RIGIDLY COLLINEAR PAIRS OF STRUCTURAL PROJECTIONS ON A JBW*-TRIPLE

C. MARTIN EDWARDS

ABSTRACT. Pre-symmetric complex Banach spaces have been proposed as models for state spaces of physical systems. A neutral GL-projection on a pre-symmetric space represents an operation on the corresponding system, and has as its range a further pre-symmetric space which represents the state space of the resulting system. Every L-projection is a neutral GL-projection, and such a projection represents a classical operation. Two neutral GL-projections R and S on the pre-symmetric space A_* represent decoherent operations when their ranges are rigidly collinear. It is shown that if R and S each satisfy a condition, a possible physical interpretation of which is that the information lost in their measurement is partially recoverable, then R and S have as supremum $R+S$ and the operations corresponding to R, S and $R+S$ are simultaneously performable. Furthermore, it is shown that the smallest L-projection majorizing R, S and $R+S$ coincide, and the greatest L-projection majorized by $R+S$ is identified.

1. INTRODUCTION

A complex Banach space A_* is said to be pre-symmetric if the open unit ball in its Banach dual space A is a bounded symmetric domain. Pre-symmetric spaces have been proposed as models for the state spaces of physical systems [31], [32], [33], [34], operations on the physical system corresponding to the pre-symmetric space A_* being represented by contractive projections R on A_*. The range RA_* of a contractive projection R is a pre-symmetric space which can be regarded as representing the state space of the filtered system [40], [46].

A contractive projection R on the pre-symmetric space A_* is said to be neutral if each element x in A_* for which $\|Rx\|$ and $\|x\|$ coincide lies in the range RA_* of R, and is said to be a GL-projection if the set

$$(RA_*)^\circ = \{x \in A_* : \|x \pm y\| = \|x\| + \|y\|, \forall y \in RA_*\}$$

of elements L-orthogonal to all those in the range RA_* of R is contained in the kernel $\ker(R)$ of R. The results of [14], [16], [18], [26] show that, for each element R of the set $\mathcal{S}_*(A_*)$ of neutral GL-projections on A_*, there exists an element $R^\perp$ of $\mathcal{S}_*(A_*)$ with range equal to $(RA_*)^\circ$. In physical terms $R^\perp$ may be thought of as representing the operation complementary to that represented by R whilst the range R_1A_* of the projection R_1 on A_* defined by

$$R_1 = \mathrm{id}_{A_*} - R - R^\perp$$

Date: December 22nd, 2005.

1991 *Mathematics Subject Classification.* Primary 46L70; Secondary 17C65 81P15.

Key words and phrases. JBW*-triple, pre-symmetric space, contractive projection, inner ideal, decoherence.

Received January 13, 2006.

isomorphism from the complete lattice $\mathcal{S}(A)$ of structural projections on A onto the complete lattice $\mathcal{I}(A)$ of weak*-closed inner ideals in A. More recently, in [16], it was shown that the mapping $R \mapsto R^*$ is an order isomorphism from the set $\mathcal{S}_*(A_*)$ of neutral GL-projections on A_* onto the complete lattice $\mathcal{S}(A)$, thereby linking the purely physical and geometric properties of the pre-symmetric space A_* with the purely algebraic properties of A.

For each element J of $\mathcal{I}(A)$, the kernel $\mathrm{Ker}(J)$ of J is defined to be the set of elements a in A for which the triple product $\{J\ a\ J\}$ is equal to zero, and the annihilator $J^\perp$ of J is defined to be the set of elements a in A for which $\{J\ a\ A\}$ is equal to zero. For each element J in $\mathcal{I}(A)$, the annihilator $J^\perp$ also lies in $\mathcal{I}(A)$, and A enjoys the generalized Peirce decomposition

$$A = J_0 \oplus J_1 \oplus J_2, \tag{1.1}$$

where,

$$J_0 = J^\perp, \qquad J_2 = J, \qquad J_1 = \mathrm{Ker}(J) \cap \mathrm{Ker}(J^\perp). \tag{1.2}$$

The structural projections onto J and $J^\perp$ are denoted by $P_2(J)$ and $P_0(J)$, respectively, and the projection $\mathrm{id}_A - P_2(J) - P_0(J)$ onto J_1 is denoted by $P_1(J)$. Furthermore,

$$\{A\ J_0\ J_2\} = \{0\}, \quad \{A\ J_2\ J_0\} = \{0\}. \tag{1.3}$$

and, for j, k, and l equal to 0, 1, or 2, the Peirce arithmetical relations,

$$\{J_j\ J_k\ J_l\} \subseteq J_{j+l-k}, \tag{1.4}$$

when $j + l - k$ is equal to 0, 1, or 2, and

$$\{J_j\ J_k\ J_l\} = \{0\}, \tag{1.5}$$

otherwise, hold, except in the cases when (j, k, l) is equal to $(0, 1, 1)$, $(1, 1, 0)$, $(1, 0, 1)$, $(2, 1, 1)$, $(1, 1, 2)$, $(1, 2, 1)$, or $(1, 1, 1)$. For j equal to 0, 1, or 2, writing $P_j(J)_*$ for the pre-adjoint of $P_j(J)$ and J_{*j} for its range, it is clear that A_* also enjoys a Peirce decomposition

