

Decoherence in Pre-symmetric Spaces

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ABSTRACT

Pre-symmetric complex Banach spaces have been proposed as models for state spaces of physical systems. A structural projection on a pre-symmetric space A_* represents an operation on the corresponding system, and has as its range a further pre-symmetric space which represents the state space of the resulting system and symmetries of the system are represented by elements of the group $\text{Aut}(A_*)$ of linear isometries of A_* . Two structural projections R and S on the pre-symmetric space A_* represent decoherent operations when their ranges are rigidly collinear. It is shown that, for decoherent elements x and y of A_* , there exists an involutive element ϕ_* in $\text{Aut}(A_*)$ which conjugates the structural projections corresponding to x and y , and conditions are found for ϕ_* to exchange x and y . The results are used to investigate when certain subspaces of A_* are the ranges of contractive projections and, therefore, represent systems arising from filtering operations.

Key words: JBW*-triple, pre-symmetric space, decoherence, involutive grading, exchange automorphism.

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Introduction

This paper presents the results of further investigations into the structure and properties of pre-symmetric complex Banach spaces. A complex Banach space A_* is said to be pre-symmetric if the open unit ball in its Banach dual space A is a bounded symmetric domain. In this case A possesses a natural triple product with respect to which it forms a JBW*-triple with unique predual A_* . Hence, there exists a bijection between the set of pre-symmetric spaces and the set of JBW*-triples. An

important property of a JBW*-triple A is that the group of linear isometries of A to itself coincides with the group of triple automorphisms $\text{Aut}(A)$ of A , and, because of the uniqueness of the pre-dual A_* of A , each such mapping is weak*-continuous, which implies that there exists an isomorphism $\phi \mapsto \phi_*$ from $\text{Aut}(A)$ onto the group $\text{Aut}(A_*)$ of linear isometries of A_* . An element ϕ of $\text{Aut}(A)$ is said to be involutive if its square is equal to the identity id_A . Clearly, this property is preserved under the mapping $\phi \mapsto \phi_*$. There is a bijection $\phi \mapsto (B_+^\phi, B_-^\phi)$ between the family of involutive automorphisms of A and that of involutive gradings of A , the properties of which were discussed in [17]. A further important property of a pre-symmetric space A_* is that its image QA_* under a contractive projection Q is also a pre-symmetric space [29, 40].

In some approaches to the theory of statistical physical systems, a pre-symmetric space A_* represents the state space of the system, the linear isometries representing the symmetries of the system and the contractive projections on A_* representing filters on the system [19–22].

A contractive projection R on the pre-symmetric space A_* is said to be *structural* if, for each element x of A_* such that Rx and x have equal norm it follows that Rx and x coincide, and, if x is an element of A_* for which, for all elements y in A_*

$$\|x + Ry\| = \|x\| + \|Ry\|,$$

then x lies in the kernel of R . Such a projection may more naturally be termed a geometric structural projection, but, in the interest of brevity, will simply be termed a structural projection. The range RA_* of a structural projection is said to be a *structural subspace* of A_* . When ordered by set inclusion the family $\mathcal{I}_*(A_*)$ of structural subspaces of A_* forms a complete lattice, as does the family $\mathcal{S}_*(A_*)$ of structural projections with respect to the order inherited from $\mathcal{I}_*(A_*)$. The adjoint R^* of an element R of $\mathcal{S}_*(A_*)$ is an element of the complete lattice $\mathcal{S}(A)$ of algebraic structural projections on the JBW*-triple A the range R^*A of which is an element of the complete lattice $\mathcal{I}(A)$ of weak*-closed inner ideals in A , the structural subspace RA_* being the predual of R^*A [10]. The topics discussed in this paper may be viewed as properties of any one of the four order isomorphic complete lattices $\mathcal{S}_*(A_*)$, $\mathcal{I}_*(A_*)$, $\mathcal{S}(A)$, and $\mathcal{I}(A)$. However, when considering pre-symmetric spaces as state spaces, the physical aspects are more transparent when viewed as properties of structural projections and structural subspaces of A_* .

For each element R of $\mathcal{S}_*(A_*)$, there exists a unique element R_0 of $\mathcal{S}_*(A_*)$ such that

$$R_0A_* = \{x \in A_* : \|x + y\| = \|x\| + \|y\|, \forall y \in RA_*\}.$$

Writing

$$R_2 = R, \quad R_1 = \text{id}_{A_*} - R_2 - R_0,$$

R_2 , R_1 , and R_0 are pairwise orthogonal projections, and

$$\text{id}_{A_*} = R_2 + R_1 + R_0$$

is the *generalized Peirce decomposition* of id_{A_*} corresponding to R . Similarly,

$$A_* = R_2A_* \oplus R_1A_* \oplus R_0A_*, \quad A = R_2^*A \oplus R_1^*A \oplus R_0^*A,$$

are the *generalized Peirce decompositions* of A_* and A corresponding to the structural subspace R_2A_* of A_* and the weak*-closed inner ideal R_2^*A of A , respectively. Observe that, although R_2A_* and R_0A_* are pre-symmetric spaces, with dual spaces the weak*-closed inner ideals R_2^*A and R_0^*A , since R_1 need not be contractive, R_1^*A need not be a JBW*-triple and certainly need not be a subtriple of A . In the special case in which R_1 is contractive, the structural projection R is said to be a *Peirce structural projection*, the same adjective being applied to the corresponding structural subspace RA_* , algebraic structural projection R^* , and inner ideal R_2^*A . Remarkably, this condition is equivalent to the purely algebraic condition that the Peirce spaces R_2^*A , R_1^*A , and R_0^*A satisfy all of the Peirce arithmetical relations [16].

Returning to physical systems, it is clear that the two conditions that a contractive projection R must satisfy in order to be structural have powerful physical motivations. In this case R represents a measurement with two outcomes in which R_2A_* represents the set of states with a positive outcome, R_0A_* represents the set of states with a negative outcome, and R_1A_* represents the information lost in the measurement. A possible interpretation of the condition that R is Peirce is that the lost information is recoverable by means of the filter represented by the contractive projection R_1 .

A pair R and S of structural projections on the pre-symmetric space A_* is said to be *compatible* if, for j and k equal to 0, 1, and 2, the Peirce projections R_j and S_k corresponding to R and S commute. In a recent paper, the properties of a pair R and S of structural projections on the pre-symmetric space A_* which are *rigidly collinear* in that

$$R_2A_* \subseteq S_1A_*, \quad S_2A_* \subseteq R_1A_*,$$

were studied [9]. Physically, such pairs may be thought to represent decoherent operations. The main results of the paper show that if R and S are both Peirce structural projections then the supremum $R \vee S$ in the complete lattice $\mathcal{S}_*(A_*)$ is equal to $R + S$, and that the three structural projections R , S , and $R + S$ are compatible, results which have clear physical interpretations.

The main purpose of this paper is to study pairs x and y of elements of the pre-symmetric space A_* of norm one which are *decoherent* in the sense that the smallest structural projections S_x and S_y which, respectively, leave x and y fixed, are rigidly collinear. In this case there exists an involutive automorphism ϕ of A the pre-adjoint ϕ_* of which conjugates the structural projections S_x and S_y . A central result of the paper shows that ϕ_* actually maps x to y if and only if

$$\|x + y\|^2 = 2.$$

Such pairs x and y of elements of A_* of norm one are said to be *strongly decoherent*.

In the theory of Banach spaces and in the study of state spaces of physical systems, much effort has been devoted to the investigation of when particular closed subspaces of a Banach space are the ranges of contractive projections. Since the range of a contractive projection on a pre-symmetric space is itself a pre-symmetric space, this problem is particularly relevant in the case of pre-symmetric spaces. The main results are used to determine under what circumstances the closed subspace of A_* generated by a family $\{x_j : j \in \Lambda\}$ of elements of A_* of norm one which are pairwise decoherent is the range of a contractive projection. Remarkably, this occurs if and only if the family is strongly decoherent.

The results of the paper depend crucially upon the detailed study of pairs of rigidly collinear tripotents in the JBW^* -triple. As a consequence, some of the preliminary results given are rather more general than strictly needed to produce the result described above, but may well be of independent interest. Furthermore, the proofs of many of the results are purely algebraic, and hold in the case in which A is an anisotropic Jordan * -triple.

The paper is organized as follows. Section 1 is devoted to a statement of definitions and a description of known results about JBW^* -triples. In section 2 a collection of results relating to the exchange automorphisms induced by collinear pairs of tripotents are presented. Many of these have appeared elsewhere, and only those proofs that are relevant to the discussion in hand are given in detail. In section 3, the main results of the paper are proved, and the final section is devoted to their applications.

1. Preliminaries

A Jordan * -algebra A which is also a complex Banach space such that, for all elements a and b in A , $\|a^*\| = \|a\|$, $\|a \circ b\| \leq \|a\| \|b\|$ and $\|\{a \ a \ a\}\| = \|a\|^3$, where

$$\{a \ b \ c\} = a \circ (b^* \circ c) + (a \circ b^*) \circ c - b^* \circ (a \circ c)$$

is the Jordan triple product on A , is said to be a *Jordan C^* -algebra* [43] or *JB^* -algebra* [44]. A Jordan C^* -algebra which is the dual of a Banach space is said to be a *Jordan W^* -algebra* [8] or a *JBW^* -algebra* [44]. Examples of JB^* -algebras are C^* -algebras and examples of JBW^* -algebras are W^* -algebras, in both cases equipped with the Jordan product

$$a \circ b = \frac{1}{2}(ab + ba).$$

The self-adjoint parts of JB^* -algebras and JBW^* -algebras are said to be *JB -algebras* and *JBW -algebras* respectively. For the properties of C^* -algebras and W^* -algebras the reader is referred to [38, 39] and for the algebraic properties of Jordan algebras to [24, 31, 33, 37].

A complex vector space A equipped with a triple product $(a, b, c) \mapsto \{a \ b \ c\}$ from $A \times A \times A$ to A which is symmetric and linear in the first and third variables, conjugate

linear in the second variable and, for elements a, b, c , and d in A , satisfies the identity

$$[D(a, b), D(c, d)] = D(\{a b c\}, d) - D(c, \{d a b\}), \tag{1}$$

where $[\ , \]$ denotes the commutator, and D is the mapping from $A \times A$ to the algebra of linear operators on A defined by

$$D(a, b)c = \{a b c\}, \tag{2}$$

is said to be a *Jordan*-triple*. For each element a in A , the conjugate linear mapping $Q(a)$ from A to itself is defined, for each element b in A , by

$$Q(a)b = \{a b a\}. \tag{3}$$

For details about the properties of Jordan*-triples the reader is referred to [30].

