Economic Sciences, Department of Operations Research, H–1828 Budapest, Hungary (e-mail: forgo@nero.bke.hu)

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Abstract. Convexity spaces defined in the paper are generated by families of continuous functions. Without imposing any explicitly stated linear structure on the spaces, Browder's, Brouwer's and Kakutani's fixed point theorems are proved and used for deriving generalized Fan inequalities and two-function minimax theorems. The existence of Nash equilibria in noncooperative games is also established under more general conditions than known before. The convexity spaces studied in the paper allow for unusual (generalized) convex sets and functions.

Key words: Convexity, Fixed points, Minimax, Nash equilibrium

1. Introduction

Generalizations of the usual convexity in $\mathbf{R}^n$ have taken numerous directions during the past fifty years. For an inspirational source and overview we refer to Danzer et al. (1993), Guddler (1997), Soltan (1984) and Van de Vel (1993). Beginning with such classics as von Neumann and Morgenstern's (1944) utility theory and Stone's (1949) axiomatization of convex structures an interesting avenue of research has been the extension of important features of convexity to spaces without an explicitly stated linear structure. An important milestone along the way was the ground breaking work of Fan (1952, 1953) who introduced, what was later termed, convex (concave)-like functions. The minimax theorem he proved for convex–concave-like functions launched an avalanche of one- and two-function minimax theorems. For an excellent review see Simons (1995). Danzer et al. (1963) have captured the very essence of generalized convexity when they say: 'The usual procedure in defining a generalized convexity is to select a property of convex sets in $\mathbf{R}^n$ or $\mathbf{E}^n$ which is either characteristic of convexity or essential in the proof of some important theorem about convex sets, and to formulate that property or a suitable variant in other settings.' When it comes to applications in game theory and economics, the real test of the power of any sort of convexity is whether the basic fixed point theorems (Brouwer's, Browder's, Kakutani's) can be carried over under reasonable conditions. This has led Joó (1989) to generalizing Komiya's (1981) convex spaces to pseudoconvex spaces where he could prove Browder's fixed-point theorem and a Nikaido–Isoda type theorem for the existence of Nash equilibrium.

To prove the existence of Nash equilibrium without any explicitly stated linear structure by applying Fan-convexity in its original form fails as is shown by the counterexample of Joó (1986). By adding continuity to Fan-convexity, calling it CF-convexity, Forgó (1994) proved a Nikaido–Isoda type existence theorem for Nash equilibrium in n-person noncooperative games.

In this paper, we bring the two ideas, pseudoconvexity and CF-convexity together since, inherently, they are very close to each other. In one, convex sets are defined as sets invariant under an abstract convex hull operation and then convex (quasi-convex) functions are defined in the usual way over these convex sets. In the other, convex hulls of sets are generated by special classes of continuous functions. It turns out that the basic fixed point theorems remain valid in very general spaces (more general than pseudoconvex). Based on these theorems, generalizations of Fan's inequality, Nash equilibria in noncooperative games and two-function minimax theorems are proved. These theorems can both be stated in appropriate convexity spaces and in terms of the generating functions.

Examples are also provided to show that even in finite dimension, there are convex sets in generalized pseudoconvex spaces generated by continuous functions that are not homeomorphic to any traditional convex set.

2. Generalized pseudoconvex spaces

For the simplicity of exposition, whenever we say topological space we always mean a Hausdorff space in which every point has a denumerable neighborhood base (M_1, T_2-space) even if less would be enough at certain places.

Let $X \neq \emptyset$ be a topological space and $\langle\cdot\rangle$ the usual convex hull operation in $\mathbf{R}^n$. Denote the standard simplex in $\mathbf{R}^n$ by $\triangle^n := \langle e_1, \ldots, e_n \rangle$ where e_i $(i = 1, \ldots, n)$ are the unit vectors.

Let a continuous function $\Phi_F : \triangle^n \longrightarrow X$ be defined for any finite set $F := \{x_1, \ldots, x_n\} \subset X$. We will call the family

$$\mathcal{F} := \{\Phi_F : F \subset X \text{ is finite}\}$$

a Φ-*family* and its members Φ-*functions*. Several Φ-families can be defined. If just one function is changed for a particular F, we will have a different family. A Φ-function Φ_F assigns 'generalized convex combinations' of the points in F to any n-tuple of weights in $\triangle^n$.

A Φ-family $\mathcal{G}$ is called a Ψ-*family* and its members Ψ-*functions*, if the functions Ψ_F in $\mathcal{G}$ are 'properly synchronized', i.e. for any $F := \{x_1, \ldots, x_n\}$ and for each subsimplex $\left\langle e_{i_1}, \ldots, e_{i_k} \right\rangle \subset \triangle^n$,

$$\Psi_F \left(\left\langle e_{i_1}, \ldots, e_{i_k} \right\rangle \right) = \Psi_G \left(\langle e_1, \ldots, e_k \rangle \right)$$

where $G := \{x_{i_1}, \ldots, x_{i_k}\} \subset F,\ e_{i_j} \in \triangle^n, e_j \in \triangle^k,\ (j = 1, \ldots, k)$.

Define a map $h : 2^X \longrightarrow 2^X$ which may have some (or all) of the following properties:

(H1) $h(\emptyset) = \emptyset$;

(H2) $h(A) \neq \emptyset$ if $A \neq \emptyset,\ A \subset X$;

(H2′) $h(\{x\}) = x, \quad x \in X$;

(H3) $h(F)$ is compact for any finite sets $F \subset X$ and
$h(A) = \cup\{h(F) : F \subset A$ is a finite set$\}, \quad A \subset X$,

(H4) $h(h(A)) \subset h(A), \quad A \subset X$.

We will use the following terminology:

h_1-*operation*: h satisfies H1, H2, H3;

h_2-*operation*: h satisfies H1, H2, H3, H4;

h_3-*operation*: h satisfies H1, H2′, H3, H4.

h_3 is the usual *convex hull operation* as defined by Komiya (1981) and adopted by Joó (1989). This is so because H2′ and H4 imply $h(h(A)) = h(A)$ for any $A \subset X$.

For each operation h_j $(j = 1, 2, 3)$, a set $A \subset X$ is said to be *semiconvex* if $h_j(A) \subset A$ and *convex* if $h_j(A) = A$.

It is easily seen that if A and B are semiconvex for the operation h_1, then so is $A \cap B$ for any $A, B \subset X$. This does not hold for convex sets because H2′ is not required. Of course, the intersection of convex sets is convex for operation h_3.

Define for each $j = 1, 2, 3$:

$\mathcal{F}_{1,j}$: a Φ-family such that for any n, $F := \{x_1, \ldots, x_n\}$ and for each subsimplex $\langle e_{i_1}, \ldots, e_{i_k}\rangle \subset \triangle^n$ we have

$$\Phi_F\left(\left\langle e_{i_1}, \ldots, e_{i_k}\right\rangle\right) \subset h_j\left(\left\{x_{i_1}, \ldots, x_{i_k}\right\}\right), \tag{1}$$

$\mathcal{F}_{2,j}$: same as $\mathcal{F}_{1,j}$ with the inclusion in (1) replaced by equality:

$$\Phi_F\left(\left\langle e_{i_1}, \ldots, e_{i_k}\right\rangle\right) = h_j\left(\left\{x_{i_1}, \ldots, x_{i_k}\right\}\right). \tag{2}$$

Similarly, we define $\mathcal{G}_{1,j}$ and $\mathcal{G}_{2,j}$ $(j = 1, 2, 3)$ for Ψ-families, when in (1) and (2) Φ_F is replaced by Ψ_F.

For any $i = 1, 2$; $j = 1, 2, 3$; the triple $(X, h_j, \mathcal{F}_{i,j})$ will be called a $P_{i,j}$-*space*, while the triple $(X, h_j, \mathcal{G}_{i,j})$ a $Q_{i,j}$-*space*.

Note that $P_{2,3}$ is the pseudoconvex space introduced by Joó (1989) and studied by Joó (1989) and Joó and Kassay (1995). It is easy to see that $Q_{2,3}$ is also a pseudoconvex space with a Φ-family having a special property.

The following diagram is meant to clarify the relationships among spaces defined above ($P \Longrightarrow P'$ means that P is a P'-space).

$$\begin{array}{ccccccccccc}
P_{1,1} & \Longleftarrow & P_{1,2} & \Longleftarrow & P_{1,3} & \qquad & Q_{1,1} & \Longleftarrow & Q_{1,2} & \Longleftarrow & Q_{1,3} \\
\Uparrow & & \Uparrow & & \Uparrow & & \Uparrow & & \Uparrow & & \Uparrow \\
P_{2,1} & \Longleftarrow & P_{2,2} & \Longleftarrow & P_{2,3} & & Q_{2,1} & \Longleftarrow & Q_{2,2} & \Longleftarrow & Q_{2,3}
\end{array}$$

$$Q_{i,j} \Longrightarrow P_{i,j} \qquad i = 1, 2; \quad j = 1, 2, 3.$$

We say that a Φ-family (Ψ-family) *generates* the map h if $h(F) = \Phi_F(\triangle^n)$ (or $h(F) = \Psi_F(\triangle^n)$) for any n and finite set $F := \{x_1, \dots, x_n\}$ with the usual convention $h(\emptyset) = \emptyset$.

Denote $G(x, n)$, $(n \geqslant 1)$ a denumerable neighborhood base of $x \in X$ which is kept fixed throughout and define the neighborhood base of any set $A \subset X$ as

$$G(A, n) := \bigcup_{x \in A} G(x, n)\,.$$

The case when the Φ-family (Ψ-family) is equicontinuous deserves special attention. Define for any $n \in \mathbf{N}$, $\lambda \in \triangle^n$, $\Phi^n_\lambda : X^n \longrightarrow X,\ \Phi^n_\lambda(x_1, \dots, x_n) := \Phi_F(\lambda)$ if $F := \{x_1, \dots, x_n\}$. The Φ-family is said to be *equicontinuous* (relative to the fixed base G), if for any $k \in \mathbf{N}$, there is an $\ell = \ell(k)$ such that

$$y \in \prod_{j=1}^{n} G(x_j, \ell) \Longrightarrow \Phi^n_\lambda(y) \in G\left(\Phi^n_\lambda(x), k\right)$$

for all $n \in \mathbf{N},\ x := (x_1, \dots, x_n) \in X^n,\ \lambda \in \triangle^n$. Note that ℓ is not allowed to depend on n, x and λ. The definition is analogous for Ψ-families.

At certain junctures, continuity of operation h will play an important role. h is said to be *continuous* if for any $A \subset X$, $n \in \mathbf{N}$, there exists a $k = k(A, n)$, such that

$$B \subset G(A, k) \Longrightarrow h(B) \subset G(h(A), n)\,.$$

PROPOSITION 1. *If the map h is generated by an equicontinuous Φ-family (or Ψ-family), then it is continuous.*

Proof. It is enough to prove the proposition for Φ-families. Let $A \subset X$ be arbitrary. Based on the definition of h and Φ^n_λ,

$$h(A) := \bigcup_{n \in \mathbf{N}} \bigcup_{\lambda \in \triangle^n} \bigcup_{x \in A^n} \Phi^n_\lambda(x)\,.$$

By the definition of equicontinuity, to any $k \in \mathbf{N}$, there is an $\ell = \ell(k)$, such that

$$y \in \prod_{j=1}^{n} G(x_j, \ell) \Longrightarrow \Phi^n_\lambda(y) \in G\left(\Phi^n_\lambda(x), k\right)$$

holds for any $n \in \mathbf{N},\ \lambda \in \triangle^n$ and $x \in A^n$.