$$A_* = J_{*0} \oplus J_{*1} \oplus J_{*2},$$

and that $P_2(J)_*$ is a neutral GL-projection such that $P_2(J)_*^\perp$ coincides with $P_0(J)_*$. In general, however, J_1 is not a JBW*-triple, and $P_1(J)$ and, hence, $P_1(J)_*$ is not contractive. A remarkable result, proved in [22], shows that the Peirce-one projections $P_1(J)$ and $P_1(J)_*$ are contractive if and only if the Peirce arithmetical relations (1.4) and (1.5) hold in all cases. In this case J is said to be a Peirce inner ideal. It follows that the mapping $R \mapsto R^*A$ is a bijection from the set $\mathcal{S}_*^p(A_*)$ of Peirce neutral GL-projections on A_* onto the set $\mathcal{I}^p(A)$ of Peirce weak*-closed inner ideals in A.

Two weak*-closed inner ideals J and K in the JBW*-triple A are said to be compatible when, for j and k equal to 0, 1, or 2, the Peirce projections $P_j(J)$ and $P_k(K)$ commute [17]. It follows that J and K are compatible elements of $\mathcal{I}(A)$ if and only if $P_2(J)_*$ and $P_2(K)_*$ are compatible elements of $\mathcal{S}_*(A_*)$. A weak*-closed inner ideal I in A is said to be an ideal if I_1 is equal to zero, or, equivalently, if I is compatible with all weak*-inner ideals in A, or, equivalently, if $P_2(I)_*$ is an L-projection on A_* [17]. The sets $\mathcal{ZI}(A)$ of weak*-closed ideals in A and $\mathcal{ZS}(A)$ of corresponding central elements of $\mathcal{S}(A)$, or M-projections, form order isomorphic Boolean sub-complete lattices of $\mathcal{I}(A)$ and $\mathcal{S}(A)$, respectively, and both are order

(NOT SAID TO BE AN IDEAL)

a JBW*-triple. The second dual A^{**} of a JB*-triple A is a JBW*-triple. For details of these results the reader is referred to [3], [4], [10], [11], [35], [38], [39], [40], [47] and [48]. Examples of JB*-triples are JB*-algebras, and examples of JBW*-triples are JBW*-algebras, for the properties of which the reader is referred to [12], [36], [49] and [50].

An element u in a JBW*-triple A is said to be a *tripotent* if $\{u\ u\ u\}$ is equal to u. The set of tripotents in A is denoted by $\mathcal{U}(A)$. For each tripotent u in A, the weak*-continuous linear operators $P_0(u)$, $P_1(u)$ and $P_2(u)$, defined by

$$P_0(u) = \mathrm{id}_A - 2D(u,u) + Q(u)^2, \quad P_1(u) = 2(D(u,u) - Q(u)^2),$$

$$P_2(u) = Q(u)^2, \tag{2.1}$$

are mutually orthogonal projection operators on A with sum id_A. For j equal to 0, 1 or 2, the range of $P_j(u)$ is the weak*-closed eigenspace $A_j(u)$ of $D(u,u)$ corresponding to the eigenvalue $\frac{1}{2}j$ and

$$A = A_0(u) \oplus A_1(u) \oplus A_2(u) \tag{2.2}$$

is the *Peirce decomposition* of A relative to u. Moreover, $A_0(u)$ and $A_2(u)$ are inner ideals in A, $A_1(u)$ is a subtriple of A and $A_j(u)$ is said to be the *Peirce j-space* corresponding to the tripotent u. Furthermore,

$$\{A\ A_2(u)\ A_0(u)\} = \{A\ A_0(u)\ A_2(u)\} = \{0\} \tag{2.3}$$

and, for j, k and l equal to 0, 1 or 2,

$$\{A_j(u)\ A_k(u)\ A_l(u)\} \subseteq A_{j+l-k}(u) \tag{2.4}$$

when $j + l - k$ is equal to 0, 1 or 2, and

$$\{A_j(u)\ A_k(u)\ A_l(u)\} = \{0\} \tag{2.5}$$

otherwise.

A pair a and b of elements in a JBW*-triple A is said to be *orthogonal* when $D(a,b)$ is equal to zero. For a subset L of A, the subset $L^\perp$ of A consisting of all elements which are orthogonal to all elements of L is a weak*-closed inner ideal in A which is known as the *annihilator* of L in A. For subsets L, M of A, $L^\perp \cap L \subseteq \{0\}$, $L \subseteq L^{\perp\perp}$, $L \subseteq M$ implies that $M^\perp \subseteq L^\perp$, and $L^\perp$ and $L^{\perp\perp\perp}$ coincide.

