for $p(1-\alpha)=1$,

$$f'(z) \ll \frac{pz^{p-1}(1+z)}{(1-z)^3}, \tag{3.10}$$

for $p(1-\alpha)>1$,

$$f'(z) \ll \frac{pz^{p-1}(1+z)}{(1-z)^{2(1-\alpha)+1}}. \tag{3.11}$$

Proof. Without loss of generality we assume that

$$f'(z)=\frac{\phi(z)\,pz^{p-1}}{(1-z)^{2p(1-\alpha)}}\cdot\frac{(1+az)}{(1-z)},$$

where $|\phi(z)|<1$ in D and $|a|=1$. We know $(1+az)/(1-z) \ll (1+z)/(1-z)$. Thus the conclusion follows from Theorem 7.

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Some Banach Spaces on Which All Biholomorphic Automorphisms Are Linear

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1

The question of whether biholomorphic maps are linear has been treated in various forms by several authors. In particular, Harris [1] has shown that a biholomorphic map of the unit ball of one space to another which takes 0 to 0 is a restriction of a linear isometry between the two spaces. He then showed that if the unit ball of a Banach space is a homogeneous domain, then it is holomorphically equivalent to the unit ball of another Banach space if and only if the two spaces are isometrically isomorphic. He asked whether this result would hold without the assumption about a homogeneous domain. Kaup and Upmeier [2] gave an answer to this question by showing that two complex Banach spaces are isometrically equivalent if and only if their open unit balls are biholomorphically equivalent. In a recent paper, Stacho [5] gave a short proof of the fact that all biholomorphic automorphisms of the unit ball in certain L^p-spaces are linear. In the present note, we show how Stacho's method can be used to obtain the same result for the C_p-spaces and the spaces $L^p(\Omega, E)$, where E is an arbitrary Banach space, Ω is σ-finite, and $1 \leqslant p < +\infty$, $p \neq 2$.

In particular, for the discrete case, we get the result that $l_p(E)$ has the linear biholomorphic property whether E has the property or not. On the other hand, we show that $c_0(E)$ has the property if and only if E has it.

A function ϕ on the open unit ball $B(E)$ of a Banach space is said to be *holomorphic* in $B(E)$ if the Frechet derivative $D\phi(x,\cdot)$ of ϕ at x exists as a bounded linear map of E into E for each $x \in B(E)$. A function ϕ from $B(E)$ to $B(E)$ is *biholomorphic* if ϕ^{-1} exists and both ϕ and ϕ^{-1} are holomorphic.

The proofs of Theorems 1 and 2 below as well as the theorem in [5] are based on the following lemma proved by Stacho in [5].

Lemma (Stacho). *If E is a Banach space with dual E^*, then every* *biholomorphic automorphism of the unit ball is linear if and only if condition* (1) *is satisfied*:

> $\langle q(x,x),\phi\rangle = -\overline{\langle z,\phi\rangle}$ for all $x\in E$, $\phi\in E^*$ with $\|x\| = \|\phi\| = 1 = \langle x,\phi\rangle$ implies $z=0$ whenever $z\in E$, q is a bilinear form from $E\times E$ to E, and $\langle\ ,\ \rangle$ denotes the pairing between E and E^*. (1)

We will say that a space for which every biholomorphic automorphism of the unit ball is linear has the linear biholomorphic property.

2

If H is a Hilbert space and T a compact operator on H, let

$$\|T\|_p = [\operatorname{trace}(T^*T)^{p/2}]^{1/p}, \qquad 1\leqslant p<\infty. \tag{2}$$

The Von Neumann–Schatten p-class $C_p = \{T\colon \|T\|_p < +\infty\}$ is a Banach space with norm given by (2). For information on C_p-spaces, see the book of Ringrose [4]. We will find it convenient to use the notation $|T| = \sqrt{T^*T}$, so that the polar decomposition of T can be written $T = V|T|$ where V is a partial isometry. We note that if $1\leqslant p<\infty$ and $1/p+1/p'=1$, then for $T\in C_p$ and $S\in C_{p'}$, $\langle T,S\rangle = \operatorname{Trace}(TS)$ defines a continuous linear functional on C_p and $C_{p'}$ is isometric to the dual of C_p with respect to this identification.