Let now $B \subset G(A, \ell)$ be arbitrary. Then the above implication holds for all $y \in B^n,\ n \in \mathbf{N},\ \lambda \in \triangle^n$ and $x \in A^n$. Thus by the definition of h, we have

$$B \subset G(A, \ell) \Longrightarrow h(B) \subset G(h(A), k)$$

that is, h is continuous. $\square$

REMARK 1. If A is compact, then it is not hard to show that ℓ can be chosen not to depend on x and λ. Independence of n still remains to be verified in actual cases.

EXAMPLE 1. Let $X := [-1, 1] \subset \mathbf{R}$ with the Euclidean topology, $F := \{x_1, \ldots, x_n\} \subset X$, $\lambda := (\lambda_1, \ldots, \lambda_n) \in \triangle^n$,

$$\psi_F(\lambda) := \sqrt{\lambda_1 x_1^2 + \cdots + \lambda_n x_n^2}\,,$$
$$h(F) := \Psi_F(\triangle^n)\,.$$

Thus, the Ψ-family generates a $Q_{2,1}$-space. We will verify that h is continuous by showing that the Ψ-family is equicontinuous.

Let $A \subset X$ be arbitrary, $\varepsilon > 0$, $x := (x_1, \ldots, x_n) \in A^n$, $\lambda := (\lambda_1, \ldots, \lambda_n) \in \triangle^n$ and $y := (y_1, \ldots, y_n) \in X^n$ such that $|y_j - x_j| \leqslant \varepsilon$ for all $j = 1, \ldots, n$. By simple algebra, we can see that

$$\left|\sqrt{\lambda_1 x_1^2 + \cdots + \lambda_n x_n^2} - \sqrt{\lambda_1 y_1^2 + \cdots + \lambda_n y_n^2}\right| \leqslant \varepsilon$$

where the right hand side does not depend on n, x and λ.

In the following, when referring to spaces $P_{i,j}$, $Q_{i,j}$ with a continuous h-operation, we will use the notation $\overline{P}_{i,j}$, $\overline{Q}_{i,j}$.

For the map h to be an h_j-operation, the generating families must satisfy certain conditions.

PROPOSITION 2. *The space $(X, h, \mathcal{G})$ generated by a Ψ-family is a $Q_{2,1}$-space.*

Proof. H1, H2, H3 is always required and (2) obviously follows from $h\left(\{x_{i_1}, \ldots, x_{i_k}\}\right) = \Psi_G\left(\langle e_{i_1}, \ldots, e_{i_k}\rangle\right)$. □

The Φ-family $\mathcal{F}$ (or Ψ-family $\mathcal{G}$) is said to have the *composition property* if for any finite sets and simplexes

$$\begin{aligned}
&F := \{x_1, \ldots, x_k\}, \qquad \lambda := (\lambda_1, \ldots, \lambda_k) \in \triangle^k;\\
&F_1 := \left\{z_1^1, \ldots, z_{\ell_1}^1\right\}, \qquad \mu^1 := \left(\mu_1^1, \ldots, \mu_{\ell_1}^1\right) \in \triangle^{\ell_1};\\
&F_1 \subset h(F), \quad x_1 = \Phi_{F_1}(\mu^1)\\
&\quad \ldots\\
&F_k := \left\{z_1^k, \ldots, z_{\ell_k}^k\right\}, \qquad \mu^k := \left(\mu_1^k, \ldots, \mu_{\ell_k}^k\right) \in \triangle^{\ell_k};\\
&F_k \subset h(F),\ x_k = \Phi_{F_k}(\mu^k)\\
&G := F_1 \cup \cdots \cup F_k := \{y_1, \ldots, y_r\}
\end{aligned}$$

and with $\gamma := (\gamma_1, \ldots, \gamma_r) \in \triangle^r$ defined as

$$\gamma_j := \sum_{t=1}^{k} \sum_{s=1}^{\ell_t} \left(\lambda_t \cdot \mu_s^t\right) \delta_{sj}^t$$

where

$$\delta_{sj}^t := \begin{cases} 1 & \text{if } y_j = z_s^t \\ 0 & \text{otherwise} \end{cases}$$

we have

$$\Phi_F(\lambda) = \Phi_G(\gamma)$$
$$(\text{or } \Psi_F(\lambda) = \Psi_G(\gamma))\,.$$

PROPOSITION 3. *If $\mathcal{G}$ has the composition property, then the space $(X, h, \mathcal{G})$ generated by a Ψ-family is a $Q_{2,2}$-space.*

Proof. By Proposition 2, we only have to show that (H4) is satisfied. Let $d \in h(h(A))$. By H3, and since h is generated by a Ψ-family, $d = \Psi_F(\lambda)$ for a finite set $F := \{x_1, \dots, x_k\} \subset h(A)$, $\lambda \in \triangle^k$. Since for any $i \in (1, \dots, k)$, $x_i \in F \subset h(A)$, therefore $x_i = \Psi_{F_i}(\mu^i)$, $\mu^i \in \triangle^i$, where $F_i := \left\{z_1^i, \dots, z_{\ell_i}^i\right\} \subset A$. Let $G = \cup_{i=1}^k F_i := \{y_1, \dots, y_r\}$. Clearly, $G \subset A$. By defining $\gamma := (\gamma_1, \dots, \gamma_r) \in \triangle^r$ as

$$\gamma_j := \sum_{t=1}^{k} \sum_{s=1}^{\ell_t} (\lambda_t \mu_s^t) \delta_s^t\,, \qquad \delta_s^t := \begin{cases} 1 & \text{if } y_j = z_s^t \\ 0 & \text{otherwise}\,, \end{cases}$$

we have, by the composition property, that

$$d = \Psi_F(\lambda) = \Psi_G(\gamma) \subset h(A)$$

which was to be proved. □

PROPOSITION 4. *If $\Psi_{\{x\}}(\triangle^1) = \{x\}$ for any $x \in X$, and $\mathcal{G}$ has the composition property, then the space $(X, h, \mathcal{G})$ generated by a Ψ-family is a $Q_{2,3}$-space (pseudoconvex space).*

Proof. Since $x = h(\{x\})$ for any $x \in X$, therefore $A \subset h(A)$, $A \subset X$ which implies $h(A) \subset h(h(A))$. By Proposition 3, $h(h(A)) \subset h(A)$ implying $h(h(A)) = h(A)$. Thus H1, H2′, H3 and H4 are satisfied which means that h is an h_3-operation. □

For all practical purposes and in order to establish connection between (generalized) pseudoconvexity and CF-convexity, two-point generation of P- and Q-spaces deserves special attention. We will call a two-point generating function *w-function* which is a continuous function $w : X^2 \times \triangle^2 \longrightarrow X$ satisfying some (or all) of the following conditions:

(W1) $w(x_1, x_2; \lambda_1, \lambda_2) = w(x_2, x_1; \lambda_2, \lambda_1)$, for all $x_1, x_2 \in X$, $(\lambda_1, \lambda_2) \in \triangle^2$;

(W2) $w(x_1, x_2; 1, 0) = w(x_1, x_3; 1, 0)$, for all $x_1, x_2, x_3 \in X$,
$w(x_1, x_2; 0, 1) = w(x_3, x_2; 0, 1)$, for all $x_1, x_2, x_3 \in X$,

(W2′) $w(x_1, x_2; 1, 0) = x_1$ for all $x_1, x_2 \in X$,

$w(x_1, x_2; 0, 1) = x_2 \quad$ for all $x_1, x_2 \in X$,

(W3) $w(x, x; \lambda_1, \lambda_2) = w(x, x; \lambda_3, \lambda_4)$,
for all $x \in X$, $\quad (\lambda_1, \lambda_2) \in \triangle^2$, $(\lambda_3, \lambda_4) \in \triangle^2$,

(W3′) $w(x, x; \lambda_1, \lambda_2) = x$, $\quad$ for all $x \in X$, $\quad (\lambda_1, \lambda_2) \in \triangle^2$,

(W4) $w\left[w\left(x_1, x_2; \dfrac{\lambda_1}{\lambda_1+\lambda_2}, \dfrac{\lambda_2}{\lambda_1+\lambda_2}\right), x_3; \lambda_1+\lambda_2, \lambda_3\right] =$

$$\begin{cases} w\left[x_1, w\left(x_2, x_3; \dfrac{\lambda_2}{\lambda_2+\lambda_3}, \dfrac{\lambda_3}{\lambda_2+\lambda_3}\right); \lambda_1, \lambda_2+\lambda_3\right] & \text{if } \lambda_2+\lambda_3 \neq 0\,, \\ \omega(x_1, x_3; 1, 0) \text{ if } \lambda_2+\lambda_3 = 0\,, \end{cases}$$

$$\text{for all } x_1, x_2, x_3 \in X, \quad (\lambda_1, \lambda_2, \lambda_3) \in \triangle^3, \quad \lambda_1+\lambda_2 \neq 0\,.$$

Clearly, W2′ and W3′ imply W2 and W3 respectively.

Two-point generation of P- and Q-spaces, though different, but closely resembles the generation of convex structures by “mixtures”. For details see Gudder (1977).

PROPOSITION 5. *If w satisfies W1, W2, W3 and W4, then the recursion*

$$\Psi_k(x_1, \ldots, x_k; \lambda_1, \ldots, \lambda_k) :=$$

$$\begin{cases} w\left(\Psi_{k-1}\left(x_1, \ldots, x_{k-1}; \dfrac{\lambda_1}{\sum_{i=1}^{k-1}\lambda_i}, \ldots, \dfrac{\lambda_{k-1}}{\sum_{i=1}^{k-1}\lambda_i}\right), \quad x_k; \displaystyle\sum_{i=1}^{k-1}\lambda_i, \lambda_k\right) \\ \qquad\qquad \textit{if } \sum_{i=1}^{k-1}\lambda_i > 0, \\ w(x_k, x_1; 1, 0) \textit{ if } \sum_{i=1}^{k-1}\lambda_i = 0 \end{cases}$$

for $k \geqslant 3$, and

$$\Psi_2(x_1, x_2; \lambda_1, \lambda_2) := w(x_1, x_2; \lambda_1, \lambda_2)$$
$$\Psi_1(x, 1) := w(x, x, 1, 0)$$

produces a Ψ-family by the projection

$$\Psi_F(\lambda_1, \ldots, \lambda_k) := w[\Psi_k(x_1, \ldots, x_k; \lambda_1, \ldots, \lambda_k), x_1; 1, 0] \text{ if}$$
$$F := \{x_1, \ldots, x_k\}, \; k \geqslant 1\,.$$

Proof. First we show that if $F := \{x_1, \ldots, x_k\}$ is a finite set of distinct points from X, then Ψ_F is well-defined, i.e. Ψ_F does not depend on how the points of F are indexed. If $|F| = 1$, or $|F| = 2$, then by W1, W2 and W3, Ψ_F is well-defined. If $|F| \geqslant 3$, then easy but lengthy calculation can show that the way the recursion is defined, W1, W2 and W4 together ensure that Ψ_F is independent of the order in which w is applied successively to points of F. Continuity of Ψ_F comes from the continuity of w.