For each non-empty subset J of the JBW*-triple A, the *kernel* $\mathrm{Ker}(J)$ of J is the weak*-closed subspace of elements a in A for which $\{J\ a\ J\}$ is equal to $\{0\}$. It follows that the annihilator $J^\perp$ of J is contained in $\mathrm{Ker}(J)$ and that $J \cap \mathrm{Ker}(J)$ is contained in $\{0\}$. A subtriple J of A is said to be *complemented* [21] if A coincides with $J \oplus \mathrm{Ker}(J)$. It can easily be seen that every complemented subtriple is a weak*-closed inner ideal. A linear projection P on the JBW*-triple A is said to be a *structural projection* [42] if, for each element a in A,

$$PQ(a)P = Q(Pa).$$

The main results of [18], [20] and [21] show that the range PA of a structural projection P is a complemented subtriple, that the kernel $\mathrm{ker}P$ of the map P coincides with $\mathrm{Ker}(PA)$, that every structural projection is contractive and weak*-continuous, and, most significantly, that every weak*-closed inner ideal is complemented.

Let $\mathcal{I}(A)$ denote the complete lattice of weak*-closed inner ideals in the JBW*-triple A and let $\mathcal{S}(A)$ denote the set of structural projections on A. The results where $f(J)$ coincides with $k(J)^{\perp} \cap J$.

3. Rigidly collinear pairs of weak*-closed inner ideals

In this section some properties of a rigidly collinear pair J and K of weak*-closed inner ideals in a JBW*-triple A are investigated. The main results relate to the conditions under which such pairs are compatible and to the existence of a supremum of J and K in the complete lattice $\mathcal{I}(A)$ of weak*-closed inner ideals in A. It turns out that, in complete generality, it is possible to reveal some facts about their central structure.

Theorem 3.1. *Let A be a JBW*-triple, and let J and K form a rigidly collinear pair of weak*-closed inner ideals in A, having Peirce spaces J_0, J_1, and J_2, and K_0, K_1, and K_2, respectively. Then, the following results hold.*

(i) *The weak*-closed inner ideals J and K are faithful.*

(ii) *The central hulls $c(J)$, $c(K)$, $c(J_1)$ and $c(K_1)$ coincide.*

(iii) *The central hull $c(J \vee K)$ of the smallest weak*-closed inner ideal $J \vee K$ containing J and K coincides with $c(J)$ and $c(K)$.*

Proof. (i) Since K_2 is contained in J_1, it follows that

$$(J_1)^{\perp} \subseteq K_2^{\perp} = K_0.$$

Hence, by [27], Theorem 3.14, and since J_2 is contained in K_1,

$$k(J) = (J_1)^{\perp} \cap J_2 \subseteq K_0 \cap K_1 = \{0\}.$$

It follows that J and, similarly, K is faithful.

(ii) Since J is compatible with each weak*-closed ideal, the set of which forms a complete Boolean lattice, it can be seen that

$$\begin{aligned} J &= J \cap (c(J) \cap c(K) \oplus c(J) \cap c(K)^{\perp} \oplus c(J)^{\perp} \cap c(K) \oplus c(J)^{\perp} \cap c(K)^{\perp}) \\ &= J \cap c(K) \oplus J \cap c(K)^{\perp} \oplus J \cap c(J)^{\perp} \cap c(K) \oplus J \cap c(J)^{\perp} \cap c(K)^{\perp}. \quad (3.1) \end{aligned}$$

Since J is contained in $c(J)$, it follows that $c(J)^{\perp}$ is contained in J_0, and, hence, $J \cap c(J)^{\perp}$ is equal to zero. Furthermore, $c(K)^{\perp}$ is contained in K_0, and, therefore,

$$J \cap c(K)^{\perp} \subseteq J_2 \cap K_0 \subseteq K_1 \cap K_0 = \{0\}.$$

From (3.1), J and $J \cap c(K)$ coincide, from which it can be seen that J is contained in $c(K)$. From the definition of central kernel, it follows that $c(J)$ is contained in $c(K)$. By exchanging J and K in the argument above, $c(K)$ is also contained in $c(J)$, as required. By [27], Corollary 3.12, and since K is contained in J_1,

$$c(J_1) \subseteq c(J) = c(K) \subseteq c(J_1),$$

and $c(J_1)$ coincides with $c(J)$ and $c(K)$. The same clearly applies to $c(K_1)$.

(iii) Since J and K are contained in $J \vee K$, it follows that

$$c(J) \subseteq c(J \vee K), \quad c(K) \subseteq c(J \vee K),$$

and, hence,

$$c(J) \vee c(K) \subseteq c(J \vee K).$$

Therefore, by (3.3) and (3.4), for j equal to 0, 1, and 2,

$$P_1(K)P_j(J) = P_j(J)P_1(K). \tag{3.5}$$

Exchanging J and K it follows that

$$P_1(J)P_j(K) = P_j(K)P_1(J). \tag{3.6}$$

Since J_2 is contained in K_1,

$$P_1(K)P_2(J) = P_2(J). \tag{3.7}$$

Since K is Peirce, by (1.4), K_1 is a subtriple of A, and it follows from from [18], Lemma 3.12 that J_{*2} is contained in K_{*1} which implies that

$$P_1(K)_*P_2(J)_* = P_2(J)_*.$$

Taking adjoints,

$$P_2(J)P_1(K) = P_2(J), \tag{3.8}$$

and then, from (3.7) and (3.8),

$$P_1(K)P_2(J) = P_2(J) = P_2(J)P_1(K). \tag{3.9}$$

Hence, from (3.9),

$$P_2(J)P_2(K) = P_2(J)P_1(K)P_2(K) = 0 = P_2(K)P_1(K)P_2(J) = P_2(K)P_2(J). \tag{3.10}$$

Since

$$\mathrm{id}_A = P_0(J) + P_1(J) + P_2(J) = P_0(K) + P_1(K) + P_2(K),$$

it follows from (3.5),(3.6) and (3.10) that, for j and k equal to 0, 1 and 2,

$$P_j(J)P_k(K) = P_k(K)P_j(J),$$

and the proof is complete. □

It is now possible to investigate the structure of the supremum $J \vee K$ of the rigidly collinear pair J and K of Peirce weak*-closed inner ideals in the JBW*-triple A. Observe that, by Theorem 3.4, J and K are compatible, their intersection table being given by