A Jordan*-triple A which is also a Banach space such that D is continuous from $A \times A$ to the Banach algebra $B(A)$ of bounded linear operators on A , and, for each element a in A , $D(a, a)$ is hermitian in the sense of [3, Definition 5.1], with non-negative spectrum, and satisfies

$$\|D(a, a)\| = \|a\|^2, \tag{4}$$

is said to be a *JB*-triple*. A subspace J of a JB*-triple A is said to be a *subtriple* if $\{J J J\}$ is contained in J , is said to be an *inner ideal* if $\{J A J\}$ is contained in J . Every norm-closed subtriple of a JB*-triple A is a JB*-triple [28], and a norm-closed subspace J of A is an ideal if and only if $\{J J A\}$ is contained in J [4]. A JB*-triple A which is the dual of a Banach space A_* is said to be a *JBW*-triple*. In this case the *predual* A_* of A is unique and, for elements a and b in A , the operators $D(a, b)$ and $Q(a)$ are weak*-continuous. It follows that a weak*-closed subtriple J of a JBW*-triple A is a JBW*-triple. The second dual A^{**} of a JB*-triple A is a JBW*-triple. For details of these results the reader is referred to [1, 2, 6, 7, 23, 25, 28, 29, 41, 42]. Examples of JB*-triples are JB*-algebras, and examples of JBW*-triples are JBW*-algebras.

A pair a and b of elements in a JBW*-triple A is said to be *orthogonal* when $D(a, b)$ is equal to zero. For a subset L of A , the subset L^\perp of A consisting of all elements which are orthogonal to all elements of L is a weak*-closed inner ideal in A which is known as the *annihilator* of L in A . For subsets L, M of A , $L^\perp \cap L \subseteq \{0\}$, $L \subseteq L^{\perp\perp}$, $L \subseteq M$ implies that $M^\perp \subseteq L^\perp$, and L^\perp and $L^{\perp\perp\perp}$ coincide.

For each non-empty subset J of the JBW*-triple A , the *kernel* $\text{Ker}(J)$ of J is the weak*-closed subspace of elements a in A for which $\{J a J\}$ is equal to $\{0\}$. It follows that the annihilator J^\perp of J is contained in $\text{Ker}(J)$ and that $J \cap \text{Ker}(J)$ is contained in $\{0\}$. A subtriple J of A is said to be *complemented* [15] if A coincides with $J \oplus \text{Ker}(J)$. It can easily be seen that every complemented subtriple is a weak*-closed inner ideal. A linear projection P on the JBW*-triple A is said to be an *algebraic structural projection* [31] if, for each element a in A ,

$$PQ(a)P = Q(Pa). \tag{5}$$

The main results of [12, 14, 15] show that the range PA of a structural projection P is a complemented subtriple, that the kernel $\ker P$ of the map P coincides with $\text{Ker}(PA)$, that every structural projection is contractive and weak*-continuous, and, most significantly, that every weak*-closed inner ideal is complemented.

Let $\mathcal{I}(A)$ denote the complete lattice of weak*-closed inner ideals in the JBW*-triple A and let $\mathcal{S}(A)$ denote the set of structural projections on A . The results of [12] can be used to show that the set $\mathcal{S}(A)$ of structural projections on A is a complete lattice with respect to the ordering defined, for elements P and Q , by $P \leq Q$ if QP is equal to P and the mapping $P \mapsto PA$ is an order isomorphism from $\mathcal{S}(A)$ onto the complete lattice $\mathcal{I}(A)$ of weak*-closed inner ideals in A . For an element P in $\mathcal{S}(A)$, the pre-adjoint P_* is an element of the complete lattice $\mathcal{S}_*(A_*)$ of structural projections on the pre-symmetric space A_* , and the mappings $P \mapsto P_*$ and $P \mapsto P_*A_*$ are order isomorphisms from $\mathcal{S}(A)$ onto the complete lattices $\mathcal{S}_*(A_*)$ and $\mathcal{I}_*(A_*)$, respectively [10].

For each element J of $\mathcal{I}(A)$, the annihilator J^\perp also lies in $\mathcal{I}(A)$ and A enjoys the *generalized Peirce decomposition*

$$A = J_0 \oplus J_1 \oplus J_2 \tag{6}$$

where

$$J_0 = J^\perp, \quad J_2 = J, \quad J_1 = \text{Ker}(J) \cap \text{Ker}(J^\perp). \tag{7}$$

The structural projections onto J and J^\perp are denoted by $P_2(J)$ and $P_0(J)$, respectively, and the projection $\text{id}_A - P_2(J) - P_0(J)$ onto J_1 is denoted by $P_1(J)$. Furthermore,

$$\{A J_0 J_2\} = \{0\}, \quad \{A J_2 J_0\} = \{0\}. \tag{8}$$

and, for j, k , and l equal to 0, 1, or 2, the Peirce arithmetical relations,

$$\{J_j J_k J_l\} \subseteq J_{j+l-k}, \tag{9}$$

when $j + l - k$ is equal to 0, 1, or 2, and

$$\{J_j J_k J_l\} = \{0\}, \tag{10}$$

otherwise, hold, except in the cases when (j, k, l) is equal to $(0, 1, 1)$, $(1, 1, 0)$, $(1, 0, 1)$, $(2, 1, 1)$, $(1, 1, 2)$, $(1, 2, 1)$, or $(1, 1, 1)$. The relations (9), (10) hold in all cases if and only if $P_1(J)$ is contractive in which case J is said to be a *Peirce inner ideal*. The word Peirce is also used to describe the corresponding algebraic structural projection on A , structural projection on A_* , and structural subspace of A_* . A pair J and K of elements of $\mathcal{I}(A)$ is said to be *compatible* if, for j and k equal to 0, 1, or 2,

$$[P_j(J), P_k(K)] = 0,$$

or, equivalently, if

$$A = \bigoplus_{j,k=0,1,2} J_j \cap K_k$$

[11]. The pair J and K is said to be *rigidly collinear* if J_2 is contained in K_1 and K_2 is contained in J_1 . If J and K are Peirce and form a rigidly collinear pair then it can be shown that the subspace $J + K$ of A is a weak*-closed inner ideal in A , the Peirce spaces corresponding to $J + K$ are given by

$$\begin{aligned} (J + K)_2 &= J + K, \\ (J + K)_1 &= J_1 \cap K_1 \oplus J_1 \cap K_0 \oplus J_0 \cap K_1, \\ (J + K)_0 &= J_0 \cap K_0, \end{aligned}$$

with corresponding Peirce projections

$$\begin{aligned} P_2(J + K) &= P_2(J) + P_2(K), \\ P_1(J + K) &= P_1(J)P_1(K) + P_1(J)P_0(K) + P_0(J)P_1(K), \\ P_0(J + K) &= P_0(J)P_0(K), \end{aligned}$$

and the weak*-closed inner ideals J , K , and $J + K$ form a pairwise compatible family [9].

Let A be a JBW*-triple with predual A_* and let $\text{Aut}(A)$ denote the group of linear isometries from A to itself. This coincides with the group of triple automorphisms of A . Each element ϕ of $\text{Aut}(A)$ is weak*-continuous and, if ϕ_* denotes the pre-adjoint of ϕ then the mapping $\phi \mapsto \phi_*$ is an isomorphism from the group $\text{Aut}(A)$ onto the group $\text{Aut}(A_*)$ of linear isometries of A .

Recall that a pair (B_+, B_-) of weak*-closed subtriples of a JBW*-triple A is said to be an *involutive grading* of A if

$$\begin{aligned} A &= B_+ \oplus B_-, \\ \{B_+ B_- B_+\} &\subseteq B_-, & \{B_- B_+ B_-\} &\subseteq B_+, \\ \{B_+ B_+ B_-\} &\subseteq B_-, & \{B_- B_- B_+\} &\subseteq B_+. \end{aligned}$$

Observe that, by symmetry, if (B_+, B_-) is an involutive grading then so also is (B_-, B_+) which, in this case, is said to be the *opposite grading*. An element ϕ of the group $\text{Aut}(A)$ of triple automorphisms of A which satisfies the condition that ϕ^2 coincides with the identity id_A , is said to be an *involutive automorphism* of A . Notice that if ϕ is an involutive automorphism of A then so also is $-\phi$. Let A be a JBW*-triple, let ϕ be an involutive automorphism of A , and let

$$B_+^\phi = \{a \in A : \phi a = a\}, \quad B_-^\phi = \{a \in A : \phi a = -a\}.$$

Then, (B_+^ϕ, B_-^ϕ) is an involutive grading and the mapping $\phi \mapsto (B_+^\phi, B_-^\phi)$ is a bijection from the set of involutive automorphisms of A onto the set of involutive gradings of A , such that $(B_+^{-\phi}, B_-^{-\phi})$ coincides with (B_-^ϕ, B_+^ϕ) . Furthermore, the pre-adjoint ϕ_* of an involutive automorphism gives rise to an involutive grading $(B_{*+}^{\phi_*}, B_{*-}^{\phi_*})$ of the predual A_* of A in which $B_{*+}^{\phi_*}$ and $B_{*-}^{\phi_*}$ may be identified with the preduals of B_+^ϕ

and B_-^ϕ , respectively. It is clear that, for an involutive automorphism ϕ of the JBW*-triple A , with corresponding involutive grading (B_+^ϕ, B_-^ϕ) , the weak*-continuous linear mapping T_ϕ , defined by

$$T_\phi = \frac{1}{2}(\text{id}_A + \phi), \tag{11}$$

is the linear projection onto the weak*-closed subtriple B_+^ϕ and $T_{-\phi}$ is the linear projection onto the weak*-closed subtriple B_-^ϕ . Clearly, the projections T_ϕ and $T_{-\phi}$ are orthogonal. Similarly, the orthogonal projections T_{ϕ_*} and $T_{-\phi_*}$ map onto $B_{**+}^{\phi_*}$ and $B_{*-}^{\phi_*}$, respectively. The reader is referred to [17, 36] for details.

2. Exchange automorphisms

An element u in a JBW*-triple A is said to be a *tripotent* if $\{u u u\}$ is equal to u . The set of tripotents in A is denoted by $\mathcal{U}(A)$. Observe that, by (4), non-zero tripotents are of norm one. For each tripotent u in A , the weak*-continuous linear operators $P_0(u)$, $P_1(u)$ and $P_2(u)$, defined using (2), (3) by

$$\begin{aligned} P_0(u) &= \text{id}_A - 2D(u, u) + Q(u)^2, & P_1(u) &= 2(D(u, u) - Q(u)^2), \\ P_2(u) &= Q(u)^2, \end{aligned}$$

are mutually orthogonal projection operators on A with sum id_A . For j equal to 0, 1, or 2, the range of $P_j(u)$ is the weak*-closed eigenspace $A_j(u)$ of $D(u, u)$ corresponding to the eigenvalue $\frac{1}{2}j$ and

$$A = A_0(u) \oplus A_1(u) \oplus A_2(u)$$

is the *Peirce decomposition* of A relative to u . Moreover, $A_2(u)$ is a Peirce weak*-closed inner ideal in A , and, consequently, the Peirce arithmetical relations (8)–(10) hold. Observe that when endowed with the product $(a, b) \mapsto a \circ_u b = \{a u b\}$ and the involution $a \mapsto a^\dagger = \{u a u\}$ the Peirce space $A_2(u)$ forms a JBW*-algebra with unit u .