Lastly, if $G \subset F$ (for simplicity, without loss of generality, we may assume that $G := \{x_1, \ldots, x_{k-1}\}$, $\quad F := \{x_1, \ldots, x_{k-1}, x_k\}$), then

$$\begin{aligned}\Psi_F(\lambda_1, \ldots, \lambda_{k-1}, 0) &= \Psi_k(x_1, \ldots, x_{k-1}, x_k; \lambda_1, \ldots, \lambda_{k-1}, 0)\\ &= w(\Psi_{k-1}(x_1, \ldots, x_{k-1}; \lambda_1, \ldots, \lambda_{k-1}), x_k; 1, 0)\\ &= w(\Psi_{k-1}(x_1, \ldots, x_{k-1}; \lambda_1, \ldots, \lambda_{k-1}), x_1; 1, 0) = \Psi_G(\lambda_1, \ldots, \lambda_{k-1}).\end{aligned}$$

Thus, since $(\lambda_1, \ldots, \lambda_{k-1}) \in \triangle^{k-1}$, $\Psi_F(\langle e_1, \ldots, e_{k-1}\rangle) = \Psi_G(\langle e_1, \ldots, e_{k-1}\rangle)$ and Ψ_F is a Ψ-function. □

Since in all the P- and Q-spaces we have discussed so far convex (semiconvex) sets are well-defined, following Joó and Kassay (1995), convex and quasiconvex functions can be defined in a natural way. If $(X, h, \mathcal{F})$ is a P-space, or $(X, h, \mathcal{G})$ is a Q-space, then $f : X \longrightarrow \mathbf{R}$ is said to be (*quasi*)*concave* if the composite function $f \circ \Phi_F$ (or $f \circ \Psi_F$) is (quasi)concave in the usual sense for any finite set F. f is said to be (*quasi*)*convex* if $-f$ is quasi(concave). In other words, if for any finite set $F := \{x_1, \ldots, x_n\}$ and $(\lambda_1, \ldots, \lambda_n) \in \triangle^n$,

$$f(\Phi_F(\lambda_1, \ldots, \lambda_n)) \geqslant \lambda_1 f(x_1) + \cdots + \lambda_n f(x_n), \tag{3}$$

then f is concave, if

$$f(\Phi_F(\lambda_1, \ldots, \lambda_n)) \geqslant \min\{f(x_1), \ldots, f(x_n)\},$$

then it is quasiconcave. (Φ_F should be replaced by Ψ_F if we work with Q-spaces).

PROPOSITION 6. *If f is quasiconcave in a $P_{2,1}$-space, then $L_a := \{x \in X : f(x) > a\}$ and $\overline{L}_a := \{x \in X : f(x) \geqslant a\}$ are semiconvex. If L_a (or $\overline{L}_a$) is semiconvex in a $P_{1,1}$-space for any $a \in \mathbf{R}$, then f is quasiconcave.*

Proof. To prove the first assertion, we have to show that $h(L_a) \subset L_a$. Let $x \in h(L_a)$. Then there is a finite set $F := \{x_1, \ldots, x_n\} \subset L_a$ and continuous function Φ_F such that $x = \Phi_F(\lambda_1, \ldots, \lambda_n)$ for some $(\lambda_1, \ldots, \lambda_n) \in \triangle^n$ and

$$f(\Phi_F(\lambda_1, \ldots, \lambda_n)) \geqslant \min\{f(x_1), \ldots, f(x_n)\} > a$$

which is exactly what was to be proved.

To prove the second assertion, let $F := \{x_1, \ldots, x_n\}$ and $\lambda := (\lambda_1, \ldots, \lambda_n) \in \triangle^n$. Let $a := \min\{f(x_1), \ldots, f(x_n)\}$. Then the following inclusions hold true by (1), $F \subset L_a$ and the semiconvexity of L_a

$$\Phi_F(\lambda) \subset h(F) \subset h(L_a) \subset L_a.$$

$\Phi_F(\lambda) \subset L_a$ is exactly the definition of quasiconcavity. □

The case when a Q-space is generated by an w-function is especially important since, in this case, a direct link can be established between CF-concavity and the concavity defined above. We recall the definition of CF-concavity from Forgó (1994).

Let X be a topological space and $f : X \longrightarrow \mathbf{R}$. f is said to be *CF-concave* if there is a continuous function w such that for any $x_1, x_2 \in X$ and $(\lambda_1, \lambda_2) \in \triangle^2$

$$f(w(x_1, x_2; \lambda_1, \lambda_2)) \geqslant \lambda_1 f(x_1) + \lambda_2 f(x_2)\,. \tag{4}$$

f is said to be *CF-quasiconcave* if instead of (4) we require

$$f(w(x_1, x_2; \lambda_1, \lambda_2)) \geqslant \min\{f(x_1), f(x_2)\}\,. \tag{5}$$

Note that beyond continuity, nothing else is assumed about w.

PROPOSITION 7. *If the w-function in* (4) *satisfies W1, W2, W3 and W4 and w generates a $Q_{2,2}$-space through the Ψ-family defined in Proposition 5, then any CF-(quasi)concave function is also (quasi)concave in $Q_{2,2}$.*

Proof. For concave functions the proposition was proved in Forgó (1994, Lemma 1). For quasiconcave functions, we use the same method of proof. By induction, we assume that for each $j \leqslant k-1$ $(k \geqslant 3)$ there is a continuous function Ψ_j generated by w recursively as in Proposition 5 and for which

$$f(\Psi_j(x_1, \ldots, x_j; \lambda_1, \ldots \lambda_j)) \geqslant \min\{f(x_1), \ldots, f(x_j)\}\,.$$

For $k = 3$, the assertion obviously holds by (5).

For $k \geqslant 4$, let

$$\begin{aligned}
&\Psi_k(x_1, \ldots, x_k; \lambda_1, \ldots, \lambda_k)\\
&\quad = w\left(\left(\Psi_{k-1}(x_1, \ldots, x_{k-1}; \frac{\lambda_1}{\sum_{i=1}^{k-1}\lambda_i}, \ldots, \frac{\lambda_{k-1}}{\sum_{i=1}^{k-1}\lambda_i}\right), x_k;\ \sum_{i=1}^{k-1}\lambda_i, \lambda_k\right)
\end{aligned}$$

assuming $\sum_{i=1}^{k-1}\lambda_i \neq 0$. By W2 and W4, it does not matter how $x_1, \ldots, x_k$ are indexed. By induction, and since f is CF-quasi-concave, we have

$$\begin{aligned}
&f(\Psi_k(x_1, \ldots, x_k; \lambda_1, \ldots, \lambda_k))\\
&\quad = f\left[\left(\Psi_{k-1}\left(x_1, \ldots, x_{k-1}; \frac{\lambda_1}{\sum_{i=1}^{k-1}\lambda_i}, \ldots, \frac{\lambda_{k-1}}{\sum_{i=1}^{k-1}\lambda_i}\right), x_k\right); \sum_{i=1}^{k-1}\lambda_i, \lambda_k\right]\\
&\quad \geqslant \min\left[f\left(\Psi_{k-1}\left(x_1, \ldots, x_{k-1}, \frac{\lambda_1}{\sum_{i=1}^{k-1}\lambda_i}, \ldots, \frac{\lambda_{k-1}}{\sum_{i=1}^{k-1}\lambda_i}\right)\right), f(x_k)\right]\\
&\quad \geqslant \min\{f(x_1), \ldots, f(x_{k-1}), f(x_k)\}\,. \qquad \square
\end{aligned}$$

Now, we will show that even if nothing but continuity is assumed of w, a $P_{1,1}$-space can be generated by the proper choice of h. Let π_k be a particular order of the elements of the finite set $F := \{x_1, \ldots, x_k\}$ and denote Π_k all the $k!$ possible orders. By the recursion in Proposition 5, a continuous function $\Phi_k^{\pi_k} : \triangle^k \longrightarrow X$ can be defined with the repeated application of w. Let

$$h := \bigcup_{\pi_k \in \Pi_k} \Phi_k^{\pi_k}$$

and fix a particular order π_0. Define

$$\Phi_F := \Phi_k^{\pi_0}$$

and

$$\mathcal{F} := \{\Phi_F : F \subset X \ \text{ is finite } \}.$$

PROPOSITION 8. *$(X, h, \mathcal{F})$ is a $P_{1,1}$-space.*

Proof. The only thing we have to show is that (1) holds for any Φ_F $(F := \{x_1, \ldots, x_k\})$ and $\langle e_{i_1}, \ldots, e_{i_k} \rangle \subset \triangle^n$. The index set $\{i_1, \ldots, i_k\}$ can be supplemented to produce an order π_1 on F. By the way the recursion is defined (whether a particular $x_j \notin F$ or it carries a weight $\lambda_j = 0$ results in the same function value) we have

$$\begin{aligned} \Phi_F \left(\langle e_{i_1}, \ldots, e_{i_k} \rangle\right) = \Phi_n^{\pi_1} \left(\langle e_{i_1}, \ldots, e_{i_k} \rangle\right) &\subset \bigcup_{\pi_n \in \Pi_n} \Phi_n^{\pi_n} \left(\langle e_{i_1}, \ldots, e_{i_k} \rangle\right) \\ &= h\left(x_{i_1}, \ldots, x_{i_k}\right) . \qquad \square \end{aligned}$$

We give two simple examples to show that even in $\mathbf{R}^1$ and $\mathbf{R}^2$ with the Euclidean topology, there are semiconconvex and convex sets not homeomorphic to any usual convex set in $\mathbf{R}^1$ or $\mathbf{R}^2$.

EXAMPLE 2. Let $X := [-1, 1]$ and $F := \{x_1, \ldots, x_n\} \subset X$, $\lambda := (\lambda_1, \ldots, \lambda_n) \in \triangle^n$,

$$\Psi_F(\lambda) = \sqrt{\lambda_1 x_1^2 + \cdots + \lambda_n x_n^2},$$

and $Q_{2,1}$ the space generated by the Ψ-family.

Let $[a, b]$ be a line segment, $0 \leqslant a \leqslant b \leqslant 1$ and $A \subset [-b, -a]$ an arbitrary set. Then $B := A \cup [a, b]$ is a semiconvex set since $h(B) = [a, b] \subset B$.