$\cap$	J_2	J_1	J_0
K_2	$\{0\}$	K_2	$\{0\}$
K_1	J_2	$J_1 \cap K_1$	$J_0 \cap K_1$
K_0	$\{0\}$	$J_1 \cap K_0$	$J_0 \cap K_0$

and, by [17], §3,

$$A = \bigoplus_{j,k=0}^{2} J_j \cap K_k. \tag{3.11}$$

Theorem 3.5. *Let A be a JBW*-triple, and let J and K form a rigidly collinear pair of Peirce weak*-closed inner ideals in A having corresponding Peirce spaces J_0, J_1, and J_2, and K_0, K_1, and K_2, and Peirce projections $P_0(J)$, $P_1(J)$, and $P_2(J)$, and $P_0(K)$, $P_1(K)$, and $P_2(K)$. Then, the following results hold.*

(i) *The subspace $J + K$ of A is a weak*-closed inner ideal in A.*

and that the inner ideal $J+K$ is complemented. By [18], Lemma 3.2, the inner ideal $J+K$ is weak*-closed.

(ii) Observe that the weak*-closed inner ideal $(J+K)_0$ is equal to $J_0 \cap K_0$. Moreover,

$$\begin{aligned}\{J_0 \cap K_0\ J_2 \oplus K_2 \oplus J_1 \cap K_1 \oplus J_1 \cap K_0 \oplus J_0 \cap K_1\ J_0 \cap K_0\} \\ \subseteq\ \{0\} \oplus \{0\} \oplus \{J_0\ J_1\ J_0\} \cap \{K_0\ K_1\ K_0\} \oplus \{J_0\ J_1\ J_0\} \cap \{K_0\ K_0\ K_0\} \\ \oplus \{J_0\ J_0\ J_0\} \cap \{K_0\ K_1\ K_0\} \\ =\ \{0\}.\end{aligned}$$

Hence,

$$\begin{aligned} A &= J_0 \cap K_0 \oplus J_2 \oplus K_2 \oplus J_1 \cap K_1 \oplus J_1 \cap K_0 \oplus J_0 \cap K_1 \\ &\subseteq (J+K)_0 \oplus \mathrm{Ker}((J+K)_0) \\ &\subseteq A. \end{aligned}$$

It follows that

$$\mathrm{Ker}((J+K)_0) = J_2 \oplus K_2 \oplus J_1 \cap K_1 \oplus J_1 \cap K_0 \oplus J_0 \cap K_1, \tag{3.13}$$

and, using the compatibility of J and K, and (3.12)-(3.13),

$$\begin{aligned} (J+K)_1 &= \mathrm{Ker}(J+K) \cap \mathrm{Ker}((J+K)_0) \\ &= J_1 \cap K_1 \oplus J_1 \cap K_0 \oplus J_0 \cap K_1, \end{aligned}$$

as required.

Observe that $P_2(J)+P_2(K)$ is a projection on A with range $J+K$ and kernel equal to $\mathrm{Ker}(J+K)$. Therefore, by [18], Theorem 3.4, $P_2(J)+P_2(K)$ is the structural projection $P_2(J+K)$ onto the weak*-closed inner ideal $J+K$. Similarly, $P_0(J)P_0(K)$ is a projection on A with range $J_0 \cap K_0$ and kernel equal to the kernel $\mathrm{Ker}(J_0 \cap K_0)$ of the weak*-closed inner ideal $J_0 \cap K_0$, and it follows that $P_0(J+K)$ is equal to $P_0(J)P_0(K)$. Finally,

$$\begin{aligned} P_1(J+K) &= \mathrm{id}_A - P_2(J+K) - P_0(J+K) \\ &= P_1(J)P_1(K) + P_1(J)P_0(K) + P_0(J)P_1(K), \end{aligned}$$

as required.

(iii) Since J and K are compatible, their corresponding Peirce projections form a commuting family, and it follows from (ii) that these also commute with the Peirce projections corresponding to $J+K$. This completes the proof the theorem. □

This theorem has the following corollary, which is an immediate consequence of [13], Corollary 4.5.

Corollary 3.6. *Under the conditions of Theorem 3.4,*

$$\{J, K, J+K, J^{\perp}, K^{\perp}, J^{\perp\perp}, K^{\perp\perp}, J^{\perp\perp} \cap J_1, K^{\perp\perp} \cap K_1\}$$

forms a family of pairwise compatible weak-closed inner ideals in A.*

It is worth observing that it also follows from [13], Corollary 3.5, that all the weak*-closed inner ideals in the set above are Peirce, with the possible exception of $J+K$. A discussion of whether or not $J+K$ is also Peirce will be postponed until the next section. Suffice to comment that, at this stage, there is no obvious reason to believe that $J+K$ is Peirce.