THEOREM 1. *If $1<p<\infty$ and $p\neq 2$, then C_p has the linear biholomorphic property.*

Proof. We show that C_p satisfies condition (1). To that end suppose q is a bilinear map on $C_p\times C_p$ to C_p, $W\in C_p$ and

$$\langle q(T,T),S\rangle = -\overline{\langle W,S\rangle} \tag{3}$$

for all $T\in C_p$, $S\in C_{p'}$, and with $\|T\|_p = \|S\|_{p'} = 1 = \langle T,S\rangle$, where $1/p+1/p'=1$. We wish to show that $W=0$.

For each $T\in C_p$, let $\hat{T} = |T|^{p-1}V^*$ where $T=|T|V$ is the polar decomposition of T. First we observe that $\hat{T}\in C_{p'}$. It is clear from the definition that $|\hat{T}| = V|T|^{p-1}V^*$ and by properties of the trace and the fact that V^*V is a projection on the range of $|T|$ we obtain

$$\begin{aligned}\operatorname{Trace}(|\hat{T}|^{p'}) &= \operatorname{Trace}(V|T|^{(p-1)p'}V^*) = \operatorname{Trace}(V|T|^pV^*)\\ &= \operatorname{Trace}(V^*V|T|^p) = \operatorname{Trace}(|T|^p).\end{aligned}$$

For a given $T\neq 0$, it is easy to show that $(T/\|T\|_p)\hat{} = (1/\|T\|_p^{p-1})\hat{T}$ and that

$$1 = \left\|\left(\frac{T}{\|T\|_p}\right)\right\| = \left\|\left(\frac{T}{\|T\|_p}\right)^{\wedge}\right\|_{p'} = \left\langle\frac{T}{\|T\|_p}, \frac{\hat{T}}{\|T\|_p^{p-1}}\right\rangle.$$

Therefore, by (3) we have

$$\langle q(T,T),\hat{T}\rangle = -\|T\|_p^2\,\overline{\langle W,T\rangle} \tag{4}$$

for all $T\in C_p$.

Next we choose hermitian projections P, Q in C_p so that $PQ=0$ and we let $T = P+\lambda Q$ for a nonzero constant λ. Then $T^*T = P+|\lambda|^2Q$ and $|T| = P+|\lambda|\,Q$. If $V = P+(\lambda/|\lambda|)Q$, it is easily verified that V is an isometry on the range of $|T|$ and $T = V|T|$ is the polar decomposition of T. Next we see that $\hat{T} = P+|\lambda|^{p-2}\bar{\lambda}Q$ and the right-hand side of (4) becomes

$$\begin{aligned}-\|T\|_p^2\overline{\langle W,\hat{T}\rangle} = -&[\operatorname{Trace}(P)\\ &+|\lambda|^p\operatorname{Trace}(Q)]^{2/p}\{\overline{\operatorname{Trace} WP} + \lambda|\lambda|^{p-2}\,\overline{\operatorname{Trace} WQ}\}.\end{aligned} \tag{5}$$

Substituting for T and $\hat{T}$ in the left-hand side of (4) and equating that to the result in (5), we obtain

$$\begin{aligned}&\sum_{i=0}^{2}\alpha_i\lambda^i + \bar{\lambda}|\lambda|^{p-2}\sum_{i=0}^{2}\beta_i\lambda^i\\ &\quad= -[\mu_1+|\lambda|^p\mu_2]^{2/p}\{\gamma_1+\lambda|\lambda|^{p-2}\gamma_2\}\end{aligned}$$

where

$$\begin{aligned}&\alpha_0 = \langle q(P,P),P\rangle, \qquad \alpha_1 = \langle q(Q,P),P\rangle + \langle q(P,Q),P\rangle,\\ &\alpha_2 = \langle q(Q,Q),P\rangle, \qquad \beta_0 = \langle q(P,P),Q\rangle,\\ &\beta_1 = [\langle q(Q,P),Q\rangle + \langle q(P,Q),Q\rangle],\\ &\beta_2 = \langle q(Q,Q),Q\rangle,\\ &\qquad \mu_1 = \operatorname{Trace} P, \qquad \mu_2 = \operatorname{Trace} Q,\\ &\qquad \gamma_1 = \overline{\operatorname{Trace}(WP)}, \qquad \gamma_2 = \overline{\operatorname{Trace}(WQ)}.\end{aligned} \tag{6}$$

Equation (6) is exactly the equation obtained by Stacho [5, p. 383] and it holds for all nonzero λ. We proceed exactly as in [5] to conclude that either $p=2$ or $\gamma_1 = 0$. Hence if $p\neq 2$, we have

$$\overline{\operatorname{Trace}(WP)} = 0 \tag{7}$$

for all choices of P as a hermitian projection.