EXAMPLE 3. Let $X \subset \mathbf{R}^2$, $X := \{x = (x_1, x_2) : x_1^2 + x_2^2 \leqslant 1\}$ and $w : X^2 \times \triangle^2 \longrightarrow X$

$$w(x, y; \lambda_1, \lambda_2) := \left(\sqrt{\lambda_1 x_1^2 + \lambda_2 y_1^2},\ \sqrt{\lambda_1 x_2^2 + \lambda_2 y_2^2}\right) .$$

Then easy calculation shows that the recursion in Proposition 5 produces the Ψ-functions:

$$\Psi_{\{x^{(1)}\}}(1) = \left(|x_1^{(1)}|, |x_2^{(1)}|\right)$$

$$\dots$$

$$\Psi_{\{x^{(1)},\dots,x^{(k)}\}}(\lambda_1, \dots, \lambda_k)$$
$$= \left(\sqrt{\lambda_1 x_1^{(1)^2} + \dots + \lambda_k x_1^{(k)^2}}, \sqrt{\lambda_1 x_2^{(1)^2} + \dots + \lambda_k x_2^{(k)^2}}\right),$$
$$(\lambda_1, \cdots, \lambda_k) \in \triangle^k .$$

Thus the space generated by the Ψ-family through (2) (and eventually through w) is a $Q_{2,1}$-space. In this space, any circle line $L(r) := \{(x_1, x_2) : x_1^2 + x_2^2 = r\}$ $(0 < r \leqslant 1)$ is a semiconvex set since for any finite set $F := \left\{x^{(1)}, \dots, x^{(k)}\right\} \subset L(r)$, $\quad h_1(F) \subset L(r)$. The circle line is known to be not homeomorphic to any convex set in $\mathbf{R}^1$ or $\mathbf{R}^2$.

Now define, for any $k \geqslant 1$, $F := \left\{x^{(1)}, \dots, x^{(k)}\right\}$ and $(\lambda_1, \dots, \lambda_k) \in \triangle^k$

$$\Psi_F^{++}(\lambda_1, \dots, \lambda_k) := (a, b)$$
$$\Psi_F^{+-}(\lambda_1, \dots, \lambda_k) := (a, -b)$$
$$\Psi_F^{-+}(\lambda_1, \dots, \lambda_k) := (-a, b)$$
$$\Psi_F^{--}(\lambda_1, \dots, \lambda_k) := (-a, -b)$$

where

$$a = \sqrt{\lambda_1 x_1^{(1)^2} + \dots + \lambda_k x_1^{(k)^2}}$$
$$b = \sqrt{\lambda_1 x_2^{(1)^2} + \dots + \lambda_k x_2^{(k)^2}} .$$

Let

$$h(F) := \Psi^{++}(\triangle^k) \cup \Psi^{+-}(\triangle^k) \cup \Psi^{-+}(\triangle^k) \cup \Psi^{--}(\triangle^k)$$

and

$$\mathcal{G} := \left\{\Psi_F^{++} : F \subset X \text{ is finite}\right\} .$$

Then $(X, h, \mathcal{G})$ is a $Q_{1,1}$-space in which any circle-line $L(r)$ is a convex set.

In subsequent sections, occasionally, we will be working with Cartesian products of P- and Q-spaces. We only define the Cartesian product of two P-spaces, the extension to finitely many spaces and Q-spaces is straightforward.

Let $P^{(1)} := (X^{(1)}, h^{(1)}, \mathcal{F}^{(1)})$ and $P^{(2)} := (X^{(2)}, h^{(2)}, \mathcal{F}^{(2)})$ be two P-spaces. Define $X^{(1,2)} := X^{(1)} \times X^{(2)}$ and $\mathcal{F}^{(1,2)} := \{\Phi_F^{(1,2)} : F \subset X^{(1,2)}$ is finite}, where

$$\Phi_F^{(1,2)} := \Phi_{F_1}^{(1)} \times \Phi_{F_2}^{(2)} : \triangle^n \longrightarrow X^{(1,2)}\,,$$
$$F := \{(x_1, y_1), \dots , (x_n, y_n)\}\,,$$
$$F_1 := \{(x_1, \dots , x_n)\} \subset X^{(1)}\,,$$
$$F_2 := \{(y_1, \dots , y_n)\} \subset X^{(2)}\,.$$

Furthermore,

$$h^{(1,2)}(A) := \cup\{h^{(1,2)}(F) : F \subset A \text{ is a finite set }\}\,,$$

where $h^{(1,2)}(F) := h^{(1)}(F_1) \times h^{(2)}(F_2)$. Now, the Cartesian product of $P^{(1)}$ and $P^{(2)}$ is defined as

$$P^{(1,2)} := (X^{(1,2)}, h^{(1,2)}, \mathcal{F}^{(1,2)})\,.$$

3. Fixed point theorems

Fixed point theorems in P- and Q-spaces, as generalizations of classical theorems of Brouwer, Browder and Kakutani, deserve special attention in their own right but also as vehicles for proving a host of important theorems in several areas of mathematics, game theory and economics.

The first significant result demonstrating the potential of pseudoconvex spaces was the proof of Browder's fixed point theorem by Joó (1989). Browder's theorem, however, remains valid in more general spaces, in particular, in the most general space we defined in Section 2.

THEOREM 1 (Browder). *Let $(X, h, \mathcal{F})$ be a compact $P_{1,1}$-space and $T : X \longrightarrow 2^X$ a map for which*

$$T(x) \neq \emptyset \text{ and semiconvex for all } x \in X\,;$$

$$T^{-1}(y) = \{x \in X : y \in T(x)\} \text{ is open in } X \text{ for all } \; y \in X\,.$$

Then there is a point $x_0 \in X$ for which $x_0 \in T(x_0)$.

Proof. $\cup_{y\in X}\{T^{-1}(y) : y \in X\} \supset X$ is an open covering of X. Since X is compact, there is a finite subcovering $\cup_{i=1}^n T^{-1}(y_i)$. Let $F := \{y_1, \dots , y_n\}$ and $A_i := T^{-1}(y_i) \cap h(F)$, $(i = 1, \dots , n)$. Then $A_1, \dots , A_n$ is an open covering of $h(F)$ which is compact by assumption H3. Therefore, there exists a partition of unity subordinate to this covering, i.e. there exist continuous functions $\beta_i : h(F) \longrightarrow \mathbf{R}$, such that $\beta_i \geqslant 0$, supp $\beta_i \subset A_i$ for all $i = 1, \dots , n$ and $\sum_{i=1}^n \beta_i = 1$. Define $g : h(F) \longrightarrow \triangle^n$, $g(z) := \sum_{i=1}^n \beta_i(z)e_i$. The composite function $g \circ \Phi_F : \triangle^n \longrightarrow \triangle^n$ is continuous and, by Brouwer's fixed point theorem in $\mathbf{R}^n$, has a fixed point

$\lambda^* \in \triangle^n$, i.e. $\lambda^* = g(\Phi_F(\lambda^*))$. Denoting $x^* := \Phi_F(\lambda^*)$, we have $x^* = \Phi_F(g(x^*))$. Let $\beta_{i_1}(x^*) > 0, \dots, \beta_{i_k}(x^*) > 0$, while the other coordinates are 0. By property (1), $x^* \in h(\{y_{i_1}, \dots, y_{i_k}\})$. Since supp $\beta_i \subset A_i$, $x^* \in A_{i_1} \cap \cdots \cap A_{i_k}$ from which, by the definition of T^{-1}, $y_{i_1}, \dots, y_{i_k} \in T(x^*)$. Since $T(x^*)$ is semiconvex, $x^* \in h(\{y_{i_1}, \dots, y_{i_k}\}) \subset T(x^*)$. □

To generalize Kakutani's fixed point theorem we will need the continuity of the h-operation.

THEOREM 2 (Kakutani). *Let $(K, h, \mathcal{F})$ be a compact $\overline{P}_{1,1}$-space and $f : K \longrightarrow 2^K$ a multifunction for which $f(x) \neq \emptyset$ and semiconvex for any $x \in K$ and the graph $G_f := \{(x, y) : x \in K, y \in f(x)\}$ is closed. Then there is an $x^* \in K$ for which $x^* \in f(x^*)$.*

Proof. For the proof, without loss of generality, we will assume that for any $x \in X$, the neighborhood base $G(x, n)$ has been 'synchronized' i.e. for any $N \in \mathbf{N}$, $x \in X$ and $x_n \longrightarrow x$, there is an n_0 such that

$$G(x_n, n) \subset G(x, N), \quad n \geqslant n_0. \tag{6}$$

Furthermore

$$y \in G(x, n) \Longleftrightarrow x \in G(y, n), \quad x, y \in X, \quad n \in \mathbf{N}. \tag{7}$$

We also need the following lemma.

LEMMA 1. For any $x \in K$, $n \in \mathbf{N}$, there is an $N \in \mathbf{N}$, such that

$$x' \in G(x, N) \Longrightarrow f(x') \subset G(f(x), n).$$

Proof of lemma. Suppose that Lemma 1 is not true. Then there are $x_0 \in X$, $n_0 \in \mathbf{N}$ such that there is an $x'_N \in G(x_0, N)$, $y_N \in f(x'_N)$ but $y_N \notin G(f(x_0), n_0)$ for any $N \in \mathbf{N}$. Let $x'_N \longrightarrow x_0$. By compactness, there is a sequence $\{y_{N_k}\}$ converging to an $y^* \in K$. Since G_f is closed, we have $x'_{N_k} \longrightarrow x_0$, $y_{N_k} \longrightarrow y^*$, $y_{N_k} \in f(x_{N_k}) \Longrightarrow y^* \in f(x_0)$. On the other hand, $K \setminus G(f(x_0), n)$ being closed and $y_N \in K \setminus G(f(x_0), n)$, we have $y^* \notin G(f(x_0), n_0) \Longrightarrow y^* \notin f(x_0)$, a contradiction. □

Now turning to the proof of the theorem, let $N \in \mathbf{N}$ be arbitrary. Since K is compact, the covering

$$K \subset \bigcup_{x \in K} G(x, N)$$

has a finite subcovering:

$$K \subset \bigcup_{i=1}^{m} G(x_i, N).$$

Take a partition of unity subordinate to this covering, i.e. continuous functions $w_i : K \longrightarrow \mathbf{R}$ to satisfy

$$w_i \geqslant 0\,, \quad \text{supp}\, w_i \subset G(x_i, N) \quad (i = 1, \dots, m), \quad \sum_{i=1}^{m} w_i = 1\,.$$

Let $y_i \in f(x_i)$ be arbitrary, $F := \{y_1, \dots, y_m\}$, $g : h(F) \longrightarrow \triangle^m$, $g(x) := \sum_{i=1}^{m} w_i(x)e_i$. Furthermore, let $g \circ \Phi_F : \triangle^m \longrightarrow \triangle^m$. By the continuity of w_i and Φ_F, $g \circ \Phi_F$ is a continuous mapping of $\triangle^m$ into itself and thus, by Brouwer's fixed point theorem, there is a fixed-point $\lambda_N^* = g \circ \Phi_F(\lambda_N^*)$.

Let $x_N^* := \Phi_F(\lambda_N^*)$. Then $\lambda_N^* = g(x_N^*)$. In λ_N^* only those components are non-zero for which $x_N^* \in G(x_i, N)$. Denote these indexes by $i_1, \dots, i_r$. On the other hand, by (1), we have

$$\lambda_N^* \in \langle e_{i_1}, \dots, e_{i_r} \rangle \Longrightarrow x_N^* = \Phi_F(\lambda_N^*) \in h(\{y_{i_1}, \dots, y_{i_r}\})\,.$$

By (7), $x_{i_j} \in G(x_N^*, N)$, $(j = 1, \dots, r)$.