Elements u and v in $\mathcal{U}(A)$ are said to be *compatible* if the weak*-closed inner ideals $A_2(u)$ and $A_2(v)$ are compatible. Observe that u and v are compatible if u lies in a Peirce j -space of v [32]. For compatible tripotents u and v , and, for j and k equal to 0, 1, or 2, denote by $A_{jk}(u, v)$ the intersection of $A_j(u)$ and $A_k(v)$. The corresponding orthogonal decomposition of A is said to be the *joint Peirce decomposition* of A corresponding to u and v . Two tripotents u and v are said to be *orthogonal* if one of the following equivalent conditions hold: $D(u, v) = 0$; $D(v, u) = 0$; $\{u u v\} = 0$; $\{v v u\} = 0$; v lies in $A_0(u)$; u lies in $A_0(v)$. Observe that there exists a partial order on $\mathcal{U}(A)$ defined, for elements u and v in $\mathcal{U}(A)$, by $u \leq v$ if $\{u v u\}$ is equal to u , or, equivalently if $v - u$ is a tripotent orthogonal to u . The set $\tilde{\mathcal{U}}(A)$ consisting of $\mathcal{U}(A)$ with a largest element adjoined forms a complete lattice. Two non-zero tripotents u and v are said to be *collinear* if u lies in $A_1(v)$ and v lies in $A_1(u)$ and *rigidly collinear*

if $A_2(u)$ and $A_2(v)$ form a pair of rigidly collinear weak*-closed inner ideals. For more details the reader is referred to [30, 33, 34].

Let A be a JBW*-triple and let u and v be a pair of collinear elements of the partially ordered set $\mathcal{U}(A)$ of tripotents in A . Observe that, because of the symmetry of the collinearity relation any result that holds for one element of a collinear pair also holds for the other. A calculation using (6)–(10) shows that the mapping $\phi_{u,v}$ defined by

$$\phi_{u,v} = \text{id}_A - 2D(u + v, u + v) + Q(u + v)^2,$$

is an involutive automorphism of A such that

$$\phi_{u,v}u = v, \quad \phi_{u,v}v = u. \tag{12}$$

The involutive automorphism $\phi_{u,v}$ is known as the *exchange automorphism* corresponding to the collinear pair u and v . For an arbitrary element w of $\mathcal{U}(A)$, the involutive automorphism

$$\psi_w = P_2(w) - P_1(w) + P_0(w)$$

is known as the *Peirce reflection* corresponding to w . A further calculation using (9), (10) shows that $2^{-\frac{1}{2}}(u + v)$ is a non-zero element of $\mathcal{U}(A)$ such that

$$\phi_{u,v} = \psi_{2^{-\frac{1}{2}}(u+v)}.$$

Similarly, for all elements α and β of \mathbb{C} for which

$$|\alpha|^2 + |\beta|^2 = 1,$$

$\alpha u + \beta v$ is a non-zero element of $\mathcal{U}(A)$. Since non-zero tripotents are of norm one it follows that the restriction of the norm in A to the subspace $\mathbb{C}u \oplus \mathbb{C}v$ is Hilbertian, with u and v perpendicular unit vectors in this subspace. Observe that the involutive grading $(B_+^{\phi_{u,v}}, B_-^{\phi_{u,v}})$ corresponding to the involutive automorphism $\phi_{u,v}$ is given by

$$B_+^{\phi_{u,v}} = A_2(2^{-\frac{1}{2}}(u + v)) \oplus A_0(2^{-\frac{1}{2}}(u + v)), \quad B_-^{\phi_{u,v}} = A_1(2^{-\frac{1}{2}}(u + v)), \tag{13}$$

from which, using (11), it can be seen that

$$\begin{aligned} T_{\phi_{u,v}} &= \frac{1}{2}(\text{id}_A + \phi_{u,v}) = P_2(2^{-\frac{1}{2}}(u + v)) + P_0(2^{-\frac{1}{2}}(u + v)), \\ T_{-\phi_{u,v}} &= \frac{1}{2}(\text{id}_A - \phi_{u,v}) = P_1(2^{-\frac{1}{2}}(u + v)). \end{aligned} \tag{14}$$

Furthermore, since $\phi_{u,v}$ is an automorphism, by (12),

$$\phi_{u,v}D(u, u) = D(v, v)\phi_{u,v}, \quad \phi_{u,v}Q(u) = Q(v)\phi_{u,v},$$

from which it follows that, for j equal to 0, 1, or 2,

$$\phi_{u,v}P_j(u) = P_j(v)\phi_{u,v}. \tag{15}$$

Let $(\phi_{u,v})_*$ be the involutive linear isometry of the predual A_* of A which is the pre-adjoint of the involutive automorphism $\phi_{u,v}$. Then, it is clear that the corresponding involutive grading $(B_{*+}^{(\phi_{u,v})_*}, B_{*-}^{(\phi_{u,v})_*})$ of A_* satisfies

$$\begin{aligned} B_{*+}^{(\phi_{u,v})_*} &= B_{*+}^{\phi_{u,v}} = A_2(2^{-\frac{1}{2}}(u+v))_* \oplus A_0(2^{-\frac{1}{2}}(u+v))_*, \\ B_{*-}^{(\phi_{u,v})_*} &= B_{*-}^{\phi_{u,v}} = A_1(2^{-\frac{1}{2}}(u+v))_*. \end{aligned} \tag{16}$$

Moreover,

$$\begin{aligned} T_{(\phi_{u,v})_*} &= \frac{1}{2}(\text{id}_{A_*} + (\phi_{u,v})_*) = P_2(2^{-\frac{1}{2}}(u+v))_* + P_0(2^{-\frac{1}{2}}(u+v))_*, \\ T_{-(\phi_{u,v})_*} &= \frac{1}{2}(\text{id}_{A_*} - (\phi_{u,v})_*) = P_1(2^{-\frac{1}{2}}(u+v))_*. \end{aligned}$$

The action of $\phi_{u,v}$ on the joint Peirce spaces of u and v is given in the following result, the proof of which can be found in [34, Theorem 1.1].

Lemma 2.1. *Let A be a JBW^* -triple, let u and v be a pair of collinear tripotents in A with corresponding Peirce spaces $A_0(u), A_1(u),$ and $A_2(u)$ and $A_0(v), A_1(v),$ and $A_2(v)$, respectively, let $\phi_{u,v}$ be the exchange automorphism associated with u and v and, for j and k equal to 0, 1, or 2, let*

$$A_{j,k}(u, v) = A_j(u) \cap A_k(v).$$

Then, the following results hold:

- (i) $\phi_{u,v}A_{22}(u, v) = A_{22}(u, v)$ and $\phi_{u,v}|_{A_{22}(u,v)} = \text{id}_{A_{22}(u,v)}$,
- (ii) $\phi_{u,v}A_{21}(u, v) = A_{12}(u, v)$ and $\phi_{u,v}|_{A_{21}(u,v)} = 2D(v, u)|_{A_{21}(u,v)}$,
- (iii) $\phi_{u,v}A_{20}(u, v) = A_{02}(u, v)$ and $\phi_{u,v}|_{A_{20}(u,v)} = Q(v)Q(u)|_{A_{20}(u,v)}$,
- (iv) $\phi_{u,v}A_{11}(u, v) = A_{11}(u, v)$ and $\phi_{u,v}|_{A_{11}(u,v)} = (4D(u, v)D(v, u) - \text{id}_A)|_{A_{11}(u,v)}$,
- (v) $\phi_{u,v}A_{10}(u, v) = A_{01}(u, v)$ and $\phi_{u,v}|_{A_{10}(u,v)} = -2D(v, u)|_{A_{10}(u,v)}$,
- (vi) $\phi_{u,v}A_{00}(u, v) = A_{00}(u, v)$ and $\phi_{u,v}|_{A_{00}(u,v)} = \text{id}_{A_{00}(u,v)}$.

The proof of the following result may be found in [32, Lemma 3.4].

Lemma 2.2. *Let A be a JBW^* -triple, let u and v be a pair of collinear tripotents in A with corresponding Peirce spaces $A_0(u), A_1(u),$ and $A_2(u)$ and $A_0(v), A_1(v),$ and $A_2(v)$, respectively. Then, the following are equivalent.*

- (i) u and v form a rigidly collinear pair,
- (ii) $A_2(u) \subseteq A_1(v)$,
- (iii) $A_2(v) \subseteq A_1(u)$,
- (iv) $A_2(u) \cap A_2(v) = \{0\}$.

A combination of Lemmas 2.1 and 2.2 yields the following result.

Lemma 2.3. *Let A be a JBW^* -triple, let u and v be a pair of rigidly collinear tripotents in A with corresponding Peirce spaces $A_0(u)$, $A_1(u)$, and $A_2(u)$ and $A_0(v)$, $A_1(v)$, and $A_2(v)$, respectively, let $\phi_{u,v}$ be the exchange automorphism associated with u and v and, for j and k equal to 0, 1, or 2, let*

$$A_{j,k}(u, v) = A_j(u) \cap A_k(v).$$

Then, the following results hold:

- (i) $\phi_{u,v}A_2(u) = A_2(v)$ and $\phi_{u,v}|_{A_2(u)} = 2D(v, u)|_{A_2(u)}$,
- (ii) $\phi_{u,v}A_{11}(u, v) = A_{11}(u, v)$ and $\phi_{u,v}|_{A_{11}(u,v)} = (4D(u, v)D(v, u) - \text{id}_A)|_{A_{11}(u,v)}$,
- (iii) $\phi_{u,v}A_{10}(u, v) = A_{01}(u, v)$ and $\phi_{u,v}|_{A_{10}(u,v)} = -2D(v, u)|_{A_{10}(u,v)}$,
- (iv) $\phi_{u,v}A_{00}(u, v) = A_{00}(u, v)$ and $\phi_{u,v}|_{A_{00}(u,v)} = \text{id}_{A_{00}(u,v)}$.

The following result is an immediate consequence of [9, Theorem 3.5].

Lemma 2.4. *Let A be a JBW^* -triple, let u and v be a pair of rigidly collinear tripotents in A with corresponding Peirce spaces $A_0(u)$, $A_1(u)$, and $A_2(u)$ and $A_0(v)$, $A_1(v)$, and $A_2(v)$, and Peirce projections $P_0(u)$, $P_1(u)$, and $P_2(u)$ and $P_0(v)$, $P_1(v)$, and $P_2(v)$, respectively, and, for j and k equal to 0, 1, or 2, let*

$$A_{j,k}(u, v) = A_j(u) \cap A_k(v).$$

Then, the following results hold.

- (i) $A_2(u) \oplus A_2(v)$ is a weak*-closed inner ideal in A with corresponding Peirce decomposition

$$\begin{aligned} A &= (A_2(u) \oplus A_2(v))_2 \oplus (A_2(u) \oplus A_2(v))_1 \oplus (A_2(u) \oplus A_2(v))_0 \\ &= (A_2(u) \oplus A_2(v)) \oplus (A_{11}(u, v) \oplus A_{1,0}(u, v) \oplus A_{0,1}(u, v) \oplus A_{00}(u, v)). \end{aligned}$$

- (ii) The corresponding Peirce projections are given by

$$\begin{aligned} P_2(A_2(u) \oplus A_2(v)) &= P_2(u) + P_2(v), \\ P_1(A_2(u) \oplus A_2(v)) &= P_1(u)P_1(v) + P_1(u)P_0(v) + P_0(u)P_1(v), \\ P_0(A_2(u) \oplus A_2(v)) &= P_0(u)P_0(v). \end{aligned}$$

- (iii) The weak*-closed inner ideals $A_2(u)$, $A_2(v)$, and $A_2(u) \oplus A_2(v)$ are pairwise compatible.

