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# NORMAL CONTRACTIVE PROJECTIONS PRESERVE TYPE

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Abstract. Given a JBW*-triple $Z$ and a normal contractive projection $P : Z \to Z$, we show that the (Murray-von Neumann) type of each summand of $P(Z)$ is dominated by the type of $Z$.

Keywords: *Contractive projection, JB*-triple, von Neumann algebra, Murray–von Neumann classification.*

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## 0. INTRODUCTION

Contractive projections play a useful role in the theory of operator algebras and Banach spaces. The ranges of contractive projections on $C^*$-algebras form an important subclass of those complex Banach spaces whose open unit balls are bounded symmetric domains. An important feature of these spaces is that they are equipped with a Jordan *triple* product, induced by the Lie algebra of the automorphism group of the open unit ball. Known as JB*-triples, they have been shown to be the appropriate category in which to study contractive projections; indeed the fact that the category of JB*-triples is stable under contractive projections played a key role in their structure theory.

Recently, contractive projections on von Neumann algebras have arisen in the study of operator spaces as well as in the theory of harmonic functions on locally compact groups. In [24], a family of Hilbertian operator spaces were studied and used to classify, in an appropriate sense, the ranges of contractive projections on $B(H)$ which are atomic as Banach spaces. In [6], it was shown that the Banach space of bounded matrix-valued harmonic functions on a locally compact group is the range of a contractive projection on a type I finite von Neumann algebra.

It has also been shown in [8] that the Banach space of harmonic functionals on the Fourier algebra of a locally compact group $G$ is the range of a contractive projection on the group von Neumann algebra VN($G$).

There is a Murray-von Neumann type classification for JBW*-triples, that is, JB*-triples which are the dual of a Banach space. In view of the fact, noted above, that the range of a contractive projection on a JB*-triple is again a JB*-triple ([14], [23], [30]) the above investigations point to a natural and important question, namely, how is the Murray-von Neumann classification of the domain affected by a contractive projection? More precisely, given a JBW*-triple $Z$ of type $X$, where $X =$ I, II or III, is the range of a normal contractive projection on $Z$ of type $Y$ with $Y \leqslant X$, meaning each summand of the range is of type $\leqslant X$? In this paper, we answer this question affirmatively. We shall see that it suffices to prove this for JW*-triples, that is, for JBW*-triples which are linearly isometric to a weak operator closed subspace of $B(H)$, stable for the triple product $xy^*z + zy^*x$, where $B(H)$ is the von Neumann algebra of bounded operators on a Hilbert space $H$.

Tomiyama ([34]) has analysed the type structure of the range of a contractive projection which is a von Neumann subalgebra of the domain. His arguments depend on the crucial fact that the range is a *subalgebra*. In our investigation, the range, which automatically has an algebraic structure, need not be a subalgebra nor even a *subtriple*. This adds both generality and complexity to our question.

This paper is organized as follows. Section 1 is devoted to background and motivation for the problem. In Section 2 we consider, as a preliminary tool, contractive projections on JW*-algebras. Propositions 2.7 and 2.8 show that if the image of a normal contractive projection on a JW*-algebra is a JW*-subalgebra (not necessarily with the same identity), then the properties of being semifinite or of type I are passed on from the domain to the image. In Section 3 we study normal contractive projections on a von Neumann algebra of type I and show in Proposition 3.5 that the image is isometric to a JW*-triple of type I. It is necessary first to prove this (in Proposition 3.1) in the special case when the projection is the Peirce 2-projection with respect to a partial isometry. Our main results, that normal contractive projections on JW*-triples preserve both type I and semifiniteness, appear in Section 4 as Theorems 4.2 and 4.4. Again, Propositions 4.1 and 4.3 deal with the special case of a Peirce 2-projection. Although Propositions 2.7 through 4.3 are each a special case of Theorems 4.2 or 4.4, they are essential steps in the proofs of these theorems and they are new and of interest. In Section 5, we extend Theorems 4.2 and 4.4 to arbitrary JBW*-triples, and consider the case of atomic JBW*-triples.

## 1. MOTIVATION AND BACKGROUND

Let $M$ be a von Neumann algebra and let $N$ be a von Neumann subalgebra of $M$ containing the identity element of $M$. A positive linear map $E : M \to N$ satisfying $Ex = x$ for $x \in N$ and $E(axb) = aE(x)b$ for $x \in M$ and $a, b \in N$ is called a *conditional expectation*. Conditional expectations have played some fundamental roles in the theory of von Neumann algebras, for instance in V. Jones' theory of subfactors. Work in the 1950s of Tomiyama and Nakamura-Takesaki-Umegaki established that conditional expectations are idempotent, contractive, and completely positive mappings, and they preserve type when normal; see the survey paper of Størmer ([32]). Conversely ([21], 10.5.85), a unital contractive projection from one $C^*$-algebra onto a unital $C^*$-subalgebra extends to a normal conditional expectation on the universal enveloping von Neumann algebra, and is in particular a conditional expectation on the $C^*$-algebra.

A type theory for weakly closed Jordan operator algebras, based on modularity of the lattice of projections, and parallel to the type classification theory for von Neumann algebras, was introduced and developed in the 1960s by Topping ([35]) and Størmer ([31]). In particular, Størmer showed that a JW-algebra is of type I if and only if its enveloping von Neumann algebra is of type I. This was extended to types II and III by Ayupov in 1982 ([1]). In some cases, the JW-algebra in these results is required to be reversible. This is discussed in Remark 2.2 of [2] and in Sections 1.1–1.3 of [3].