Since K is compact, there exists a subsequence $x_{N_K}^* \longrightarrow x^* \in K$. Let now $M \in \mathbf{N}$ be arbitrary. By the continuity of h, there is an $M_1 \in N$ such that

$$B \subset G(f(x^*), M_1) \Longrightarrow h(B) \subset G(h(f(x^*), M) \subset G(f(x^*), M) \tag{8}$$

since $f(x^*)$ is semiconvex.

By Lemma 1, there exists an $M_2 \in \mathbf{N}$ for which

$$x_{i_j} \in G(x^*, M_2) \Longrightarrow f(x_{i_j}) \subset G(f(x^*), M_1)\,. \tag{9}$$

Since $x_{N_k}^* \longrightarrow x^*$, by (6), if k is large enough, we have

$$x_{i_j} \in G(x_{N_k}^*, N_k) \subset G(x^*, M_2)\,.$$

Therefore, by (8) and (9)

$$y_{i_j} \in f(x_{i_j}) \subset G(f(x^*), M_1) \Longrightarrow h(\{y_{i_1}, \dots, y_{i_r}\}) \subset G(f(x^*), M)\,.$$

Since $x_{N_k}^* \in h(\{y_{i_1}, \dots, y_{i_r}\})$, we have

$$x_{N_k}^* \in G(f(x^*), M)\,.$$

Take a $y_{N_k}^* \in f(x^*)$. Clearly, $x_{N_k}^* \in G(y_{N_k}^*, M)$ for $k \geqslant k_0$, where we can suppose that $k_0 = k_0(M) \longrightarrow \infty$ as $M \longrightarrow \infty$.

Finally, let $N^* \in \mathbf{N}$ be arbitrary. By (7), $y_{N_{k_0}}^* \in G(x_{N_{k_0}}^*, M)$. Since $x_{N_{k_0}}^* \longrightarrow x^*$ as $M \longrightarrow \infty$, by (6), we have for large enough M

$$y_{N_{k_0}}^* \in G(x_{N_{k_0}}^*, M) \subset G(x^*, N)\,.$$

We have thus obtained that for any $N^* \in \mathbf{N}$, there is an $y^{(N^*)} \in f(x^*) \cap G(x^*, N^*)$, implying $y^{(N^*)} \longrightarrow x^*$ as $N^* \longrightarrow \infty$. Since $f(x^*)$ is closed, $x^* \in f(x^*)$. $\square$

In vector spaces, Brouwer's fixed point theorem is a straightforward consequence of Kakutani's since if f is a function (single-valued), then $f(x)$ being a singleton is (semi) convex. In Theorem 2, however, $\overline{P}_{1,1}$-space should be changed to $\overline{P}_{1,3}$ if we want to derive Brouwer's theorem from it because we need to have H2$'$ satisfied. (Note that H4 was not needed in the proof of Theorem 2 thus it can be dispensed with if we only want Brouwer's fixed point theorem to hold.) Nevertheless, we have the fixed point theorem:

THEOREM 3 (Brouwer). *Let K be a compact $\overline{P}_{1,3}$-space and $f : K \longrightarrow K$ a continuous function. Then, there is an $x^* \in K$ such that $x^* = f(x^*)$.*

Another generalization of Brouwer's theorem is the Markov–Kakutani theorem (see Dugundji and Granes 1982: 75).

THEOREM 4 (Markov–Kakutani). *Let K be a compact $\overline{P}_{1,3}$-space and $\mathcal{H}$ a family of continuous functions $f : K \longrightarrow K$ satisfying the following conditions:*

(i) *for any $f \in \mathcal{H}$, the set of fixed points* $\mathrm{Fix}(f)$ *is semiconvex,*

(ii) *for any $f, g \in \mathcal{H}$, $\quad f \circ g = g \circ f$.*

Then there is an $x^* \in K$ such that $f(x^*) = x^*$ for any $f \in \mathcal{H}$.

Proof. We have to prove that $\cap_{f\in\mathcal{H}}\mathrm{Fix}(f) \neq \emptyset$. Since f is continuous and K is compact, so is $\mathrm{Fix}(f)$. Therefore it is enough to prove the finite intersection property. The proof goes by induction. For $n = 1$, the finite intersection property holds since $\mathrm{Fix}(f) \neq \emptyset$ for any $f \in \mathcal{H}$ by Theorem 3. Assume now that it holds for any integer $1 \leqslant k \leqslant n-1$. Let $\overline{x} \in \mathrm{Fix}(f_k)$, $1 \leqslant k \leqslant n-1$. Then $f_n(\overline{x}) \in \mathrm{Fix}(f_k)$, $1 \leqslant k \leqslant n-1$, since $f_k(f_n(\overline{x})) = f_n(f_k(\overline{x})) = f_n(\overline{x})$ by (ii). Therefore we have the inclusion

$$f_n\left(\bigcap_{k=1}^{n-1}\mathrm{Fix}(f_k)\right) \subset \bigcap_{k=1}^{n-1}\mathrm{Fix}(f_k)\,. \tag{10}$$

The set $B_{n-1} := \cap_{k=1}^{n-1}\mathrm{Fix}(f_k)$ is nonempty by the inductive hypothesis, it is compact and semiconvex since $\mathrm{Fix}(f_k)$, $\quad 1 \leqslant k \leqslant n-1$ is compact and semiconvex. (Note that the intersection of semiconvex sets is also semiconvex.) Then the continuous map $f_n : B_{n-1} \longrightarrow B_{n-1}$ has a fixed point x^* by Theorem 3 i.e. $f_n(x^*) = x^*$. By (10), $x^* \in \mathrm{Fix}(f_k)$, $1 \leqslant k \leqslant n$, thus $\cap_{k=1}^{n}\mathrm{Fix}(f_k) \neq \emptyset$. □

The next example shows that continuity of operation h in Theorem 2 and 3 cannot be dispensed with.

EXAMPLE 4. Let $X := \{(x, y) \in \mathbf{R}^2 : x = \cos w,\ y = \sin w,\ 0 \leqslant w < 2\pi\}$ be the unit circle line and $F := \{w_1, \ldots, w_n\}$ the polar coordinate representation of a finite subset of X. Define the function $\Psi_F : \triangle^n \longrightarrow X$ as

$$\Psi_F(\lambda_1, \ldots, \lambda_n) := [\cos(\lambda_1 w_1 + \cdots + \lambda_n w_n),\ \sin(\lambda_1 w_1 + \cdots + \lambda_n w_n)]$$

which, in turn, generates a $Q_{2,3}$-space via $h(F) := \Psi_F(\triangle^n)$. In this space, any singleton $\{(x, y)\}$, $(x, y) \in X$ is a convex set. Operation h is obviously not continuous at $(0, 0)$.

Consider now the map $T : X \longrightarrow X$,

$$T(x(w), y(w)) := [\cos(w + \pi),\ \sin(w + \pi)]$$

which is continuous and maps each point in X to a convex set (singleton). T, however, has no fixed points. Thus, in $Q_{2,3}$, not only Kakutani's but also Brouwer's fixed point theorem fails.

4. Fan inequalities

Fan-type inequalities are very useful tools in nonlinear analysis, game theory and economics (see Aubin 1979; Fan 1952, 1953). In situations where there is no explicitly stated linear structure present one needs the Fan inequality in a more general setting. The following theorem is a generalization of Fan's inequality in a P-space.

THEOREM 5. *Let $(X, h, \mathcal{F})$ be a compact $P_{2,1}$-space, $f : X^2 \longrightarrow \mathbf{R}$ and $G : X \longrightarrow 2^X$ with the following properties:*

(1) *for each $x \in X$, $x \in G(x)$ and $G(x)$ is semiconvex;*
(2) *for each $y \in X$, $G^{-1}(y) := \{x \in X : y \in G(x)\}$ is open;*
(3) *for each $x \in X$, $f(x, \cdot)$ is quasiconcave on $G(x)$;*
(4) *for each $y \in X$, $f(\cdot, y)$ is l.s.c. on X;*
(5) *for each $x \in X$, $f(x, x) \leqslant 0$.*

Then, there is an $x^* \in X$ such that $f(x^*, y) \leqslant 0$ for all $y \in G(x^*)$.

Proof. Assume the result is false. Then, for any $x \in X$, the set

$$S(x) := \{y \in X : f(x, y) > 0\} \cap G(x)$$

is not empty. Define

$$\alpha := \min_{x \in X} \sup_{y \in G(x)} f(x, y)\,.$$

α exists because of the lower semicontinuity of $f(\cdot, y)$ and $\alpha > 0$ since $S(x) \neq \emptyset$ for all $x \in X$. ($\alpha = \infty$ is possible). Let $\beta > 0$ be arbitrary and define the map $x \longrightarrow T(x)$ for each $x \in X$

$$T(x) := \{y \in X : f(x, y) > \min\{\beta, \alpha/2\}\} \cap G(x)\,.$$

For any $x \in X$, $T(x) \neq \emptyset$, and it is semiconvex by Proposition 6, since $f(x, \cdot)$ is quasiconcave and $G(x)$ is semiconvex by assumption. Furthermore,

$$T^{-1}(y) := \{x \in X : f(x, y) > \min\{\beta, \alpha/2\}\} \cap G^{-1}(y)$$

is open by the lower semicontinuity of $f(\cdot, y)$ and by property (2). Then, by Theorem 1, there is a point $x^* \in T(x^*)$ i.e.

$$f(x^*, x^*) > \min\{\beta, \alpha/2\} > 0\,,$$

which contradicts the assumption $f(x^*, x^*) \leqslant 0$. □

If $G(x) \equiv X$ for any $x \in X$, then we obtain the classical version of Fan's inequality. Inclusion of the "neighborhood" function will be of use in proving existence theorems in game theory.

The following theorem, under somewhat different conditions, is a strict-inequality version of Theorem 4.

THEOREM 6. *Let* $(X, h, \mathcal{F})$ *be a compact* $\overline{P}_{2,1}$*-space,* $f : X^2 \longrightarrow \mathbf{R}$ *and* $G : X \longrightarrow 2^X$ *with the following properties:*
(1) *for each* $x \in X$, $x \in G(x)$ *and* $G(x)$ *is a closed semiconvex set;*
(2) *for each* $x \in X$, $f(x, \cdot)$ *is quasiconcave on* $G(x)$*;*
(3) f *is jointly continuous in both variables on* X^2*;*
(4) *for each* $x \in X$, $f(x, x) < 0$.
Then, there is an $x^* \in X$ *such that* $f(x^*, y) < 0$ *for all* $y \in G(x^*)$.