Recall that it need not be the case that the weak*-closed inner ideal $A_2(u) \oplus A_2(v)$ that appears in the previous lemma, is Peirce. Indeed, it is easy to find examples for which it is not. It is not true in general that the tripotent $2^{-\frac{1}{2}}(u+v)$ is compatible with u and v . However, it is possible to make some remarks about the relationship between the Peirce spaces corresponding to $2^{-\frac{1}{2}}(u+v)$ and those corresponding to u and v .

Lemma 2.5. *Under the conditions of Lemma 2.3, let $A_0(2^{-\frac{1}{2}}(u+v))$, $A_1(2^{-\frac{1}{2}}(u+v))$, and $A_2(2^{-\frac{1}{2}}(u+v))$ and $P_0(2^{-\frac{1}{2}}(u+v))$, $P_1(2^{-\frac{1}{2}}(u+v))$, and $P_2(2^{-\frac{1}{2}}(u+v))$ be the Peirce spaces and Peirce projections corresponding to the tripotent $2^{-\frac{1}{2}}(u+v)$ in A . Then, the following results hold.*

- (i) $A_2(2^{-\frac{1}{2}}(u+v)) \subseteq A_2(u) \oplus A_2(v)$,
- (ii) $A_0(2^{-\frac{1}{2}}(u+v)) \cap (A_2(u) \oplus A_2(v)) = \{0\}$,
- (iii) $2^{-\frac{1}{2}}(u+v)$ is a maximal tripotent in the JBW*-triple $A_2(u) \oplus A_2(v)$,
- (iv) $(\text{id}_A + \phi_{u,v})A_2(u) \subseteq A_2(2^{-\frac{1}{2}}(u+v))$,
- (v) $P_0(2^{-\frac{1}{2}}(u+v))P_2(u) = 0$,
- (vi) $P_2(2^{-\frac{1}{2}}(u+v))P_2(u) = \frac{1}{2}(\text{id}_A + \phi_{u,v})P_2(u)$,
- (vii) $P_2(u)P_2(2^{-\frac{1}{2}}(u+v))P_2(u) = \frac{1}{2}P_2(u)$,
- (viii) $P_2(2^{-\frac{1}{2}}(u+v))P_2(u)P_2(2^{-\frac{1}{2}}(u+v)) = \frac{1}{2}P_2(2^{-\frac{1}{2}}(u+v))$.

Proof. (i) It can be seen that the tripotent $2^{-\frac{1}{2}}(u+v)$ lies in $A_2(u) \oplus A_2(v)$ which, by Lemma 2.4, is a weak*-closed inner ideal in A . Since $A_2(2^{-\frac{1}{2}}(u+v))$ is the smallest weak*-closed inner ideal containing $2^{-\frac{1}{2}}(u+v)$, the result follows.

(ii) Let a be an element of $A_0(2^{-\frac{1}{2}}(u+v)) \cap (A_2(u) \oplus A_2(v))$ and let b and c be the unique elements of $A_2(u)$ and $A_2(v)$, respectively, such that

$$a = b + c.$$

Then, using (8)–(10) and Lemma 2.3 (i),

$$\begin{aligned} 0 &= 2\{2^{-\frac{1}{2}}(u+v) 2^{-\frac{1}{2}}(u+v) a\} = \{u u a\} + \{u v a\} + \{v u a\} + \{v v a\} \\ &= \{u u b\} + \{u v b\} + \{v u b\} + \{v v b\} \\ &\quad + \{u u c\} + \{u v c\} + \{v u c\} + \{v v c\} \\ &= b + 0 + D(v, u)b + \frac{1}{2}b + \frac{1}{2}c + D(u, v)c + 0 + c \\ &= (\frac{3}{2}b + \frac{1}{2}\phi_{u,v}c) + (\frac{3}{2}c + \frac{1}{2}\phi_{u,v}c). \end{aligned}$$

It follows that

$$3b + \phi_{u,v}c = -3c - \phi_{u,v}b \in A_2(u) \cap A_2(v) = \{0\},$$

by Lemma 2.2. Hence,

$$9b = -3\phi_{u,v}c = \phi_{u,v}^2b = b,$$

and b and, similarly, c , is equal to zero, as required.

- (iii) This is immediate from the previous two results.
- (iv) By Lemma 2.3 (i) it can be seen that

$$(\text{id}_A + \phi_{u,v})A_2(u) \subseteq A_2(u) \oplus A_2(v).$$

Let a be an element of $A_2(u)$. Then, by (14), the element $(\text{id}_A + \phi_{u,v})a$ lies in the weak*-closed subtriple $B_+^{\phi_{u,v}}$ which, by (13), coincides with $A_2(2^{-\frac{1}{2}}(u+v)) \oplus A_0(2^{-\frac{1}{2}}(u+v))$. Therefore,

$$(\text{id}_A + \phi_{u,v})a = P_2(2^{-\frac{1}{2}}(u+v))(\text{id}_A + \phi_{u,v})a + P_0(2^{-\frac{1}{2}}(u+v))(\text{id}_A + \phi_{u,v})a,$$

From above, it follows that $P_0(2^{-\frac{1}{2}}(u+v))(\text{id}_A + \phi_{u,v})a$ is the difference of two elements of $A_2(u) \oplus A_2(v)$ which simultaneously lies in $A_0(2^{-\frac{1}{2}}(u+v))$. By (ii) it is therefore equal to zero and

$$(\text{id}_A + \phi_{u,v})a = P_2(2^{-\frac{1}{2}}(u+v))(\text{id}_A + \phi_{u,v})a,$$

which lies in $A_2(2^{-\frac{1}{2}}(u+v))$, as required.

- (v) Using (14) and (iv) above, for each element a in A ,

$$2(P_2(2^{-\frac{1}{2}}(u+v)) + P_0(2^{-\frac{1}{2}}(u+v)))P_2(u)a = (\text{id}_A + \phi_{u,v})P_2(u)a,$$

which lies in $A_2(2^{-\frac{1}{2}}(u+v))$. Hence, the element

$$P_0(2^{-\frac{1}{2}}(u+v))P_2(u)a \in A_2(2^{-\frac{1}{2}}(u+v)) \cap A_0(2^{-\frac{1}{2}}(u+v)) = \{0\},$$

as required.

- (vi) Using (14) and (v) above,

$$\begin{aligned} \frac{1}{2}(\text{id}_A + \phi_{u,v})P_2(u) &= P_2(2^{-\frac{1}{2}}(u+v))P_2(u) + P_0(2^{-\frac{1}{2}}(u+v))P_2(u) \\ &= P_2(2^{-\frac{1}{2}}(u+v))P_2(u), \end{aligned}$$

as required.

- (vii) By (15), Lemma 2.2, the compatibility of u and v , and (v), (vi) above,

$$\begin{aligned} P_2(u)P_2(2^{-\frac{1}{2}}(u+v))P_2(u) &= P_2(u)\left(\frac{1}{2}(\text{id}_A + \phi_{u,v}) - P_0(2^{-\frac{1}{2}}(u+v))\right)P_2(u) \\ &= P_2(u)\left(\frac{1}{2}(\text{id}_A + \phi_{u,v})\right)P_2(u) \\ &= \frac{1}{2}P_2(u) + \frac{1}{2}P_2(u)P_2(v)\phi_{u,v} = \frac{1}{2}P_2(u), \end{aligned}$$

as required.

(viii) By (13), (14), (15), and (i) and (vi) above,

$$\begin{aligned} P_2(2^{-\frac{1}{2}}(u+v))P_2(u)P_2(2^{-\frac{1}{2}}(u+v)) &= \frac{1}{2}(\text{id}_A + \phi_{u,v})P_2(u)P_2(2^{-\frac{1}{2}}(u+v)) \\ &= \frac{1}{2}P_2(u)P_2(2^{-\frac{1}{2}}(u+v)) \\ &\quad + \frac{1}{2}P_2(v)\phi_{u,v}P_2(2^{-\frac{1}{2}}(u+v)) \\ &= \frac{1}{2}(P_2(u) + P_2(v))P_2(2^{-\frac{1}{2}}(u+v)) \\ &= \frac{1}{2}P_2(2^{-\frac{1}{2}}(u+v)) \end{aligned}$$

as required. □

It is now possible to prove the main result of this section which shows that, for a pair of rigidly collinear tripotents u and v in a JBW*-triple A , the involutive automorphism $\phi_{u,v}$ from $A_2(u)$ to $A_2(v)$ may, using projections, be factored in a transparent manner through $A_2(2^{-\frac{1}{2}}(u+v))$.

Theorem 2.6. *Let A be a JBW*-triple, let u and v be a pair of rigidly collinear tripotents in A , let $\phi_{u,v}$ be the exchange automorphism associated with u and v and, for j equal to 0, 1, or 2, let $A_j(u)$, $A_j(v)$, and $A_j(2^{-\frac{1}{2}}(u+v))$ and $P_j(u)$, $P_j(v)$, and $P_j(2^{-\frac{1}{2}}(u+v))$ be the Peirce spaces and Peirce projections corresponding to the tripotents u , v , and $2^{-\frac{1}{2}}(u+v)$. Then, the following results hold.*

- (i) *The mappings $2^{\frac{1}{2}}P_2(2^{-\frac{1}{2}}(u+v))|_{A_2(u)} : A_2(u) \rightarrow A_2(2^{-\frac{1}{2}}(u+v))$ and $2^{\frac{1}{2}}P_2(u)|_{A_2(2^{-\frac{1}{2}}(u+v))} : A_2(2^{-\frac{1}{2}}(u+v)) \rightarrow A_2(u)$ are isometric triple isomorphisms onto the JBW*-triples $A_2(2^{-\frac{1}{2}}(u+v))$ and $A_2(u)$, respectively, and are inverses of each other.*
- (ii) *The mapping $2^{\frac{1}{2}}P_2(2^{-\frac{1}{2}}(u+v))|_{A_2(u)}$ coincides with the mapping $2^{-\frac{1}{2}}(\text{id}_A + \phi_{u,v})|_{A_2(u)}$.*
- (iii) *The composition of the mappings $2^{\frac{1}{2}}P_2(2^{-\frac{1}{2}}(u+v))|_{A_2(u)}$ and $2^{\frac{1}{2}}P_2(v)|_{A_2(2^{-\frac{1}{2}}(u+v))}$ coincides with the mapping $\phi_{u,v}|_{A_2(u)}$.*

Proof. Let a be an element of $A_2(u)$. Then, by Lemma 2.5 (vii),

$$2^{\frac{1}{2}}P_2(u)2^{\frac{1}{2}}P_2(2^{-\frac{1}{2}}(u+v))a = 2P_2(u)P_2(2^{-\frac{1}{2}}(u+v))P_2(u)a = P_2(u)a = a.$$

For an element a in $A_2(2^{-\frac{1}{2}}(u+v))$, by Lemma 2.5 (viii),

$$\begin{aligned} 2^{\frac{1}{2}}P_2(2^{-\frac{1}{2}}(u+v))2^{\frac{1}{2}}P_2(u)a &= 2P_2(2^{-\frac{1}{2}}(u+v))P_2(u)2^{\frac{1}{2}}P_2(2^{-\frac{1}{2}}(u+v))a \\ &= 2P_2(2^{-\frac{1}{2}}(u+v))a = a. \end{aligned}$$

Hence, the mappings described in (i) are bijections between $A_2(u)$ and $A_2(2^{-\frac{1}{2}}(u + v))$, and are inverses of each other. Moreover, by Lemma 2.5 (vi),

$$2^{\frac{1}{2}}P_2(2^{-\frac{1}{2}}(u + v))|_{A_2(u)} = 2^{-\frac{1}{2}}(\text{id}_A + \phi_{u,v})|_{A_2(u)},$$

which implies that (ii) holds.