If W is a compact hermitian, then $W=\Sigma\lambda_i P_i$, where $\{P_i\}$ is a family of hermitian projections which are pairwise orthogonal. From (7) we have $0=\text{Trace}(WP_k)$ for each k implies $\lambda_k=0$ for each k, so that $W=0$. Finally, if W is an arbitrary member of C_p, $W=A_1+iA_2$, where A_1 and A_2 are hermitian. Then $0=\text{Trace}(WP)=\text{Trace}(A_2P)+i\ \text{Trace}(A_2P)$ for all hermitian projections P. Since the trace of a product of hermitian operators is necessarily real, we conclude that $\text{Trace}(A_1P)=\text{Trace}(A_2P)=0$ for all hermitian P. By our previous argument, $A_1=A_2=0$ so that we have succeeded in showing that $W=0$.

3

For $1\leqslant p<\infty$, let $L^p(\Omega, E)$ denote the Banach space of weakly measurable functions F defined on a measure space (Ω, Σ, μ) taking values in a Banach space E for which

$$\|F\|_p=\left(\int_\Omega |F(\omega)|^p\,d\mu(\omega)\right)^{1/p}<+\infty.$$

By adding a restriction on the measure space we can extend the theorem of Stacho [5] to the more general $L^p(\Omega, E)$ spaces.

THEOREM 2. *Let (Ω, Σ, μ) be a σ-finite measure space which does not consist of a single atom, let E be a Banach space and suppose $1\leqslant p<\infty$, $p\neq 2$. Then $L^p(\Omega, E)$ has the linear biholomorphic property.*

Proof. The proof follows the same lines as that of Theorem 1. For $F\in L^p(\Omega, E)$, define $\hat{F}\colon \Omega\to E^*$ by

$$\hat{F}(\omega)(z)=[z, F(\omega)]\,\|F(\omega)\|^{p-2},$$

where $z\in E$ and $[\ ,\ ]$ is a semi-inner product compatible with the norm of E. (For information on semi-inner products see Lumer [3].) Then $\hat{F}$ is an element of $L_{p'}(\Omega, E^*)$ which can be identified with a subspace of $(L_p(\Omega, E))^*$.

Now let q be a bilinear form and $U\in L^p(\Omega, E)$ be given as in condition (1). Then

$$\int_\Omega [q(F,F)(\omega), F(\omega)]\,\|F(\omega)\|^{p-2}\,d\mu(\omega)$$

$$=-\|F\|_p^2\int_\Omega \overline{[U(\omega), F(\omega)]}\,\|F(\omega)\|^{p-2}\,d\mu(\omega) \tag{8}$$

for all $F\in L^p(\Omega, E)$. Assume first that $\mu(\Omega)$ is finite. Let F_1 be an element of $L^p(\Omega, E)$ for which $\|F_1(\omega)\|=1$ for all ω such that $F_1(\omega)\neq 0$. Let $z_2\in E$ with $\|z_2\|=1$. Next choose Ω_1 and Ω_2 as disjoint subsets of Ω, each having positive measure and let $F(\omega)=\chi_{\Omega_1}F_1(\omega)+\lambda\chi_{\Omega_2}(\omega)z_2$ for $\omega\varepsilon\Omega$, where χ_{Ω_1}, χ_{Ω_2} are characteristic functions and λ is a scalar.

Using the fact that for each $\omega\in\Omega$,

$$\|F(\omega)\|=\chi_{\Omega_1}\|F_1(\omega)\|+|\lambda|\,\chi_{\Omega_2}\|z_2\|$$

and

$$[U(\omega), F(\omega)]=[U(\omega), F_1(\omega)]\chi_{\Omega_1}(\omega)+[U(\omega), z_2]\chi_{\Omega_2}(\omega)\bar{\lambda},$$

we can conclude, after substitution into (8) and suitable calculations, that

$$\int_{\Omega_1}\overline{[U(\omega), F_1(\omega)]}\,d\mu(\omega)=0.$$

Since this holds for all Ω_1 with $\mu(\Omega_1)=0$, we must have

$$[U(\omega), F_1(\omega)]=0 \qquad \text{a.e.}$$

It now follows by a proper choice of F_1 that $U(\omega)=0$ a.e. A standard argument will lead to the same conclusion if Ω is σ-finite.