Proof. Again, the proof is indirect. If the result were false, then for any $x \in X$ the set

$$U(x) := \{y \in X : f(x, y) \geqslant 0\} \cap G(x)$$

is not empty and it is semiconvex by Proposition 6, since $f(x, \cdot)$ is quasiconcave and $G(x)$ is semiconvex. The graph

$$G_U := \{(x, y) \in X^2 : x \in X\,;\, y \in U(x)\} = \{(x, y) \in X^2 : f(x, y) \geqslant 0\}$$

is closed by the upper semicontinuity of $f(\cdot, \cdot)$ and because $G(x)$ is closed. Then, by Theorem 2, the map $x \longrightarrow U(x)$ has a fixed point $x^* \in U(x^*)$, i.e. $f(x^*, x^*) \geqslant 0$, contradicting the assumption $f(x^*, x^*) < 0$. □

5. Two-function minimax theorems

Two-function minimax theorems are straightforward generalizations of classical minimax theorems. Given two sets X, Y and two functions $f, g : X \times Y \longrightarrow \mathbf{R}$, a two-function minimax inequality is

$$\max_X \min_Y g \geqslant \min_Y \max_X f\,, \tag{11}$$

where $f \leqslant g$ is usually assumed. This inequality holds under various sets of conditions imposed on X, Y, f and g. For a good review see Simons (1995). Simons

(1981) proved (11) for quasiconvex–quasiconcave functions of suitable continuity properties in topological vector spaces. Joó and Kassay (1995) extended Simons' results to pseudoconvex spaces. In the following, we will show that (11) holds in spaces even more general than pseudoconvex.

THEOREM 7. *Let* $P_{2,1}^{(1)} := \left(X, h_1^{(1)}, \mathcal{F}_{2,1}^{(1)}\right)$ *and* $P_{2,1}^{(2)} := \left(Y, h_1^{(2)}, \mathcal{F}_{2,1}^{(2)}\right)$ *be two compact* $P_{2,1}$*-spaces,* $f, g : X \times Y \longrightarrow \mathbf{R}$ *with* $f \leqslant g$ *such that*

(i) f *is l.s.c. in its second variable, u.s.c. and quasiconcave in its first variable,*

(ii) g *is u.s.c. in its first variable, l.s.c. and quasiconvex in its second variable.*

Then

$$\max_{x\in X}\min_{y\in Y} g(x, y) \geqslant \min_{y\in Y}\max_{x\in X} f(x, y) .$$

Proof. If the result were false, there would be an $r \in \mathbf{R}$ such that

$$\min_{y\in Y}\max_{x\in X} f(x, y) > r > \max_{x\in X}\min_{y\in Y} g(x, y) . \tag{12}$$

Let $P_{2,1}^{(1,2)} := \left(Z, h_1^{(1,2)}, \mathcal{F}_{2,1}^{(1,2)}\right)$ be the product space of $P_{2,1}^{(1)}$ and $P_{2,1}^{(2)}$, where $Z = X \times Y$. Define $T : Z \times Z \longrightarrow \mathbf{R}$ by

$$T((\widehat{x}, \widehat{y}), (x, y)) = \min\{f(x, \widehat{y}) - r,\ r - g(\widehat{x}, y)\} .$$

We can easily see that $P_{2,1}^{(1,2)}$ and T satisfy the conditions of Theorem 5. T is l.s.c. in the first variable since f is l.s.c. in its second variable, g is u.s.c. in the second variable and the minimum of two l.s.c. functions is also l.s.c. T is quasiconcave in the second variable because f is quasiconcave in the first, g is quasiconvex in the second variable and the minimum of quasiconcave functions is also quasiconcave. $f(x, y) \leqslant g(x, y)$ implies that $T((x, y), (x, y)) \leqslant 0$.

Thus, by Theorem 5, there is an $(x^*, y^*) \in Z$ such that

$$T((x^*, y^*), (x, y)) \leqslant 0$$

for any $(x, y) \in Z$. Hence, $f(x, y^*) \leqslant r$ or $g(x^*, y) \geqslant r$ for any $(x, y) \in Z$, which means that

$$\min_{y\in Y}\max_{x\in X} f(x, y) \leqslant r$$

or

$$\max_{x\in X}\min_{y\in Y} g(x, y) \geqslant r ,$$

contradicting (12). □

Theorem 6 can be used to prove a 'strict' two-function minimax theorem under somewhat different conditions.

THEOREM 8. *Let* $\overline{P}_{2,1}^{(1)} := \left(X, h_1^{(1)}, \mathcal{F}_{2,1}^{(1)}\right)$ *and* $\overline{P}_{2,1}^{(2)} := \left(Y, h_1^{(2)}, \mathcal{F}_{2,1}^{(2)}\right)$ *be two compact* $\overline{P}_{2,1}$*-spaces,* $f, g : X \times Y \longrightarrow \mathbf{R}$ *with* $f < g$ *such that*

(i) f *is l.s.c. in its second variable, jointly u.s.c. in both variables and quasiconcave in its first variable,*

(ii) g *is u.s.c. in its first variable, jointly l.s.c. in both variables and quasiconvex in its second variable.*

Then

$$\max_{x\in X}\min_{y\in Y} g(x, y) > \min_{y\in Y}\max_{x\in X} f(x, y) .$$

Proof. The proof goes along the lines of Theorem 7 and it is indirect again. If the result were not true, then there would be an $r \in \mathbf{R}$ such that

$$\min_{y\in Y}\max_{x\in X} f(x, y) \geqslant r \geqslant \max_{x\in X}\min_{y\in Y} g(x, y) . \tag{13}$$

Define the same map T as in the proof of Theorem 7. This time, we will show that T satisfies the conditions of Theorem 6.

T is quasiconcave in the second variable since we have the same quasiconcavity/convexity conditions as in Theorem 7. Since f is jointly u.s.c., g is jointly l.s.c. in both variables, T is the minimum of two jointly u.s.c. functions. Therefore, T is also jointly u.s.c. in both variables. Also, $f(x, y) < g(x, y)$ implies $T((x, y), (x, y)) < 0$.

Thus, by Theorem 6, there is an $(x^*, y^*) \in Z$ such that

$$T((x^*, y^*), (x, y)) < 0$$

for any $(x, y) \in Z$. Hence $f(x, y^*) < r$ or $g(x^*, y) > r$ for any $(x, y) \in Z$ which means that

$$\min_{y\in Y}\max_{x\in X} f(x, y) < r$$

or

$$\max_{x\in X}\min_{y\in Y} g(x, y) > r ,$$

contradicting (13). □

In view of Proposition 5, a version of Theorem 7 (and Theorem 8) can be stated in terms of w-functions without any reference to $P_{2,1}$ (or $\overline{P}_{2,1}$) spaces.

THEOREM 9. *Let* X *and* Y *be compact topological spaces,* $f, g : X \times Y \longrightarrow \mathbf{R}$*,* $f \leqslant g$ *and jointly continuous in both variables. If there exists a continuous function* $w_1 : X^2 \times \triangle^2 \longrightarrow X$ *such that for any* $x_1, x_2 \in X$*,* $(\lambda_1, \lambda_2) \in \triangle^2$

$$f(w_1(x_1, x_2; \lambda_1, \lambda_2), y) \geqslant \min\{f(x_1, y), f(x_2, y)\}$$

holds for any $y \in Y$*, and there exists a continuous function* $w_2 : Y^2 \times \triangle^2 \longrightarrow Y$ *such that for any* $y_1, y_2 \in Y, \quad (\mu_1, \mu_2) \in \triangle^2$

$$g(x, w_2(y_1, y_2; \mu_1, \mu_2)) \leqslant \max\{g(x, y_1), g(x, y_2)\}$$

and w_1, w_2 *have properties W1, W2. W3, W4, then*

$$\max_{x\in X} \min_{y\in Y} g(x, y) \geqslant \min_{y\in Y} \max_{x\in X} f(x, y) .$$

Proof. The result directly follows from Proposition 5 and Theorem 7. □

Theorem 8 can also be restated in the same spirit.

Theorem 2, being a generalization of Kakutani's fixed point theorem, enables us to prove a 'reverse' theorem for two-function minimax in a more general setting than in Forgó and Joó (1998, to appear).

In order to state the theorem we need a few definitions. A function $\varphi : \mathbf{R}^2 \longrightarrow \mathbf{R}$ is said to be a *submaximum* function if for any $u, v \in \mathbf{R}$, $\varphi(u, v) \leqslant \max\{u, v\}$. Let X and Y be nonempty sets and $f, g : X \times Y \longrightarrow \mathbf{R}$. Given a submaximum function φ, f is said to be φ*-concave-like* with respect to g, if for all $\lambda > 0$ and $x_1, x_2 \in X$, there exists $x_3 \in X$ such that

$$y \in Y \Longrightarrow g(x_3, y) \geqslant \varphi[f(x_1, y), f(x_2, y)] - \lambda .$$

Note that x_3 depends on $\lambda, \varphi, x_1, x_2$.

For any fixed $x_1, x_2 \in X$ and $\lambda > 0$ define $Z : X \longrightarrow 2^Y$ and for any $\widehat{x} \in X$, $d_{\widehat{x}} : X \longrightarrow \mathbf{R}$ as

$$Z(x) := \{y \in Y : g(x, y) \leqslant \varphi[f(x_1, y), f(x_2, y)] - \lambda\}$$

$$d_{\widehat{x}}(x) := \min_{y\in Z(\widehat{x})} g(x, y) .$$

THEOREM 10. *Let* $(X, h_1, \mathcal{F}_{1,1})$ *be a compact* $\overline{P}_{1,1}$*-space and* Y *a compact topological space,* $f, g : X \times Y \longrightarrow \mathbf{R}$ *continuous functions and* φ *a continuous submaximum function. If*

(i) *for any nonempty closed set* $K \subset Y$*,*

$$\max_{x\in X} \min_{y\in K} g(x, y) \geqslant \min_{y\in K} \max_{x\in X} f(x, y) ,$$

(ii) *for any nonempty closed set* $K \subset Y$*, the set of maximizers*

$$\arg\max_{x\in X} \left\{\min_{y\in K} g(x, y)\right\}$$

is semiconvex, then f *is* φ*-concave-like with respect to* g*.*

Proof. Suppose, on the contrary, that f is not φ-concave-like with respect to g. Then there exist $\lambda > 0$ and $x_1, x_2 \in X$ such that for any $x \in X$ the set $Z(x)$ is not empty. By the continuity of f, g and φ, $Z(\widehat{x})$ is compact for any $\widehat{x} \in X$ and thus $d_{\widehat{x}}$ is continuous. Therefore, for any $\widehat{x} \in X$, the set-valued mapping $\widehat{x} \longrightarrow S(\widehat{x})$ where

$$S(\widehat{x}) := \left\{ x \in X : x \in \arg\max_{x \in X} d_{\widehat{x}}(x) \right\}$$

is u.s.c. By assumption (ii), the map is also semiconvex. Thus by Theorem 2, there exists $x^* \in X$ such that $x^* \in S(x^*)$, i.e.

$$d_{x^*}(x^*) = \min_{y \in Z(x^*)} g(x^*, y) = \max_{x \in X} \min_{y \in Z(x^*)} g(x, y)\,.$$

Then for any $y \in Z(x^*)$,

$$\max_{x \in X} \min_{y \in Z(x^*)} g(x, y) \leqslant \varphi[f(x_1, y), f(x_2, y)] - \lambda\,.$$

Since φ is a submaximum function

$$\varphi[f(x_1, y), f(x_2, y)] \leqslant \max\{f(x_1, y), f(x_2, y)\}$$

implying

$$\max_{x \in X} \min_{y \in Z(x^*)} g(x, y) < \min_{y \in Z(x^*)} \max_{x \in X} f(x, y)\,,$$

which is a contradiction to assumption (i). □

6. Existence of Nash equilibria

Existence of Nash equilibria for noncooperative games in strategic form has been a central issue in game theory ever since it was first established by Nash (1950) for mixed extensions of finite games. Another milestone was Nikaido and Isoda's (1955) existence theorem for concave games. The latest contribution, extending the result to topological vector spaces and reducing the continuity requirements to bare necessities is due to Tan et al. (1995).