It follows that $P_2(K)$ is a projection from J_1 onto K_2 with kernel equal to $\mathrm{Ker}_{J_1}(K_2)$. Therefore, from [18], Theorem 3.4, $P_2(K)$ is the structural projection from J_1 onto K_2. Similarly, $P_1(J)P_0(K)$ is the structural projection onto the weak*-closed inner ideal $J_1 \cap K_0$ in J_1. It follows that $P_1(J)P_1(K)$ is the Peirce-one projection from J_1 onto the Peirce-one space $(K_2)_{J_1,1}$. Since both J and K are Peirce, by [22], Theorem 4.8, the projections $P_1(J)$ and $P_1(K)$ are contractive. The same clearly applies to their product, and the same theorem shows that K_2 is a Peirce weak*-closed inner ideal in J_1.

The same results clearly apply when the roles of J and K are reversed. □

Two lemmas are required before it is possible to prove the main result concerning the central kernel of the supremum $J+K$ of the pair of J and K of rigidly collinear Peirce weak*-closed inner ideals. The first is of a fairly general nature.

Lemma 3.8. *Let A be a JBW*-triple and let M and N be weak*-closed subtriples in A such that A coincides with $M \oplus N$ and*

$$\{M\ M\ N\} \subseteq N, \quad \{N\ N\ M\} \subseteq M, \quad \{M\ N\ M\} = \{0\}, \quad \{N\ M\ N\} = \{0\}.$$

Then, the following results hold.

(i) *The weak*-closed subtriples M and N of A are Peirce inner ideals in A.*
(ii) *The weak*-closed inner ideals $M^\perp$ and $N^\perp$ in A coincide with the central kernels $k(M)$ and $k(N)$ of N and M, respectively.*
(iii) *The Peirce decompositions of A corresponding to M and N are given by*

$$A = M_2 \oplus M_1 \oplus M_0 = M \oplus f(N) \oplus k(N),$$

$$A = N_2 \oplus N_1 \oplus N_0 = N \oplus f(M) \oplus k(M),$$

where $f(M)$ and $f(N)$ are the faithful parts of M and N, respectively.
(iv) *The weak*-closed inner ideals M and N in A form a compatible pair.*

Proof. First observe that

$$\{M\ A\ M\} = \{M\ M \oplus N\ M\} = \{M\ M\ M\} + \{M\ N\ M\} = M,$$

with the same result applying to N. Hence, M and N are inner ideals in A. Since

$$\{N\ M\ N\} = \{M\ N\ M\} = \{0\},$$

it follows that

$$M \subseteq \mathrm{Ker}(N), \quad N \subseteq \mathrm{Ker}(M).$$

Hence, using [18], Theorem 5.4,

$$A = M \oplus N \subseteq M \oplus \mathrm{Ker}(M) = A,$$

and it follows that

$$N = \mathrm{Ker}(M).$$

Similarly, M coincides with $\mathrm{Ker}(N)$. The linear projection R on A defined, for an element a of A, by

$$Ra = b,$$

where

$$a = b + c,$$

Observe that the intersection diagram corresponding to M and N is given by

$\cap$	M_2	M_1	M_0
N_2	$\{0\}$	$f(N)$	$k(N)$
N_1	$f(M)$	$\{0\}$	$\{0\}$
N_0	$k(M)$	$\{0\}$	$\{0\}$

and

$$A = \bigoplus_{j,k=0}^{2} (M_j \cap N_k).$$

By [17], §3, M and N form a compatible pair, and the proof of (iv) is complete. Finally, observe that, by [18], Corollary 3.5, there exists a unique structural projection $P_0(k(N))P_2(N)$ onto the weak*-closed inner ideal $f(N)$. It follows from [18], Theorem 3.4, that

$$\begin{aligned} P_1(M) &= \mathrm{id}_A - P_0(M) - P_2(M) = \mathrm{id}_A - P_2(k(N))P_2(N) - (\mathrm{id}_A - P_2(N)) \\ &= P_0(k(N))P_2(N). \end{aligned}$$

Therefore, being a product of contractive projections, the projection $P_1(M)$ is contractive. Hence, by [22], Theorem 4.8, M is a Peirce inner ideal in A. The same clearly applies to N, thereby completing the proof of (i). □

Lemma 3.9. *Let A be a JBW*-triple, let J and K form a rigidly collinear pair of Peirce weak*-closed inner ideals in A, having corresponding Peirce spaces J_0, J_1, and J_2, and K_0, K_1, and K_2, and let*

$$B = J_2 \oplus (J_1 \cap K_0).$$

Then, B is a weak-closed subtriple of A in which J_2 and $J_1 \cap K_0$ are compatible Peirce weak*-closed inner ideals, the corresponding Peirce decompositions being given by*

$$\begin{aligned} B &= (J_2)_{B,2} \oplus (J_2)_{B,1} \oplus (J_2)_{B,0} \\ &= J_2 \oplus f_B(J_1 \cap K_0) \oplus k_B(J_1 \cap K_0), \\ B &= (J_1 \cap K_0)_{B,2} \oplus (J_1 \cap K_0)_{B,1} \oplus (J_1 \cap K_0)_{B,0} \\ &= (J_1 \cap K_0) \oplus f_B(J_2) \oplus k_B(J_2). \end{aligned}$$