It will be shown that the mapping $2^{\frac{1}{2}}P_2(2^{-\frac{1}{2}}(u + v))|_{A_2(u)}$ is a Jordan*-isomorphism from the JBW*-algebra $A_2(u)$ to the JBW*-algebra $A_2(2^{-\frac{1}{2}}(u + v))$. To this end, let a be an element of $A_2(u)$, and, since $P_2(2^{-\frac{1}{2}}(u + v))$ is a structural projection, using (5),

$$\begin{aligned} & \{2^{\frac{1}{2}}P_2(2^{-\frac{1}{2}}(u + v))a \ 2^{-\frac{1}{2}}(u + v) \ P_2(2^{-\frac{1}{2}}(u + v))a\} \\ &= 2P_2(2^{-\frac{1}{2}}(u + v))\{a \ P_2(2^{-\frac{1}{2}}(u + v))2^{-\frac{1}{2}}(u + v) \ a\} \\ &= 2^{\frac{1}{2}}P_2(2^{-\frac{1}{2}}(u + v))\{a \ u + v \ a\} \\ &= 2^{\frac{1}{2}}P_2(2^{-\frac{1}{2}}(u + v))\{a \ u \ a\}, \end{aligned}$$

since, by (10),

$$\{a \ v \ a\} \in \{A_2(u) \ A_1(u) \ A_2(u)\} = \{0\}.$$

It follows that the mapping is a Jordan isomorphism. Moreover, since $P_2(2^{-\frac{1}{2}}(u + v))$ is a structural projection,

$$\begin{aligned} & \{2^{-\frac{1}{2}}(u + v) \ 2^{\frac{1}{2}}P_2(2^{-\frac{1}{2}}(u + v))a \ 2^{-\frac{1}{2}}(u + v)\} \\ &= (2)^{\frac{1}{2}}P_2(2^{-\frac{1}{2}}(u + v))\{2^{-\frac{1}{2}}(u + v) \ P_2(2^{-\frac{1}{2}}(u + v))a \ 2^{-\frac{1}{2}}(u + v)\} \\ &= 2^{\frac{1}{2}}\{P_2(2^{-\frac{1}{2}}(u + v))2^{-\frac{1}{2}}(u + v) \ a \ P_2(2^{-\frac{1}{2}}(u + v))2^{-\frac{1}{2}}(u + v)\} \\ &= 2^{\frac{1}{2}}\{2^{-\frac{1}{2}}(u + v) \ a \ 2^{-\frac{1}{2}}(u + v)\} \\ &= 2^{-\frac{1}{2}}(\{u \ a \ u\} + \{v \ a \ v\} + 2\{u \ a \ v\}) \\ &= 2^{-\frac{1}{2}}(\{u \ a \ u\} + 2\{u \ a \ v\}), \end{aligned} \tag{17}$$

since, by (10),

$$\{v \ a \ v\} \in \{A_2(v) \ A_1(v) \ A_2(v)\} = \{0\}.$$

Furthermore, using Lemma 2.5 (vi), the fact that $\phi_{u,v}$ is a triple automorphism, Lemma 2.3 (i), and (1),

$$\begin{aligned} 2^{\frac{1}{2}}P_2(2^{-\frac{1}{2}}(u + v))\{u \ a \ u\} &= 2^{-\frac{1}{2}}(\text{id}_A + \phi_{u,v})\{u \ a \ u\} \\ &= 2^{-\frac{1}{2}}(\{u \ a \ u\} + \{v \ \phi_{u,v}a \ v\}) \\ &= 2^{-\frac{1}{2}}(\{u \ a \ u\} + 2\{v \ \{v \ u \ a\} \ v\}) \\ &= 2^{-\frac{1}{2}}(\{u \ a \ u\} + 2\{u \ a \ v\}). \end{aligned} \tag{18}$$

It follows from (17), (18) that the mapping $2^{\frac{1}{2}}P_2(2^{-\frac{1}{2}}(u + v))$ is a Jordan $*$ -isomorphism from the JBW $*$ -algebra $A_2(u)$ onto the JBW $*$ -algebra $A_2(2^{-\frac{1}{2}}(u + v))$ and, clearly, its inverse has the same property. Since the Jordan triple product in both of these JBW $*$ -algebras coincide with the restrictions of the triple product on A , this completes the proof of (i).

By applying (i) to the tripotent v rather than u it follows that the mapping $2^{\frac{1}{2}}P_2(v)2^{\frac{1}{2}}P_2(2^{-\frac{1}{2}}(u + v))|_{A_2(u)}$ is a Jordan $*$ -isomorphism from the JBW $*$ -algebra $A_2(u)$ onto the JBW $*$ -algebra $A_2(v)$. Observe that, by Lemma 2.5 (vi), the compatibility of u and v , and (15),

$$\begin{aligned} 2P_2(v)P_2(2^{-\frac{1}{2}}(u + v))P_2(u) &= P_2(v)(\text{id}_A + \phi_{u,v})P_2(u) \\ &= P_2(v)P_2(u) + \phi_{u,v}P_2(u)P_2(u) = \phi_{u,v}P_2(u), \end{aligned}$$

and the proof of (iii) is complete. □

Since triple isomorphisms on JBW $*$ -triples are automatically isometric the result above has the following immediate corollary.

Corollary 2.7. *Under the conditions of Theorem 2.6, for all elements a in $A_2(u)$,*

$$2\|a\|^2 = \|a + \phi_{u,v}a\|^2.$$

3. Decoherent states

Let A be a JBW $*$ -triple with predual A_* , and let A_1 and A_{*1} be the closed unit balls in A and A_* , respectively. For each subset E of A_{*1} and F of A_1 , let the subsets E' and F' be defined by

$$E' = \{a \in A_1 : a(x) = 1 \ \forall x \in E\}, \quad F' = \{x \in A_{*1} : a(x) = 1 \ \forall a \in F\}.$$

Then E' is a weak $*$ -closed face of A_1 and F' is a norm-closed face of A_{*1} and the mappings $E \mapsto E'$ and $F \mapsto F'$ are anti-order isomorphisms between the complete lattices $\mathcal{F}_n(A_{*1})$ and $\mathcal{F}_{w^*}(A_1)$ of norm-closed faces of A_{*1} and weak $*$ -closed faces of A_1 and are inverses of each other. The mapping $u \mapsto \{u\}$, is an order isomorphism from the complete lattice of tripotents $\tilde{\mathcal{U}}(A)$ in A onto the complete lattice $\mathcal{F}_n(A_{*1})$. Moreover, the mapping $u \mapsto \{u\}'$ is an anti-order-isomorphism from $\tilde{\mathcal{U}}(A)$ onto the complete lattice $\mathcal{F}_{w^*}(A_1)$. Furthermore, for every element u in $\mathcal{U}(A)$,

$$\{u\}' = u + A_0(u) \cap A_1.$$

The reader is referred to [13] for details.

Lemma 3.1. *Let A be a JBW $*$ -triple with predual A_* , let ϕ be an element of the group $\text{Aut}(A)$ of triple automorphisms of A with pre-adjoint the element ϕ_* of the*

group $\text{Aut}(A_*)$ of linear isometries of A_* , and let u be an element of the partially ordered set $\mathcal{U}(A)$ of tripotents in A with $\{u\}$, the corresponding norm-closed face of the unit ball A_{*1} in A_* . Then,

$$\phi_*(\{u\}) = \{\phi^{-1}u\}.$$

Proof. For each element y in $\phi_*(\{u\})$, there exists an element x in $\{u\}$, such that

$$\phi_*x = y.$$

Then,

$$1 = x(u) = (\phi_*)^{-1}y(u) = (\phi^{-1})_*y(u) = y(\phi^{-1}u),$$

and y lies in the norm-closed face $\{\phi^{-1}u\}$. Hence, $\phi_*(\{u\})$ is contained in $\{\phi^{-1}u\}$. The reverse inclusion follows by a simple reversion of the argument above. \square

The following corollary is immediate from Lemma 3.1.

Corollary 3.2. *Let A be a JBW*-triple with predual A_* , let u and v be a pair of rigidly collinear tripotents in A , let $\phi_{u,v}$ be the exchange automorphism associated with u and v having pre-adjoint the involutive linear isometry $(\phi_{u,v})_*$ of A_* , and let $\{u\}$, and $\{v\}$, be the corresponding norm-closed faces of the unit ball A_{*1} in A_* . Then,*

$$(\phi_{u,v})_*(\{u\}) = \{v\}.$$

For every element x in A_* there exists a unique tripotent $e(x)$ in A such that x lies in $A_2(e(x))_*$ and the restriction of x to $A_2(e(x))$ is a faithful normal positive linear functional on the JBW*-algebra $A_2(e(x))$. The tripotent $e(x)$ is said to be the *support* of x . The support $e(x)$ of an element x in A_* is the least tripotent in A such that $x(e(x))$ equals $\|x\|$. If x is of norm one then the face $\{x\}'$, of A_1 coincides with the norm-closed face $\{e(x)\}$, which is the normal state space of the JBW*-algebra $A_2(e(x))$. From a physical point of view, $\{e(x)\}$, may be regarded as the set of normalized states of the local system defined by the normalized state x . For details the reader is referred to [13, 23].

Lemma 3.3. *Let A be a JBW*-triple with predual A_* , let x be an element of A_* of norm one, having support tripotent $e(x)$, and let $A_2(e(x))_*$ be the structural subspace of A_* that is the predual of the Peirce space $A_2(e(x))$ of A corresponding to $e(x)$. Then, $A_2(e(x))_*$ is the smallest structural subspace of A_* containing x .*

Proof. Since the norm-closed face $\{x\}'$, is the normal state space of the JBW*-algebra $A_2(e(x))$, it follows that x lies in $A_2(e(x))_*$. Suppose that J is a weak*-closed inner ideal in A for which x lies in the predual J_* of J . Then, by [12, Lemma 5.2 and Theorem 5.4], the tripotent $e(x)$ lies in J . Hence, $A_2(e(x))$ is contained in J and $A_2(e(x))_*$ is contained in J_* , as required. \square

Two elements x and y of norm one in the pre-symmetric space A_* are said to be *decoherent* if the pair $A_2(e(x))_*$ and $A_2(e(y))_*$ of structural subspaces are rigidly collinear. The following result is an immediate consequence of Lemma 3.3 and results of [10, 12].