With the help of fixed point theorems proved in this paper, we can further generalize theorems of Nikaido–Isoda-type to spaces with no linear structure. We will prove three theorems and also provide an example to illustrate how the existence of Nash equilibrium in a duopoly game can be proved under unusual conditions.

Let $N := \{1, \ldots, n\}$ be the set of players. A *game in strategic form* Γ is an ordered $3n$-tuple

$$\Gamma := \{X_1, \ldots, X_n; \varphi_1, \ldots, \varphi_n; f_1, \ldots, f_n\}$$

where, for each player $i \in N$, X_i is her strategy space, $\varphi_i : X := \prod_{j=1}^n X_j \longrightarrow 2^{X_i}$ is her neighborhood function ($\varphi_i(x_1, \dots, x_n)$ is the set of reachable strategy n-tuples of player i from $(x_1, \dots, x_n) \in X$), and $f_i : X \longrightarrow \mathbf{R}$ is her payoff function.

For each $i \in N$, denote $X_{-i} := \prod_{j \in N\setminus\{i\}} X_j$ and $x_{-i} \in X_{-i}$ stands for $(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$. If $x_i \in X_i$ and $x_{-i} \in X_{-i}$, we use (x_i, x_{-i}) to denote $y = (y_1, \dots, y_n) \in X$ such that $y_i = x_i$, $y_{-i} = x_{-i}$. An n-tuple $(x_1^*, \dots, x_n^*) \in X$ is a *Nash equilibrium point* of Γ if

$$(x_i^*, x_{-i}^*) \in \varphi_i(x_i^*, x_{-i}^*) \quad \text{and} \quad f_i(x_i^*, x_{-i}^*) \geqslant f_i(x_i, x_{-i}^*)$$

holds for all $x_i \in \varphi_i(x_i^*, x_{-i}^*)$ and for each $i \in N$.

To keep notation simple in this section, when we say X is a P-space (or Q-space) we mean that $(X, h, \mathcal{F})$ is a P-space (or Q-space).

THEOREM 11. *Let $\Gamma := \{X_1, \dots X_n; \varphi_1, \dots, \varphi_n; f_1, \dots, f_n\}$ be a game with the following properties:*

(1) *for each $i \in N$, X_i is a nonempty, compact $\overline{P}_{1,1}$-space,*
(2) *for each $i \in N$ and $x \in X$, $\varphi_i(x)$ is compact and $x_i \in \varphi_i(x)$,*
(3) *for each $i \in N$, f_i is u.s.c. on X,*
(4) *for each $i \in N$ and fixed $u_i \in X_i$, the function $f_i(u_i, \cdot)$ is l.s.c. on X_{-i},*
(5) *for any fixed $u \in X$, the* best reply

$$B(u) := \bigcap_{i=1}^n \{x = (x_1, \dots x_n) \in X : x_i \in \arg \max_{y \in \varphi_i(u)} f_i(y, u_{-i})\}$$

is semiconvex.

Then Γ has at least one Nash equilibrium point.

Proof. Since X is the Cartesian product of $\overline{P}_{1,1}$-spaces, it is also a $\overline{P}_{1,1}$-space. The best reply correspondence $u \longrightarrow B(u)$, $u \in X$ has nonempty values since $\varphi_i(u) \neq \emptyset$ and compact, f_i is u.s.c. By assumptions (2) and (5), $B(u)$ is a compact, semiconvex set for any $u \in X$. We will now show that $B : u \longrightarrow B(u)$ is also u.s.c. This amounts to showing that the graph

$$G_B := \{(x, y) : x \in X\,, y \in B(x)\}$$

is closed. Assume it is not. Then, there is an $(x^0, y^0) \notin G_B$, such that every neighborhood (in the product topology on X^2) of (x^0, y^0) contains a point of G_B. $x^0 \in X$, since X is closed, therefore $y^0 \notin B(x^0)$ i.e. for at least one player (say player 1), there is an $y_1^1 \in X_1$ such that

$$f_1(y_1^1, x_2^1, \dots, x_n^0) > f_1(y_1^0, x_2^0, \dots, x_n^0)\,. \tag{14}$$

Define the function $F : X^2 \longrightarrow \mathbf{R}$ as

$$F(x, y) := f_1(y_1^1, x_2, \dots, x_n) - f_1(y_1, x_2, \dots, x_n)\,.$$

By assumptions (3) and (4), F is l.s.c., therefore the set

$$C := \{(x, y) \in X^2 : F(x, y) \leqslant 0\}$$

is closed. For any $(\overline{x}, \overline{y}) \in G_B$, $F(\overline{x}, \overline{y}) \leqslant 0$ but, by (14), $F(x^0, y^0) > 0$, contradicting the closedness of C.

Now X and the map B satisfy all the conditions of Theorem 2, therefore there is an $x^* \in B(x^*)$, which means that x^* is a Nash equilibrium point of Γ. □

We note that if $\overline{P}_{1,1}$-space is replaced with $\overline{P}_{2,1}$-space, φ_i has semiconvex values and $f_i(\cdot, u_{-i})$ is quasiconcave on X_i for any fixed $u_{-i} \in X_{-i}$ and $i \in N$, then $B(u)$ is semiconvex for any $u \in X$. Thus, if X_i is a topological vector space and $\varphi_i(x) \equiv X_i$ for any $x \in X$ for each $i \in N$, then Theorem 11 reduces to the existence theorem of Tan et al. (1995, Theorem 2.1), which in turn, is a generalization of the famous Nikaido–Isoda theorem (1955).

A similar theorem can be proved if assumption (5) is replaced with a somewhat stricter condition. What we gain is that $\overline{P}_{1,1}$ can be relaxed to the more general $P_{1,1}$-space.

THEOREM 12. *Let* $\Gamma := \{X_1, \dots, X_n; \varphi_1, \dots, \varphi_n; f_1, \dots, f_n\}$ *be a game with the following properties:*

(1) *for each* $i \in N$, X_i *is a non-empty, compact* $P_{1,1}$*-space;*

(2) *for each* $i \in N$, φ_i *is u.s.c. and for any* $x \in X$, $\varphi_i(x)$ *is compact and* $x_i \in \varphi_i(x)$*;*

(3) *for each* $i \in N$, f_i *is u.s.c. on* X*;*

(4) *for each* $i \in N$ *and fixed* $u_i \in X_i$, *the function* $f_i(u_i, \cdot)$ *is l.s.c. on* X_{-i}*;*

(5) *for any fixed* $u \in X$, *the* approximate best reply

$$\begin{aligned} A(u, \lambda) := \bigcap_{i=1}^{n} \{y = (y_1, \dots, y_n) \in X : f_i(y_i, u_{-i}) \\ > \max_{z_i \in \varphi_i(u)} f_i(z_i, u_{-i}) - \lambda\} \end{aligned}$$

is semiconvex for any $\lambda > 0$.

Then Γ *has at least one Nash equilibrium point.*

For the proof we will need the following lemma.

LEMMA 2. *Let* X *and* Y *be two compact topological spaces. Let* $G : Y \longrightarrow 2^X$ *be a u.s.c. map with compact values and* $f : X \times Y \longrightarrow \mathbf{R}$ *u.s.c. on* $X \times Y$ *and for any fixed* $x \in X$, $f(x, \cdot)$ *is l.s.c. Then the function* $d : Y \longrightarrow \mathbf{R}$ *defined by* $d(y) := \max_{x \in G(y)} f(x, y)$ *is continuous.*

Proof. For any $\alpha \in \mathbf{R}$, the lower level set $L(\alpha)$ of d can be written as

$$L(\alpha) := \{y \in Y : d(y) \leqslant \alpha\} = \bigcap_{x \in G(y)} \{y \in Y : f(x, y) \leqslant \alpha\}.$$

By the lower semicontinuity of $f(x, \cdot)$, $L(\alpha)$ is the intersection of closed sets which implies that $L(\alpha)$ is also closed, i.e. d is l.s.c.

To prove upper semicontinuity of d, take $y \in Y$ and $\varepsilon > 0$. Since f is u.s.c., to any $x \in G(y)$ there is a neighborhood N_x of (x, y) (in the product topology) such that for each $(u, v) \in N_x$ we have

$$f(u, v) \leqslant f(x, y) + \varepsilon\,. \tag{15}$$

Denote $P_1 : X \times Y \longrightarrow X$ the projection to X-space. Then $P_1(N_x)$ is a neighborhood of x and

$$G(y) \subset \bigcup_{x \in G(y)} P_1(N_x)\,.$$

Since $G(y)$ is compact, there are finitely many points $x_1, \ldots, x_n$ for which

$$G(y) \subset \bigcup_{i=1}^{n} P_1\left(N_{x_i}\right)\,.$$

By the upper semicontinuity of G, there is a neighborhood V_y of y to satisfy

$$G(v) \subset \bigcup_{i=1}^{n} P_1\left(N_{x_i}\right) \tag{16}$$

for any $v \in V_y$. Consider now the following neighborhood of y

$$V := V_y \bigcap \left(\bigcap_{i=1}^{n} P_2\left(N_{x_i}\right) \right)$$

where $P_2 : X \times Y \longrightarrow Y$ is the projection to Y-space. Then, by (15) and (16), for any $(u, v) \in N_x,\ v \in V$, we get

$$f(u, v) \leqslant \max_{1 \leqslant i \leqslant n} f(x_i, y) + \varepsilon \leqslant d(y) + \varepsilon$$

for any $x \in G(v)$. This implies the inequality

$$d(v) \leqslant d(y) + \varepsilon$$

for any $v \in V$, i.e. d is u.s.c. at $y \in Y$. □

Proof of Theorem 12. Take an arbitrary positive sequence $\{\lambda_k\}$ converging to 0. For any $u \in X$, consider the corresponding sequence of approximate best replies $A(u, \lambda_k)$. $A(u, \lambda_k)$ is non-empty and semiconvex by assumptions (2) and (5). For each $y \in X$, by Lemma 2 and assumption (4), the function $u \longrightarrow f_i(y_i, u_{-i}) - \max_{z_i \in \varphi_i(u)} f_i(z_i, u_{-i})$ is l.s.c., therefore

$$A^{-1}(y, \lambda_k) := \bigcap_{i=1}^{n} \{u \in X : f_i(y_i, u_{-i}) > \max_{z_i \in \varphi_i(u)} f_i(z_i, u_{-i}) - \lambda_k\}$$

is open in X. Then all the conditions of Theorem 2 are met for the set X and the map A defined on it. Hence, there exists an $x^k \in X$ such that $x^k \in A(x^k, \lambda_k)$, i.e.

$$f_i(x_i^k, x_{-i}^k) > \max_{z_i \in \varphi_i(x^k)} f_i(z_i, x_{-i}^k) - \lambda_k$$

holds for each $k \in \mathbf{N}$.