Proof. Notice that, using (1.4)-(1.5),

$$\begin{aligned} \{B\ B\ B\} &= \{J_2 \oplus J_1 \cap K_0\ J_2 \oplus J_1 \cap K_0\ J_2 \oplus J_1 \cap K_0\} \\ &= \{J_2\ J_2\ J_2\} + \{J_2\ J_2\ J_1 \cap K_0\} + \{J_2\ J_1 \cap K_0\ J_2\} \\ &\quad + \{J_2\ J_1 \cap K_0\ J_1 \cap K_0\} + \{J_1 \cap K_0\ J_2\ J_1 \cap K_0\} \\ &\quad + \{J_1 \cap K_0\ J_1 \cap K_0\ J_1 \cap K_0\} \\ &\subseteq J_2 \oplus J_1 \cap K_0 \oplus \{0\} \oplus J_2 \oplus J_1 \cap K_0 \\ &= B. \end{aligned}$$

Hence, B is a weak*-closed subtriple of A, and, clearly, J_2 and $J_1 \cap K_0$ are weak*-closed subtriples of B such that

$$\{J_2\ J_2\ J_1 \cap K_0\} \subseteq J_1 \cap K_0, \quad \{J_1 \cap K_0\ J_1 \cap K_0\ J_2\} \subseteq J_2,$$

$$\{J_2\ J_1 \cap K_0\ J_2\} = \{J_1 \cap K_0\ J_2\ J_1 \cap K_0\} = \{0\}.$$

Therefore, by Lemma 3.7, Lemma 3.8, and [27], Theorem 3.14,

$$((J+K)_1)^{\perp} \cap J = k_{K_1}(J) \cap k_{J_2 \oplus J_1 \cap K_0}(J).$$

The same result with J and K reversed clearly holds and, using (3.20), the proof of the theorem is complete.

□

4. Examples and remarks

One possible conjecture about the supremum $J+K$ of the rigidly collinear pair J and K of Peirce weak*-closed inner ideals in a JBW*-triple A is that it is also Peirce. In order to refute this conjecture it suffices to consider the following example.

Recall that the non-associative algebra $\mathbb{O}$ of complex octonions can be represented as the complex vector space of matrices of the form

$$u = \begin{bmatrix} \alpha & \mathbf{x} \\ \mathbf{y} & \beta \end{bmatrix},$$

where α and β lie in $\mathbb{C}$, and $\mathbf{x}$ and $\mathbf{y}$ lie in $\mathbb{C}^3$. Addition is pointwise, whilst multiplication is given by

$$\begin{aligned} uu' &= \begin{bmatrix} \alpha & \mathbf{x} \\ \mathbf{y} & \beta \end{bmatrix} \begin{bmatrix} \alpha' & \mathbf{x}' \\ \mathbf{y}' & \beta' \end{bmatrix} \\ &= \begin{bmatrix} \alpha\alpha' + \mathbf{x}.\mathbf{y}' & \alpha\mathbf{x}' + \beta'\mathbf{x} + \mathbf{y} \wedge \mathbf{y}' \\ \alpha'\mathbf{y} + \beta\mathbf{y}' - \mathbf{x} \wedge \mathbf{x}' & \beta\beta' + \mathbf{x}'.\mathbf{y} \end{bmatrix}. \end{aligned}$$

Let $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ be unit basis vectors in $\mathbb{C}^3$ in the three co-ordinate directions. Then the following elements of $\mathbb{O}$ form a basis:

$$c_1^+ = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}; \quad c_1^- = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}; \quad c_2^+ = \begin{bmatrix} 0 & 0 \\ \mathbf{i} & 0 \end{bmatrix}; \quad c_2^- = \begin{bmatrix} 0 & -\mathbf{i} \\ 0 & 0 \end{bmatrix};$$

$$c_3^+ = \begin{bmatrix} 0 & 0 \\ \mathbf{j} & 0 \end{bmatrix}; \quad c_3^- = \begin{bmatrix} 0 & -\mathbf{j} \\ 0 & 0 \end{bmatrix}; \quad c_4^+ = \begin{bmatrix} 0 & 0 \\ \mathbf{k} & 0 \end{bmatrix}; \quad c_4^- = \begin{bmatrix} 0 & -\mathbf{k} \\ 0 & 0 \end{bmatrix}.$$

This basis is known as the *Cayley grid* for $\mathbb{O}$. The natural involution $u \mapsto u^\circ$ is given by

$$\begin{bmatrix} \alpha & \mathbf{x} \\ \mathbf{y} & \beta \end{bmatrix}^\circ = \begin{bmatrix} \bar{\alpha} & \bar{\mathbf{y}} \\ \bar{\mathbf{x}} & \bar{\beta} \end{bmatrix},$$

where

$$(x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k})^{-} = (\bar{x}_1\mathbf{i} + \bar{x}_2\mathbf{j} + \bar{x}_3\mathbf{k}).$$