Lemma 3.4. *Let A be a JBW^* -triple, let x and y be elements of the predual A_* of A of norm one, let $e(x)$ and $e(y)$ be the support tripotents of x and y , and let $A_2(e(x))$ and $A_2(e(y))$ and $P_2(e(x))$ and $P_2(e(y))$ be Peirce spaces and Peirce projections associated with $e(x)$ and $e(y)$ respectively. Then, the following are equivalent.*

- (i) x and y form a decoherent pair.
- (ii) The tripotents $e(x)$ and $e(y)$ form a rigidly collinear pair.
- (iii) The weak*-closed inner ideals $A_2(e(x))$ and $A_2(e(y))$ in A form a rigidly collinear pair.
- (iv) The structural projections $P_2(e(x))_*$ and $P_2(e(y))_*$ on A_* form a rigidly collinear pair.

The following result is now immediate from Lemma 2.4 and Theorem 2.6.

Theorem 3.5. *Let A be a JBW^* -triple with predual A_* , let x and y be a decoherent pair of elements of A_* of norm one, having support tripotents $e(x)$ and $e(y)$, let $\phi_{e(x),e(y)}$ be the associated exchange automorphism, and let $A_2(e(x))_*$ and $A_2(e(y))_*$ and $P_2(e(x))_*$ and $P_2(e(y))_*$ be the corresponding structural subspaces and structural projections, respectively. Then, the following results hold.*

- (i) $A_2(e(x))_* \oplus A_2(e(y))_*$ is the smallest structural subspace of A_* containing x and y .
- (ii) $A_2(e(x+y))_* \subseteq A_2(e(x))_* \oplus A_2(e(y))_*$.
- (iii) The structural subspaces $A_2(e(x))_*$, $A_2(e(y))_*$ and $A_2(e(x))_* \oplus A_2(e(y))_*$ form a pairwise compatible family.
- (iv) The linear isometry $(\phi_{e(x),e(y)})_*$ maps the structural subspace $A_2(e(x))_*$ onto the structural subspace $A_2(e(y))_*$.
- (v) The mappings

$$2^{\frac{1}{2}} P_2(2^{-\frac{1}{2}}(e(x) + e(y)))_*|_{A_2(e(x))_*} : A_2(e(x))_* \longrightarrow A_2(2^{-\frac{1}{2}}(e(x) + e(y)))_*$$

and

$$2^{\frac{1}{2}} P_2(e(x))_*|_{A_2(2^{-\frac{1}{2}}(e(x)+e(y)))_*} : A_2(2^{-\frac{1}{2}}(e(x) + e(y)))_* \longrightarrow A_2(e(x))_*$$

are isometric linear isomorphisms onto the structural subspaces $A_2(2^{-\frac{1}{2}}(e(x) + e(y)))_*$ and $A_2(e(x))_*$, respectively, and are inverses of each other.

(vi) *The equality*

$$2^{\frac{1}{2}} P_2(2^{-\frac{1}{2}}(e(x) + e(y)))_* |_{A_2(e(x))_*} = 2^{-\frac{1}{2}}(\text{id}_A + \phi_{e(x), e(y)})_* |_{A_2(e(x))_*}$$

holds.

(vii) *The equalities*

$$\begin{aligned} (\phi_{e(x), e(y)})_* |_{A_2(e(x))_*} &= 2D(e(y), e(x))_* |_{A_2(e(x))_*} \\ &= 2P_2(e(y))_* P_2(2^{-\frac{1}{2}}(e(x) + e(y)))_* |_{A_2(e(x))_*} \end{aligned}$$

hold.

Before coming to the main result the following lemma is needed.

Lemma 3.6. *Let A be a JBW^* -triple, let u and v be a pair of collinear tripotents in A , let $\{u\}$, and $\{v\}$, be the corresponding norm-closed faces of the unit ball A_{*1} in the pre-dual A_* of A , and let x and y be elements of $\{u\}$, and $\{v\}$, respectively. Then, the following results hold.*

(i) $\|x + y\| \geq 2^{\frac{1}{2}}$.

(ii) *Equality holds in the inequality above if and only the support tripotent $e(x + y)$ of $x + y$ is majorized by the tripotent $2^{-\frac{1}{2}}(u + v)$.*

Proof. (i) Observe that, being a non-zero tripotent, $2^{-\frac{1}{2}}(u + v)$ is of norm one. However,

$$\begin{aligned} (x + y)(2^{-\frac{1}{2}}(u + v)) &= 2^{-\frac{1}{2}}(x(u) + x(v) + y(u) + y(v)) \\ &= 2^{-\frac{1}{2}}(1 + P_2(u)_* x(P_1(u)v) + P_2(v)_*(P_1(v)u) + 1) = 2^{\frac{1}{2}}. \end{aligned}$$

Therefore,

$$\|x + y\| \geq 2^{\frac{1}{2}},$$

as required.

(ii) Suppose that equality holds and then, as above,

$$(x + y)(2^{-\frac{1}{2}}(u + v)) = \|x + y\|,$$

and

$$e(x + y) \leq 2^{-\frac{1}{2}}(u + v).$$

Conversely, if this holds then

$$(x + y)/\|x + y\| \in \{e(x + y)\}_\iota \subseteq \{2^{-\frac{1}{2}}(u + v)\}_\iota,$$

and

$$\|x + y\| = 2^{-\frac{1}{2}}(x + y)(u + v) = 2^{\frac{1}{2}},$$

as required. □

A decoherent pair x and y of elements of the pre-symmetric space A_* of norm one is said to be *strongly decoherent* when the exchange automorphism $(\phi_{e(x),e(y)})_*$ of A_* maps x to y . The main result of the section discussed the conditions under which this occurs. The following lemma considers a more general situation in which u and v form a rigidly collinear pair of tripotents in the JBW*-triple A and x and y are elements of the norm-closed faces $\{u\}$, and $\{v\}$, respectively.

Lemma 3.7. *Let A be a JBW*-triple, let u and v be a pair of rigidly collinear tripotents in A , let $\phi_{u,v}$ be the corresponding exchange automorphism of A , let $\{u\}$, and $\{v\}$, be the corresponding norm-closed faces of the unit ball A_{*1} in the pre-dual A_* of A , and let x and y be elements of $\{u\}$, and $\{v\}$, respectively. Then, the following conditions are equivalent.*

- (i) $\|x + y\| = 2^{\frac{1}{2}}$.
- (ii) $(\phi_{u,v})_*x = y$.
- (iii) $e(x + y) \leq 2^{-\frac{1}{2}}(u + v)$.
- (iv) $2D(u, v)_*x = y$.
- (v) $2D(v, u)_*y = x$.

Proof. (i) \Rightarrow (ii) Observe that, since y lies in $A_2(v)_*$ and u lies in $A_1(v)$,

$$y(u) = P_2(v)_*y(P_1(v)u) = 0,$$

and, similarly,

$$x(v) = 0.$$

Therefore,

$$\begin{aligned} 2^{\frac{1}{2}} &= \|x + y\| \geq \|P_2(2^{-\frac{1}{2}}(u + v))_*(x + y)\| \\ &= \sup\{|(x + y)(P_2(2^{-\frac{1}{2}}(u + v))a)| : a \in A_1\} \\ &\geq \sup\{|(x + y)a| : a \in A_2(2^{-\frac{1}{2}}(u + v))_1\} \geq |(x + y)(2^{-\frac{1}{2}}(u + v))| \\ &= |2^{-\frac{1}{2}}(x(u) + x(v) + y(u) + y(v))| = 2^{\frac{1}{2}}. \end{aligned}$$

Hence,

$$\|P_2(2^{-\frac{1}{2}}(u + v))_*(x + y)\| = \|x + y\|,$$

and, since $P_2(2^{-\frac{1}{2}}(u + v))_*$ is a structural projection,

$$P_2(2^{-\frac{1}{2}}(u + v))_*(x + y) = x + y.$$

It follows from (16) that

$$(\phi_{u,v})_*(x + y) = x + y.$$

and, using Corollary 3.2 and Lemma 2.2,

$$x - (\phi_{u,v})_*x = (\phi_{u,v})_*x - y \in A_2(u)_* \cap A_2(v)_* = \{0\}.$$

Hence,

$$(\phi_{u,v})_*x = y,$$

as required.

(ii) \Rightarrow (i) Observe that

$$(\phi_{u,v})_*(x + y) = x + (\phi_{u,v})_*x = x + y.$$

Therefore, using Lemma 2.3 (i) and (16),

$$x + y \in (A_2(u)_* \oplus A_2(v)_*) \cap (A_2(2^{-\frac{1}{2}}(u + v))_* \oplus A_0(2^{-\frac{1}{2}}(u + v))_*).$$

Therefore, there exist elements z_2 in $A_2(2^{-\frac{1}{2}}(u + v))_*$ and z_0 in $A_0(2^{-\frac{1}{2}}(u + v))_*$ such that

$$x + y = z_2 + z_0.$$

By Lemma 2.5 (i), $A_2(2^{-\frac{1}{2}}(u + v))_*$ is contained in $A_2(u)_* \oplus A_2(v)_*$, and, therefore,

$$z_0 = x + y - z_2 \in (A_2(u)_* \oplus A_2(v)_*) \cap A_0(2^{-\frac{1}{2}}(u + v))_*.$$

It follows from [12, Lemma 5.2 and Theorem 5.4], that the support tripotent $e(z_0)$ of z_0 is contained in the weak*-closed inner ideal $(A_2(u) \oplus A_2(v)) \cap A_0(2^{-\frac{1}{2}}(u + v))$. Therefore, by Lemma 2.5 (ii), $e(z_0)$ is equal to zero which implies that z_0 is equal to zero. It follows that the element $x + y$ lies in the predual $A_2(2^{-\frac{1}{2}}(u + v))_*$ of the JBW*-algebra $A_2(2^{-\frac{1}{2}}(u + v))$.