Thus we have that condition (1) is satisfied and the theorem follows from Stacho's lemma.

The above argument will go through with very little change if the single space E is replaced by a family $\{E(t)\colon t\in\Omega\}$ of Banach spaces so that $F(t)\in E(t)$ for each t. This remark applies in the next section where we wish Ω to be a countable discrete space.

4

Let $\{E_i\}$ denote a sequence of Banach spaces and let $c_0(E_i)$ denote the set of elements $x=(x_i)$ of the product space ΠE_i for which $\lim\|x_i\|=0$. Then $c_0(E_i)$ is a Banach space under the norm $\|x\|=\sup_{1\leqslant i<\infty}\|x_i\|$ whose dual space is isometric to $l_1(E_i^*)$ under the obvious pairing.

THEOREM 3. *The space $X=c_0(E_i)$ has the linear biholomorphic property if and only if each E_i has the linear biholomorphic property.*

Proof. First suppose that each E_i has the linear biholomorphic property. Let q be a bilinear mapping of $X\times X$ to X such that

$$\langle q(x,x), x^*\rangle=-\overline{\langle u, x^*\rangle} \tag{9}$$

for all x, x^* such that $1 = \langle x, x^* \rangle = \|x\| = \|x^*\|$. For each i let δ_i and π_i be the injection of E_i into ΠE_i and the projection of ΠE_i onto E_i, respectively. For a given i, let $q_i(u, v)$ be defined on $E_i \times E_i$ by

$$q_i(u, v) = \pi_i q(\delta_i(u), \delta_i(v)).$$

Then q_i is a bilinear map of $E_i \times E_i$ into E_i. Now suppose $x_i \in E_i$ with $\|x_i\| = 1$ and suppose $x_i^* \in E_i^*$ such that $1 = \langle x_i, x_i^* \rangle = \|x_i\| = \|x_i^*\|$. Since $x = \delta_i(x_i) \in X$, $x^* = \delta_i(x_i^*) \in X^*$ and $\langle x, x^* \rangle = \langle x_i, x_i^* \rangle = 1 = \|x_i\| = \|x\| = \|x^*\|$, we must have from (9) that

$$\langle q(x, x), x^* \rangle = -\overline{\langle u, x^* \rangle}. \tag{10}$$

Hence by definition of q_i and from (10) we have

$$\langle q_i(x_i, x_i), x_i^* \rangle = \langle q(x, x), x^* \rangle = -\overline{\langle u, x^* \rangle} = -\overline{\langle u_i, x_i^* \rangle}.$$

Thus condition (1) of the lemma implies $u_i = 0$ since E_i has the linear biholomorphic property. We conclude that $u = 0$, so that condition (1) is satisfied by X itself, and X has the linear biholomorphic property.

For the converse, suppose X has the linear biholomorphic property. Let i be given and let ψ be a biholomorphic automorphism of $B(E_i)$. Define Ψ on $B(X)$ by

$$\begin{aligned} \pi_j \Psi(x) &= \pi_j x, & j &\neq i \\ &= \psi(\pi_i(x)), & j &= i. \end{aligned}$$

It is straightforward to show that Ψ is a biholomorphic mapping of $B(X)$ onto itself and is therefore linear by hypothesis. It follows that ψ is necessarily linear on E_i.

It is interesting to note that $l_p(E_i)$ $(1 \leqslant p < \infty, p \neq 2)$ has the linear biholomorphic property even if some E_i fails the property, while $c_0(E_i)$ can have the property only if E_i has it for each i. In particular, since the one-dimensional complex space does not have the property, neither does the sequence space (c_0). In fact, using techniques similar to those in [1], we obtain the following result which we state without proof.

THEOREM 4. *F is a biholomorphic automorphism of $B(c_0)$ if and only if there exists a unimodular function $\alpha(\cdot)$, a permutation $\phi(\cdot)$ of the positive integers, and an $x_0 \in B(c_0)$ such that*

$$F(x)(n) = \alpha(n) \left[\frac{x(\phi(n)) - x_0(\phi(n))}{1 - \bar{x}_0(\phi(n))\, x(\phi(n))} \right]$$

for each $x \in B(c_0)$.

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