Let A denote the JBW*-triple factor $M_{1,2}(\mathbb{O})$ of 1×2 matrices over $\mathbb{O}$. The quadratic operator in A is defined, for elements $[u_1\ u_2]$ and $[v_1\ v_2]$ in A by

$$\begin{aligned} Q([u_1\ u_2])([v_1\ v_2]) &= \{[u_1\ u_2]\ [v_1\ v_2]\,[u_1\ u_2]\} \\ &= [u_1(v_1^\circ u_1) + u_2(v_2^\circ u_1)\ \ u_2(v_2^\circ u_2) + u_1(v_1^\circ u_2)]. \end{aligned}$$

It follows that for elements $[u_1\ u_2]$, $[v_1\ v_2]$ and $[w_1\ w_2]$ in A, the triple product is defined by

$$\begin{aligned} 2\{[u_1\ u_2]\ [v_1\ v_2]\ [w_1\ w_2]\} = [&u_1(v_1^\circ w_1) + w_1(v_1^\circ u_1) + u_2(v_2^\circ w_1) \\ &+ w_2(v_2^\circ u_1) \quad u_1(v_1^\circ w_2) + w_1(v_1^\circ u_2) \\ &+ u_2(v_2^\circ w_2) + w_2(v_2^\circ u_2)]. \end{aligned}$$

Boolean lattice that is the orthomodular lattice centre $\mathcal{ZP}(A)$ of $\mathcal{P}(A)$. Moreover, with respect to the Jordan triple product defined, for elements a, b and c in A, by

$$\{a\ b\ c\} = \frac{1}{2}(ab^*c + cb^*a),$$

A is a JBW*-triple. For details, the reader is referred to [44], [45] and [47].

For each element e in $\mathcal{P}(A)$, the *central support* $c(e)$ of e is defined by

$$c(e) = \bigwedge\{z \in \mathcal{ZP}(A) : e \leq z\}.$$

A pair (e, f) of elements of $\mathcal{P}(A)$ is said to be *centrally equivalent* if $c(e)$ and $c(f)$ coincide. The common central support is denoted by $c(e, f)$. When endowed with the product ordering, the set $\mathcal{CP}(A)$ of centrally equivalent pairs of elements of $\mathcal{P}(A)$ forms a complete lattice in which the lattice supremum coincides with the supremum in the product lattice, but, in general, the lattice infimum does not. The results of [19] show that the mapping $(e, f) \mapsto eAf$ is an order isomorphism from $\mathcal{CP}(A)$ onto $\mathcal{I}(A)$.

For an element (e, f) in $\mathcal{CP}(A)$, let

$$(e, f)' = (c(f')e', c(e')f'). \tag{4.1}$$

Then, the mapping $(e, f) \mapsto (e, f)'$ is order reversing, and if J is the weak*-closed inner ideal eAf in A, then the annihilator $J^{\perp}$ coincides with $c(f')e'Ac(e')f'$. It follows that the generalized Peirce decomposition of A corresponding to the weak*-closed inner ideal J is given by

$$J = J_0 \oplus J_1 \oplus J_2,$$

where

$$J_2 = eAf, \qquad J_0 = c(e')e'Ac(e')f',$$

and

$$J_1 = ec(f')Ac(e, f)f' + c(e, f)e'Ac(e')f.$$

Furthermore, every weak*-closed inner ideal J in A is Peirce.

The results of [23] show that for two elements (e, f) and (g, h) of $\mathcal{CP}(A)$ the corresponding weak*-closed inner ideals

$$J = eAf, \quad K = gAh,$$

are compatible if and only if

$$eg = ge, \quad fh = hf,$$

and are orthogonal if and only if $(e, f) \leq (g, h)'$, or, equivalently, if and only if, in $\mathcal{P}(A)$,

$$e + g \leq 1, \quad f + h \leq 1.$$

Although the complete lattice $\mathcal{CP}(A)$ is not, in general, orthomodular, it is possible to give a definition of its centre. An element (g, h) in $\mathcal{CP}(A)$ is said to be *central* if, for each element (e, f) in $\mathcal{CP}(A)$,

$$(e, f) = ((g, h) \wedge (e, f)) \vee ((g, h)' \wedge f)).$$

The results of [23] show that (g, h) is central if and only if g and h are equal and lie in $\mathcal{ZP}(A)$. Denoting by $\mathcal{ZCP}(A)$ the set of elements of $\mathcal{CP}(A)$ of the form (w, w), where w lies in $\mathcal{ZP}(A)$, the restriction of the mapping $(e, f) \mapsto eAf$ to $\mathcal{ZCP}(A)$ is

As in the proof of Theorem 3.10, using (4.9)-(4.10),

$$\begin{aligned} k_{K_1}(J) &= (J_1 \cap K_1)^{\perp} \cap J \\ &= w_1 c(f'h')' e A w_1 (c(f'h')' + c(f'h')c(e'))f \\ &\quad \oplus w_2(c(e'g')' + c(e'g')c(f'))eAw_2 c(e'g')' f, \end{aligned} \tag{4.11}$$

$$\begin{aligned} k_{J_2 \oplus J_1 \cap K_0}(J) &= (J_1 \cap K_0)^{\perp} \cap J \\ &= w_1 c(e')' e A w_1 c(e')' f \oplus w_2 c(f')' e A w_2 c(f')' f. \end{aligned} \tag{4.12}$$