Let a be an element of the JBW*-algebra $A_2(2^{-\frac{1}{2}}(u + v))$. Then, by Theorem 2.6 (i), there exists an element b of $A_2(u)$ such that

$$a = 2^{\frac{1}{2}}P_2(2^{-\frac{1}{2}}(u + v))b.$$

Since, by Lemma 2.3 (i),

$$\phi_{u,v}b \in A_2(v) \subseteq A_1(u),$$

using Lemma 2.5 (vi), (14), and Lemma 2.3 (i), it can be seen that

$$\begin{aligned} (x + y)(a) &= 2^{\frac{1}{2}}(x + (\phi_{u,v})_*x)(P_2(2^{-\frac{1}{2}}(u + v))b) \\ &= 2^{-\frac{1}{2}}(\text{id}_{A_*} + (\phi_{u,v})_*)x(\text{id}_A + \phi_{u,v})b \\ &= 2^{\frac{1}{2}}(x(b) + x(\phi_{u,v}b)) \\ &= 2^{\frac{1}{2}}x(b) + 2^{\frac{1}{2}}(P_2(u)_*x)(P_1(u)\phi_{u,v}b) = 2^{\frac{1}{2}}x(b). \end{aligned} \tag{19}$$

As in the proof of Theorem 2.6, $2^{\frac{1}{2}}P_2(2^{-\frac{1}{2}}(u+v))$ is a Jordan $*$ -isomorphism between JBW $*$ -algebras and, therefore, maps positive elements of $A_2(u)$ to positive elements of $A_2(2^{-\frac{1}{2}}(u+v))$. Since x is a state of the JBW $*$ -algebra $A_2(u)$, it follows from (19) that $x+y$ is a positive linear functional on the JBW $*$ -algebra $A_2(2^{-\frac{1}{2}}(u+v))$. Therefore, observing that, as in the proof above,

$$x(v) = y(u) = 0,$$

and, using the fact that a positive linear functional on a JBW $*$ -algebra attains its norm at the unit,

$$\|x+y\| = (x+y)(2^{-\frac{1}{2}}(u+v)) = (2^{-\frac{1}{2}})(x(u)+y(v)) = 2^{\frac{1}{2}},$$

as required.

(ii) \Leftrightarrow (iii) This is immediate from Lemma 3.6.

(ii) \Rightarrow (iv) Observe that, since u and v are compatible, using (7)–(10), for an element a in A ,

$$D(u,v)P_2(u)a = \{u v P_2(u)a\} \in \{A_2(u) A_1(u) A_2(u)\} = \{0\}, \tag{20}$$

$$D(u,v)P_2(v)a = \{u v P_2(v)a\} \in A_2(u) \cap A_1(v) = A_2(u), \tag{21}$$

$$D(u,v)P_1(u)P_1(v)a = \{u v P_1(u)P_1(v)a\} \in A_2(u) \cap A_0(v) = \{0\}, \tag{22}$$

$$D(u,v)P_1(u)P_0(v)a = \{u v P_1(u)P_0(v)a\} = 0, \tag{23}$$

$$D(u,v)P_0(u)P_1(v)a = \{u v P_0(u)P_1(v)a\} \in A_1(u) \cap A_0(v), \tag{24}$$

$$D(u,v)P_0(u)P_0(v)a = \{u v P_0(u)P_0(v)a\} = 0. \tag{25}$$

Therefore, since x lies in $A_2(u)_*$, using Lemma 2.4, (20)–(25), (15), and Lemma 2.3 (i),

$$\begin{aligned} 2D(u,v)_*x(a) &= 2x(D(u,v)a) = 2P_2(u)_*x(D(u,v)a) \\ &= 2P_2(u)_*x(D(u,v)(P_2(u) + P_2(v) + P_1(u)P_1(v) \\ &\quad + P_1(u)P_0(v) + P_0(u)P_1(v) + P_0(u)P_0(v))a) \\ &= 2x(P_2(u)(D(u,v)P_2(v)a + D(u,v)P_0(u)P_1(v))a) \\ &= 2x(D(u,v)P_2(v)a + D(u,v)P_2(v)P_0(u)P_1(v)a) = x(\phi_{u,v}P_2(v)a) \\ &= (\phi_{u,v})_*x(P_2(v)a) = y(P_2(v)a) = P_2(v)_*y(a) \\ &= y(a), \end{aligned}$$

as required.

(iv) \Rightarrow (ii) For each element a in A , by (15) and Lemma 2.3 (i),

$$\begin{aligned} (\phi_{u,v})_*x(a) &= x(\phi_{u,v}a) = (P_2(u)_*x)(\phi_{u,v}a) \\ &= x(P_2(u)\phi_{u,v}a) = x(\phi_{u,v}P_2(v)a) \\ &= 2x(D(u,v)P_2(v)a) = 2D(u,v)_*x(P_2(v)a) \\ &= y(P_2(v)a) = P_2(v)_*y(a) = y(a), \end{aligned}$$

as required.

(ii) \Leftrightarrow (v) This follows by exchanging u and v in the argument above. □

It is now possible to state the main result of the paper.

Theorem 3.8. *Let A be a JBW*-triple with predual A_* , let x and y be a decoherent pair of elements of A_* of norm one, let $e(x)$, $e(y)$, and $e(x + y)$ be the support tripotents of x , y , and $x + y$, respectively, and let $\phi_{e(x),e(y)}$ be the associated exchange automorphism of A . Then, the following conditions are equivalent.*

- (i) *The elements x and y are strongly decoherent.*
- (ii) $\|x + y\| = 2^{\frac{1}{2}}$.
- (iii) $e(x + y) \leq 2^{-\frac{1}{2}}(e(x) + e(y))$.
- (iv) $e(x + y) = 2^{-\frac{1}{2}}(e(x) + e(y))$.
- (v) $2D(e(x), e(y))_*x = y$.
- (vi) $2D(e(y), e(x))_*y = x$.

Proof. All except the implication (iii) \Rightarrow (iv) follows from Lemma 3.7. Let

$$f = 2^{-\frac{1}{2}}(e(x) + e(y)) - e(x + y).$$

Then, f is a tripotent majorized by the tripotent $2^{-\frac{1}{2}}(e(x) + e(y))$, and, therefore, is a self-adjoint idempotent in the JBW*-algebra $A_2(2^{-\frac{1}{2}}(e(x) + e(y)))$. Therefore, by Theorem 2.6, $2^{\frac{1}{2}}P_2(e(x))f$ and $2^{\frac{1}{2}}P_2(e(y))f$ are self-adjoint idempotents in the JBW*-algebras $A_2(e(x))$ and $A_2(e(y))$, respectively. Hence, again using the fact that

$$x(e(y)) = y(e(x)) = 0,$$

it follows that

$$\begin{aligned} x(2^{\frac{1}{2}}P_2(e(x))f) + y(2^{\frac{1}{2}}P_2(e(y))f) &= 2^{\frac{1}{2}}(P_2(e(x))_*x)(f) + P_2(e(y))_*y(f) \\ &= 2^{\frac{1}{2}}(x + y)(f) \\ &= (x + y)(e(x) + e(y)) \\ &\quad - 2^{\frac{1}{2}}(x + y)(e(x + y)) \\ &= x(e(x)) + y(e(y)) - 2^{\frac{1}{2}}\|x + y\| \\ &= 0. \end{aligned} \tag{26}$$

Since x and y are faithful states of the JBW*-algebras $A_2(e(x))$ and $A_2(e(y))$, respectively, by (26),

$$x(2^{\frac{1}{2}}P_2(e(x))f) = y(2^{\frac{1}{2}}P_2(e(y))f) = 0,$$

and, hence,

$$P_2(e(x))f = P_2(e(y))f = 0.$$

It follows from Theorem 2.6 that,

$$f = 2P_2(2^{-\frac{1}{2}}(e(x) + e(y))P_2(e(x))f = 0,$$

and

$$e(x + y) = 2^{-\frac{1}{2}}(e(x) + e(y)),$$

as required. □

Before examining some consequences of this result, two further preliminary results are needed.

Lemma 3.9. *Let A be a JBW^* -triple with predual A_* , let x be an element of A_* of norm one having support tripotent $e(x)$, and let ϕ be an element of the group $\text{Aut}(A)$ of triple automorphisms of A having pre-adjoint the linear isometry ϕ_* of A_* . Then,*

$$e(\phi_*x) = \phi^{-1}(e(x)).$$

Proof. Notice that $\phi^{-1}(e(x))$ is a tripotent and the element ϕ_*x of A_* is of norm one. Moreover,

$$\phi_*x(\phi^{-1}(e(x))) = x(e(x)) = 1,$$

and it follows that the tripotent $\phi^{-1}(e(x))$ is contained in the norm-closed face $\{\phi_*x\}'$ of the unit ball A_{*1} in A_* . Therefore,

$$e(\phi_*x) \leq \phi^{-1}(e(x)). \tag{27}$$

Applying the same result to the element ϕ_*x and the triple automorphism ϕ^{-1} shows that

$$e(x) = e((\phi_*)^{-1}\phi_*x) = e((\phi^{-1})_*\phi_*x) \leq \phi(e(\phi_*x)).$$

Since the restrictions of triple automorphisms of A are order isomorphisms of the partially ordered set $\mathcal{U}(A)$,

$$\phi^{-1}(e(x)) \leq \phi^{-1}\phi(e(\phi_*x)) = e(\phi_*x), \tag{28}$$

and the result follows from (27), (28). □

Lemma 3.10. *Let A be a JBW^* -triple, let u and v be a rigidly collinear pair of tripotents in A , let $\phi_{u,v}$ be the associated exchange automorphism of A , and let f be a tripotent in the Peirce space $A_2(u)$ corresponding to u . Then, f and $\phi_{u,v}f$ form a rigidly collinear pair of tripotents in A .*

Proof. Observe that, by Lemma 2.3 (i), (2), and (10),

$$\begin{aligned} \{f f \phi_{u,v}f\} &= 2\{f f D(v, u)f\} = 2D(f, f)D(v, u)f \\ &= 2D(v, u)D(f, f)f + 2D(f, \{f v u\})f - 2D(\{v u f\}, f)f \\ &= 2D(v, u)f - \{f f \phi_{u,v}f\}, \end{aligned}$$

since

$$\{f v u\} \in \{A_2(u) A_1(u) A_2(u)\} = \{0\}.$$

Again using Lemma 2.3 (i), it follows that

$$\{f f \phi_{u,v}f\} = D(v, u)f = \frac{1}{2}\phi_{u,v}f,$$

which implies that the tripotent $\phi_{u,v}f$ lies in $A_1(e)$. Furthermore, since $\phi_{u,v}$ is a triple automorphism of period two,

$$\{\phi_{u,v}f \phi_{u,v}f f\} = \phi_{u,v}\{f f \phi_{u,v}f\} = \frac{1}{2}\phi_{u,v}\phi_{u,v}f = \frac{1}{2}f,$$

and the tripotent f lies in $A_1(\phi_{u,v}f)$. Hence, f and $\phi_{u,v}$ form a collinear pair with f in $A_2(u)$ and $\phi_{u,v}f$ in $\phi_{u,v}A_2(u)$, which, by Lemma 2.3 (i), coincides with $A_2(v)$. Therefore, using Lemma 2.2,

$$A_2(f) \cap A_2(\phi_{u,v}f) \subseteq A_2(u) \cap A_2(v) = \{0\},$$

and, by the same lemma, f and $\phi_{u,v}f$ form a rigidly collinear pair. □

A further consequence of Lemmas 3.7, 3.9, and 3.10 can now be stated.