Since X is compact, there exists a subsequence $\{w^\ell\}$ of $\{x^k\}$ and $x^* \in X$ such that $w^\ell \longrightarrow x^*$. Let $w^\ell := x^{k(\ell)}$ where $k(\ell) \longrightarrow \infty$ and $\lambda_{k(\ell)} \longrightarrow 0$. By Lemma 2 and assumption (3), for each $i \in N$,

$$f_i(x_i^*, x_{-i}^*) \geqslant \lim \sup_{\ell \longrightarrow \infty} f_i(x_i^{k(\ell)}, x_{-i}^{k(\ell)}) \geqslant \\ \lim_{\ell \longrightarrow \infty} \max_{z_i \in \varphi_i(x^{k(\ell)})} f_i(z_i, x_{-i}^{k(\ell)}) = \max_{z_i \in \varphi_i(x^*)} f_i(z_i, x_{-i}^*) .$$

Therefore, x^* is a Nash equilibrium point of game Γ. □

Again, we note that if $P_{1,1}$-space in Theorem 12 is replaced by $P_{2,1}$-space, then for each $i \in N$, assuming φ_i to have semiconvex values, $f_i(\cdot, u_{-i})$ to be quasiconcave on X_i for any fixed $u_{-i} \in X_{-i}$, the approximate best reply $A(u, \lambda_k)$ will be semiconvex for any $u \in X$ and $k \in \mathbf{N}$. Thus, assumption (5) can be stated in terms of quasiconcavity of the payoff functions. It is easy to see that Theorem 12 also reduces to Theorem 2.1 in Tan et al. (1995) if we drop the neighborhood functions and restrict ourselves to topological vector spaces.

If continuity and quasiconcavity assumptions are given for the sum of the payoff functions, then we get another existence theorem.

THEOREM 13. *Let* $\Gamma := \{X_1, \ldots, X_n; \varphi_1, \ldots, \varphi_n; f_1, \ldots, f_n\}$ *be a game with the following properties:*

(1) *for each* $i \in N$*,* X_i *is a non-empty, compact* $P_{2,1}$*-space;*

(2) *for each* $i \in N$ *and any* $x \in X$*,* $x_i \in \varphi_i(x)$*,* $\varphi_i(x)$ *is semiconvex and* $\varphi_i^{-1}(y) := \{x \in X : y \in \varphi_i(x)\}$ *is open for all* $y \in X_i$*;*

(3) $\sum_{i=1}^n f_i$ *is u.s.c. on* X*;*

(4) *for each* $i \in N$ *and fixed* $u_i \in X_i$*, the function* $f_i(u_i, \cdot)$ *is l.s.c. on* X_{-i}*;*

(5) *for any fixed* $u \in X$*,*
the function $\sum_{i=1}^n f_i(\cdot, u_{-i})$ *is quasiconcave on* $\prod_{i=1}^n \varphi_i(u)$*.*

Then Γ *has a Nash equilibrium point.*

Proof. Define the function $g : X^2 \longrightarrow \mathbf{R}$ by

$$g(x, y) := \sum_{i=1}^{n} [f_i(y_i, x_{-i}) - f_i(x_i, x_{-i})]$$

and $V : X \longrightarrow 2^X$ by

$$V(x) := \prod_{i=1}^{n} \varphi_i(x) .$$

Then X, g and V satisfy all the conditions of Theorem 5:

(i) X is a $P_{2,1}$-space by assumption (1),

(ii) for any $x \in X$, $V(x)$ is semiconvex, $x \in V(x)$ and

$$V^{-1}(y) := \{x \in X : y \in V(x)\}$$

is open for all $y \in Y$ by assumption (2),

(iii) for each $y \in X$, $g(\cdot, y)$ is l.s.c. by assumptions (3) and (4),

(iv) for any fixed $x \in X$, $g(x, \cdot)$ is quasiconcave on $V(x)$ by assumption (5),

(v) $g(x, x) = 0$ for any $x \in X$ by the definition of g.

Thus, by Theorem 5, there is an $x^* \in X$ such that $g(x^*, y) \leqslant 0$ for all $y \in V(x^*)$. For each $i \in N$ and $u_i \in \varphi_i(x^*)$ set $y := (u_i, x^*_{-i}) \in V(x^*)$. Then $g(x^*, y) = f_i(u_i, x^*_{-i}) - f_i(x^*_i, x^*_{-i}) \leqslant 0$ holds for all $u_i \in \varphi_i(x^*)$ and $i \in N$ which means that x^* is a Nash equilibrium point of Γ. □

It is easy to state Theorem 13 in a neighborhood function-free form in a topological vector space setting. Then we will get Theorem 2.2 in Tan et al. (1995). The above equilibrium theorems can also be rewritten in terms of w-functions without any reference to P-spaces in analogy to how it was done in Theorem 9.

EXAMPLE 5. For an illustration, take a very simple single-product duopoly game with price differentiation. Demand for the product is unusual, it is high for low and high prices (e.g. this can be the case for vintage wines and other prestige products). The decision variable for both firms is price change which varies from -1 (lowest possible) to 1 (highest possible) continuously.

For simplicity, we will assume symmetry, therefore it is enough to consider only one firm (say firm 1). If its price change is denoted by x, while that of firm 2 by y, then the change P_1 in firm 1's profit is assumed to be given as

$$P_1(x, y) := \begin{cases} yx^2 & \text{if } -1 \leqslant y < 0\,, \\ 0 & \text{if } y = 0\,, \\ yx^2 + y & \text{if } 0 < y \leqslant 1\,. \end{cases}$$

Elementary calculation shows that the best-reply correspondence $y \longrightarrow B_1(y)$ for firm 1 is

$$B_1(y) := \begin{cases} -\dfrac{1}{2y} & \text{if } -1 \leqslant y < -\dfrac{1}{2} \\ 1 & \text{if } -\dfrac{1}{2} \leqslant y < 0 \\ [-1, 1] & \text{if } y = 0 \\ \{1, 1\} & \text{if } 0 < y \leqslant 1\,. \end{cases}$$

Compactness of the strategy sets is obvious. The graph of B_1 is closed as shown in Figure 1, therefore B_1 is u.s.c.

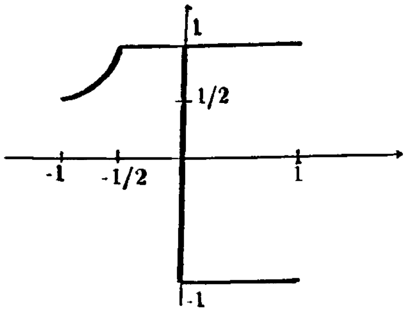

Figure 1.

The w-function (the same as the one in Example 1 and 2) $w : \triangle^2 \times [-1, 1]^2 \longrightarrow [-1, 1]$

$$w(x_1, x_2; \lambda_1, \lambda_2) := \sqrt{\lambda_1 x_1^2 + \lambda_2 x_2^2}$$

generates a $\overline{G}_{2,1}$-space, with $B_1(y)$ semiconvex for each $y \in [-1, 1]$ as was shown in Example 2. Thus, Theorem 11 applies and there exists a Nash equilibrium point.

Note that if upper semicontinuity of the best-reply correspondence can be easily established, then there is no need for checking continuity properties of the payoff functions.

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References

Aubin, J.P. (1979), *Mathematical Methods of Game and Economic Theory*. North-Holland, Amsterdam.

Danzer, L., Grünbaum, B. and Klee, V. (1963), Helly's theorem and its relatives, *Convexity, Proceedings of Symposia in Pure Mathematics*, Vol. VII, pp. 101–180. AMS, Providence, RI.

Dugundji, J. and Granas, A. (1982), *Fixed Point Theory*. PWN-Polish Scientific Publishers, Warsawa.

Fan, K. (1952), Fixed-point and minimax theorems in locally convex topological linear spaces, *Proceedings of the National Academy of Sciences, USA* 38: 121–126.

Fan, K. (1953), Minimax theorems, *Proceedings of the National Academy of Sciences, USA* 39: 42–47.

Forgó, F. (1994), On the existence of Nash-equilibrium in generalized concave games, in *Generalized Convexity, Lecture Notes in Economics and Mathematical Systems*, Vol. 405, pp. 53–61. Springer Verlag.

Forgó, F. and Joó, I. (1998), Necessary conditions for two-function minimax inequalities, to appear in *New Trends in Mathematical Programming*. Kluwer Academic Publishers, Dordrecht.

Gudder, S.P. (1977), Convexity and mixtures, *SIAM Review* 19: 221–240.

Joó, I. and Stachó, L.L (1982), A note on Ky Fan's minimax theorem, *Acta Math. Hung.* 39: 401–407.

Joó, I. (1986), Answer to a problem of M. Horváth and A. Sövegjártó, *Annales Univ. Sci. Budapest Sect. Math.* 29: 203–207.

Joó, I. (1989), On some convexities, *Acta Math. Hung.* 54: 163–172.

Joó, I. and Kassay, G. (1995), Convexity, minimax theorems and their applications, *Annales Univ. Sci. Budapest* 38: 71–93.

Komiya, H. (1981), Convexity on a topological space, *Fund. Math.* 111: 107–113.

Nash, J. (1950), Equilibrium points in finite games, *Proceedings of the National Academy of Sciences, USA* 36: 48–49.

Nikaido, H. and Isoda, K. (1955), Note on noncooperative convex games, *Pacific Journal of Mathematics* 5: 807–815.

Simons, S. (1981), Minimax and variational inequalities, are they of fixed point or Hahn–Banach type? In: *Game Theory and Mathematical Economics*, 2, pp. 379–388. North Holland, Amsterdam.

Simons, S. (1995), Minimax theorems and their proofs. In: *Minimax and Applications*, pp. 1–24 Kluwer Academic Publishers, Dordrecht.

Soltan, B.P. (1984), *Introduction to the Axiomatic Theory of Convexity* (in Russian). Stiinca, Kishinev.

Stone, N.H. (1949), Postulates for the barycentric calculus, *Ann. Math.* 29: 25–30.

Tala, J.E. and Marchi, E. (1966), Games with nonconvex strategy sets, *Optimization* 37: 177–181.

Tan, K-K., Yu, J. and Yuan, X.-Z. (1995), Existence theorems of Nash equilibria for noncooperative N-person games, *International Journal of Game Theory* 24: 217–222.

Van de Vel, M.L.J. (1993), *Theory of Convex Structures*. North-Holland, Amsterdam.

von Neumann, J. and Morgenstern, O. (1944), *Theory of Games and Economic Behavior*. Princeton University Press, Princeton, N.J.