From (4.11)-(4.12) it can be seen that

$$\begin{aligned} &k_{K_1}(J) \cap k_{J_2 \oplus J_1 \cap K_0}(J) \\ &= w_1 c(f'h')' c(e')' e A w_1 c(f'h')' c(e')' f \oplus w_2 c(e'g')' c(f')' e A w_2 c(e'g')' c(f')' f. \end{aligned}$$

Therefore, by Theorem 3.10,

$$\begin{aligned} k(J+K) &= k_{K_1}(J) \cap k_{J_2 \oplus J_1 \cap K_0}(J) \oplus k_{K_1}(J) \cap k_{J_2 \oplus J_1 \cap K_0}(J) \\ &= w_1 c(f'h')' c(e')' e A w_1 c(f'h')' c(e')' (f+h) \\ &\quad \oplus w_2 c(e'g')' c(f')' (e+g) A w_2 c(e'g')' c(f')' f, \end{aligned}$$

a conclusion that could, of course, also have been reached using (4.2) and [27], Theorem 4.1. Notice that, in the special case in which

$$w_1 f + w_1 h = w_1, \quad e w_1 = g w_1 = w_1,$$

then

$$w_1 c(f'h')' = w_1 c(e')' = w_1,$$

and the ideal $w_1 A w_1$ is contained in $k(J+K)$. It is therefore possible to give simple finite-dimensional examples in which $k(J+K)$ is non-zero.

Observe that, using [24], similar calculations to those used above apply when the W*-algebra A is replaced by any rectangular JBW*-triple.

REFERENCES

[1] E.M. ALFSEN and E.G. EFFROS, Structure in real Banach spaces I, *Ann. Math.* **96** (1972), 98-128.

[2] E.M. ALFSEN and E.G. EFFROS, Structure in real Banach spaces II, *Ann. Math.* **96** (1972), 129-174.

[3] T.J. BARTON and R.M. TIMONEY, Weak*-continuity of Jordan triple products and its applications, *Math. Scand.*, **59** (1986), 177-191.

[4] T.J. BARTON, T. DANG, and G. HORN, Normal representations of Banach Jordan triple systems, *Proc. Amer. Math. Soc.*, **102** (1987), 551-555.

[5] E. BEHRENDS, *M-structure and the Banach-Stone Theorem*, Lecture Notes in Mathematics, Vol. 736, Springer-Verlag, Berlin/Heidelberg/New York, 1979.

[6] F.F. BONSALL and J. DUNCAN, *Numerical Ranges of Operators on Normed Spaces and of Elements of Normed Algebras*, Cambridge Univ. Press, Cambridge, 1971.

[7] L.J. BUNCE and C-H. CHU, Compact operations, multipliers and the Radon Nikodym property in JB*-triples, *Pacific J. Math.*, **153** (1992), 249-265.

[8] F. CUNNINGHAM JNR., M-structure in Banach spaces, *Math. Proc. Cambridge Philos. Soc.*, **63** (1967), 613-629.

[9] F. CUNNINGHAM JNR., E.G. EFFROS, and N.M. ROY, M-structure in dual Banach spaces, *Israel J. Math.*, 14 (1973), 304-309.

[10] S. DINEEN, Complete holomorphic vector fields in the second dual of a Banach space, *Math. Scand.*, **59** (1986), 131-142.

[11] S. DINEEN, *The second dual of a JB* -triple system, in Complex Analysis, Functional Analysis and Approximation Theory*, (J. Mujica, Ed.), North Holland, Amsterdam, 1986.

[43] E. NEHER, *Jordan Triple Systems by the Grid Approach*, Lecture Notes in Mathematics, Vol. 1280, Springer-Verlag, Berlin/Heidelberg/New York, 1987.
[44] G.K. PEDERSEN. *C^*-algebras and their automorphism groups.* London Mathematical Society Monographs, Vol.14, Academic Press, London 1979.
[45] S. SAKAI, *C^*-algebras and W^*-algebras*, Springer-Verlag, Berlin/Heidelberg/New York, 1971.
[46] L.L. STACHÓ, A projection principle concerning biholomorphic automorphisms, *Acta Sci. Math.*, 44 (1982), 99-124.
[47] H. UPMEIER. *Symmetric Banach manifolds and Jordan C^*-algebras*, North Holland, Amsterdam, 1985.
[48] H. UPMEIER. *Jordan Algebras in Analysis, Operator Theory, and Quantum Mechanics*, American Mathematical Society, Providence, 1986.
[49] J.D.M. WRIGHT, Jordan C*-algebras, *Michigan Math. J.*, **24** (1977), 291-302.
[50] M.A. YOUNGSON, A Vidav theorem for Banach Jordan algebras, *Math. Proc. Cambridge Philos. Soc.*, 84 (1978), 263-272.

THE QUEEN'S COLLEGE, OXFORD, UNITED KINGDOM
E-mail address: martin.edwards@queens.ox.ac.uk