Theorem 3.11. *Let A be a JBW^* -triple, let u and v be a pair of rigidly collinear tripotents in A and let $\{u\}_l$ and $\{v\}_l$ be the corresponding norm-closed faces of the unit ball A_{*1} in the pre-dual A_* of A . Then, for each element x of $\{u\}_l$, there exists a unique element y in $\{v\}_l$, such that x and y form a strongly decoherent pair.*

Proof. Observe that, since x is contained in $\{u\}_l$,

$$e(x) \leq u. \tag{29}$$

Let $\phi_{u,v}$ be the exchange automorphism of A associated with u and v , and let y be the element $(\phi_{u,v})_*x$ of A_* . Since $(\phi_{u,v})_*$ is an isometry, y is of norm one, using Lemma 3.9, the fact that the restriction of $\phi_{u,v}$ to $\mathcal{U}(A)$ is an order isomorphism, and (29),

$$e(y) = e((\phi_{u,v})_*x) = \phi_{u,v}e(x) \leq \phi_{u,v}u = v.$$

By Lemma 3.10, it follows that $e(x)$ and $e(y)$ form a rigidly collinear pair of tripotents, which, by Lemma 3.7, are strongly decoherent.

Observe that, by Theorem 3.8,

$$(\phi_{e(x),e(y)})_*x = y.$$

Suppose that z is a further element of $\{v\}$, for which x and z form a strongly decoherent pair. Then, by Lemma 3.7,

$$z = (\phi_{u,v})_*x = y,$$

and the element y is unique. □

4. Applications and examples

As was remarked in the introduction, contractive projections play an important part not only in the theory of pre-symmetric spaces but also in their physical applications [5, 18, 29, 35, 40]. Recall that a *GL-projection* R on the pre-symmetric space A_* is a contractive projection such that, if x is an element of A_* for which, for all elements y in A_*

$$\|x + Ry\| = \|x\| + \|Ry\|,$$

then x lies in the kernel of R . Observe that such a projection is an element of the complete lattice $\mathcal{S}_*(A_*)$ of structural projections on A_* provided that, for all elements x in A_* for which $\|Rx\|$ and $\|x\|$ coincide, then so also do Rx and x . For every contractive projection R on A_* there exists a unique GL-projection with the same range as that of R . In particular, for each element x in A_* of norm one, the unique GL-projection Q_x onto the closed subspace $\mathbb{C}x$ of A_* is defined, for z in A_* , by

$$Q_x z = z(e(x))x, \tag{30}$$

where $e(x)$ is the support tripotent of x . For more details the reader is referred to [10, 26].

A family $\{x_j : j \in \Lambda\}$ of elements of norm one in the pre-symmetric space A_* is said to be *decoherent* if each pair of elements in the set is decoherent, and is said to be *strongly decoherent* if each pair of elements of the set is strongly decoherent. The following result is a consequence of the results of the previous section and makes an important connection between the concepts of strong decoherence and contractive projections.

Theorem 4.1. *Let A be a JBW*-triple with predual A_* and let $\{x_j : j \in \Lambda\}$ be a decoherent family of elements of A_* of norm one. Then, the family is strongly decoherent if and only if there exists a contractive projection on A_* with range the norm-closed linear span $\overline{\text{lin}\{x_j : j \in \Lambda\}}^n$ of the family $\{x_j : j \in \Lambda\}$.*

Proof. If Λ contains a single element the result is trivial. Suppose that Λ contains two elements j and k . Since x_j lies in $A_1(e(x_k))_*$ and x_k lies in $A_1(e(x_j))_*$, it follows that

$$2D(e(x_j), e(x_j))_*x_k = x_k, \quad 2D(e(x_k), e(x_k))_*x_j = x_j. \tag{31}$$

By Theorem 3.8, x_j and x_k form a strongly decoherent pair if and only if

$$2D(e(x_j), e(x_k))_*x_j = x_k, \quad 2D(e(x_k), e(x_j))_*x_k = x_j. \tag{32}$$

The result is now a consequence of [27, Theorem 5.2]. If Λ contains more than two elements, then, for an arbitrary pair j and k of elements of Λ , (30) automatically holds. Moreover, for distinct elements j, k , and l in Λ and an element a in A , since $e(x_j)$ and $e(x_k)$ lie in $A_1(e(x_l))$, using (8)–(10),

$$\begin{aligned} D(e(x_j), e(x_k))_*x_l(a) &= x_l(P_2(e(x_l))\{e(x_j) e(x_k) (P_2(e(x_l))a \\ &\quad + P_1(e(x_l))a + P_0(e(x_l))a)\} \\ &= x_l(P_2(e(x_l))\{e(x_j) e(x_k) P_2(e(x_l))a\} \\ &= 0, \end{aligned}$$

since

$$\begin{aligned} \{e(x_j) e(x_k) P_2(e(x_l))a\} &\in \{A_2(e(x_j)) A_1(e(x_j)) A_1(e(x_j))\} \\ &\subseteq A_2(e(x_j)) \subseteq A_1(e(x_l)). \end{aligned}$$

It follows that

$$D(e(x_j), e(x_k))_*x_l = 0. \tag{33}$$

Furthermore, by rigid collinearity, (8)–(10), and Lemma 2.2,

$$\{e(x_j) e(x_k) e(x_l)\} \in A_2(e(x_j)) \cap A_2(e(x_l)) = \{0\}. \tag{34}$$

Since (31)–(34) automatically hold, it follows from Theorem 3.8 and [27, Theorem 5.3], that the family is strongly decoherent if and only if its closed linear span is the range of a contractive projection. \square

This result has a series of corollaries.

Corollary 4.2. *Let u and v be a pair of rigidly collinear tripotents in the JBW*-triple A . Then, there exists a weak*-continuous contractive projection on A with range $\mathbb{C}u \oplus \mathbb{C}v$.*

Proof. Let x be an element of the norm-closed face $\{u\}$, of the unit ball A_{*1} in the predual A_* of A , let $\phi_{u,v}$ be the exchange automorphism corresponding to u and v , and let

$$y = (\phi_{u,v})_*x.$$

Then, by Theorem 3.11, x and y form a strongly decoherent pair. Let Q_x and Q_y be the unique GL-projections on A_* onto $\mathbb{C}x$ and $\mathbb{C}y$ defined in (30). Therefore, using [26] and Theorem 4.1, $Q_x + Q_y$ is a contractive projection on A_* with range $\mathbb{C}x \oplus \mathbb{C}y$. Furthermore, the range of the adjoint $(Q_x + Q_y)^*$ is clearly $\mathbb{C}e(x) + \mathbb{C}e(y)$. Define the bounded linear projection R on A_* , for z in A_* , by

$$Rz = z(u)x + z(v)y.$$

Then, for an element a in A ,

$$R^*a = x(a)u + y(a)v.$$

Moreover, for elements z in A_* and a in A ,

$$\begin{aligned} z(R^*(Q_x + Q_y)^*a) &= ((Q_x + Q_y)(z(u)x + z(v)y))(a) \\ &= (z(u)x + z(v)y(e(x))x + z(u)x(e(y))y + z(v)y)(a) \\ &= (z(u)x + z(u)y)(a) = z(R^*a), \end{aligned}$$

and it follows that

$$R^*(Q_x + Q_y)^* = R^*.$$

Since, by Lemma 3.10, both $e(x)$ and $e(y)$ and u and v form rigidly collinear pairs, the restriction of the norm on A to the two subspaces $\mathbb{C}e(x) \oplus \mathbb{C}e(y)$ and $\mathbb{C}u \oplus \mathbb{C}v$ is Hilbertian. From above, the restriction of R^* to $\mathbb{C}e(x) \oplus \mathbb{C}e(y)$ is an isometry between the two Hilbert spaces, and, since $(Q_x + Q_y)^*$ is contractive, it follows that R^* is contractive, as required. \square

Corollary 4.3. *Let A_* be a pre-symmetric space and let $\{x_j : j \in \Lambda\}$ be a decoherent family of extreme points of the unit ball A_{*1} in A_* . Then, the family $\{x_j : j \in \Lambda\}$ is strongly decoherent.*

Proof. Since an extreme point x of A_{*1} is a minimal norm-closed face, it follows that $e(x)$ is a minimal element of $\mathcal{U}(A)$ and $A_2(e(x))$ coincides with $\mathbb{C}e(x)$. Hence, every decoherent pair of extreme points is automatically strongly decoherent. \square

Observe that, for a rigidly collinear pair u and v of non-minimal tripotents of the JBW*-triple A , there exist distinct elements x and y in the norm-closed face $\{u\}$, of the unit ball A_{*1} in A_* for which, by Lemma 3.1, it is necessarily true that $(\phi_{u,v})_*x$ and $(\phi_{u,v})_*y$ are distinct elements of $\{v\}$. By Theorem 3.11, x and $(\phi_{u,v})_*x$ form a strongly decoherent pair but y and $(\phi_{u,v})_*x$ do not. Consequently, by Theorem 4.1, there is a contractive projection onto the space $\mathbb{C}x \oplus \mathbb{C}(\phi_{u,v})_*x$ but not onto the space $\mathbb{C}y \oplus \mathbb{C}(\phi_{u,v})_*x$. An example which illustrates this when A is the W^* -algebra $M_4(\mathbb{C})$ of 4×4 complex matrices can be found in [27]. It should also be noted that no such example can be given when A is the W^* -algebra $M_2(\mathbb{C})$ since rigidly collinear tripotents are of rank one and, therefore, minimal, in which case Corollary 4.3 applies.

Recall that a JBW*-triple is said to be *atomic* if, for each non-zero element u of $\mathcal{U}(A)$, there exists a minimal element $e(x)$ of $\mathcal{U}(A)$ for which $e(x) \leq u$. The final corollary shows that, for atomic JBW*-triples, the situation is transparent.

Corollary 4.4. *Let A be an atomic JBW*-triple with predual A_* , and let $\{u_j : j \in \Lambda\}$ be a family of pairwise rigidly collinear tripotents in A . Then, there exists a weak*-continuous contractive projection with range the weak*-closed linear span $\overline{\text{lin}\{u_j : j \in \Lambda\}}^{w^*}$ of $\{u_j : j \in \Lambda\}$*

Proof. Let k lie in Λ and let x_k be the extreme point of the unit ball A_{*1} in the predual A_* of A such that $e(x_k) \leq u_k$. Then, for j not equal to k , let

$$x_j = (\phi_{u_k, u_j})_* x_k.$$

Since linear isometries clearly map the set of extreme points of A_{*1} onto itself, by Theorem 3.11, $\{x_j : j \in \Lambda\}$ forms a decoherent family of extreme points of A_{*1} . It follows from Corollary 4.3 that the family is strongly decoherent and, by Theorem 4.1, that there exists a contractive projection on A_* with range the norm-closed linear span $\overline{\text{lin}\{x_j : j \in \Lambda\}}^n$ of the family $\{x_j : j \in \Lambda\}$. Proceeding as in the proof of Corollary 4.2, define the projections $\{R_j : j \in \Lambda\}$ on A_* , for an element z in A_* , by

$$R_j z = z(u_j)x_j.$$

As in the proof of [27, Theorem 5.4], the net $(\sum_{j \in \lambda_0} R_j)_{\lambda_0 \in \Gamma}$, where Γ is the set of finite subsets of Λ , converges in the strong operator topology to a contractive projection R on A_* , the adjoint R^* of which is the required weak*-continuous contractive projection with range $\overline{\text{lin}\{u_j : j \in \Lambda\}}^{w^*}$. □

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