A special case of a result of Choi-Effros in 1977 ([5]), of fundamental importance in the rapidly advancing theory of operator spaces, states that the range of a unital completely positive projection on a $C^*$-algebra, while not in general a subalgebra, nevertheless carries the structure of a $C^*$-algebra. The proof hinges on a conditional expectation formula (needed to prove that the abstract product is associative) which is established using the Kadison-Schwarz inequality for positive linear maps. We note that such a projection is completely contractive.

A special case of a result of Effros-Størmer in 1979 ([10]) states that the range of a unital positive projection on a $C^*$-algebra, while not in general a Jordan subalgebra, carries a natural Jordan algebra structure. As before, the proof depends on a conditional expectation formula (needed to prove that the abstract product satisfies the Jordan identity), and such a projection is contractive.

The above results raised the question of what algebraic structure exists in the range of an arbitrary contractive projection on a $C^*$-algebra. A special case of a result of Friedman and Russo in 1983 states that the range of such a projection is linearly isometric to a subspace, closed under the triple product $xy^*z + zy^*x$, of the second dual of the $C^*$-algebra. Because of the lack of an order structure and hence the unavailability of the Kadison-Schwarz inequality, new techniques were needed and developed by Friedman-Russo in their theory of "operator algebras without order" ([12]), including some conditional expectation formulas for the triple product ([13], Corollary 1).

During the 1980s, the theory of JB*-triples was developed extensively; for a summary, see the survey [27]. In particular, a type I theory was developed for JBW*-triples by Horn in his thesis in 1984. In this theory, idempotents (projections) were replaced by tripotents (which are abstraction of partial isometries), and the reduced algebra $pAp$ was replaced by the Peirce 2-space of a tripotent. Of special importance here is the algebraic fact that such a Peirce 2-space has an abstract structure of a Jordan algebra, and moreover Horn has proved that a JBW*-triple is of type I if, and only if, it contains a complete (= maximal) tripotent whose Peirce 2-space is a Jordan algebra of type I. The remarkable structure theorem of Horn states that type I JBW*-triples are isometric to direct sums of tensor products of a commutative von Neumann algebra by a Cartan factor.

We now recall some definitions. A *Jordan triple system* is a complex vector space $V$ with a *Jordan triple product* $\{\cdot,\cdot,\cdot\} : V \times V \times V \to V$ which is symmetric and linear in the outer variables, conjugate linear in the middle variable and satisfies the Jordan triple identity

$$\{a,b,\{x,y,z\}\} = \{\{a,b,x\},y,z\} - \{x,\{b,a,y\},z\} + \{x,y,\{a,b,z\}\}.$$

A complex Banach space $Z$ is called a JB*-triple if it is a Jordan triple system such that for each $z \in Z$, the linear map

$$z \,\Box\, z : v \in Z \mapsto \{z,z,v\} \in Z$$

is Hermitian, that is, $\|e^{it(z \Box z)}\| = 1$ for all $t \in \mathbb{R}$, with non-negative spectrum and $\|z \,\Box\, z\| = \|z\|^2$. A JB*-triple $Z$ is called a JBW*-*triple* if it is a dual Banach space, in which case its predual is unique, denoted by $Z_*$, and the triple product is separately weak* continuous. The second dual $Z^{**}$ of a JB*-triple is a JBW*-triple. A norm-closed subspace of a JB*-triple is called a *subtriple* if it is closed with respect to the triple product. A JBW*-triple is called a JW*-*triple* if it can be embedded as a subtriple of some $B(H)$.

The JB*-triples form a large class of Banach spaces which include $C^*$-algebras, Hilbert spaces and spaces of rectangular matrices. The triple product in a $C^*$-algebra $\mathcal{A}$ is given by

$$\{x,y,z\} = \frac{1}{2}(xy^*z + zy^*x).$$

In fact, $\mathcal{A}$ is a Jordan algebra in the product

$$x \circ y = \frac{1}{2}(xy + yx)$$

and we have $\{x,y,z\} = (x \circ y^*) \circ z + (y^* \circ z) \circ x - (z \circ x) \circ y^*$. A norm-closed subspace of a $C^*$-algebra is called a JC*-*algebra* if it is also closed with respect to the involution $*$ and the Jordan product $\circ$ given above. A JC*-algebra is called a JW*-*algebra* if it is a dual Banach space.

An element $e$ in a JB*-triple $Z$ is called a *tripotent* if $\{e,e,e\} = e$ in which case the map $e \,\Box\, e : Z \to Z$ has eigenvalues $0$, $\frac{1}{2}$ and $1$, and we have the following decomposition in terms of eigenspaces

$$Z = Z_2(e) \oplus Z_1(e) \oplus Z_0(e)$$

which is called the *Peirce decomposition* of $Z$. The $\frac{k}{2}$-eigenspace $Z_k(e)$ is called the *Peirce k-space*. The *Peirce projections* from $Z$ onto the Peirce $k$-spaces are given by

$$P_2(e) = Q^2(e), \quad P_1(e) = 2(e \,\Box\, e - Q^2(e)), \quad P_0(e) = I - 2e \,\Box\, e + Q^2(e),$$

where $Q(e)z = \{e,z,e\}$ for $z \in Z$. The Peirce projections are contractive.

In later computation, we will use frequently the *Peirce rules*

$$\{Z_i(e)\, Z_j(e)\, Z_k(e)\} \subset Z_{i-j+k}(e),$$

where $Z_l(e) = \{0\}$ for $l \neq 0,1,2$. We note that the Peirce 2-space $Z_2(e) = P_2(e)(Z)$ is a Jordan Banach algebra with identity $e$, the Jordan product $a \circ b = \{a,e,b\}$ and involution $a^\# = \{e,a,e\}$ which satisfy

$$\|a^\#\| = \|a\|, \quad \|\{a,a^\#,a\}\| = \|a\|^3,$$

where $\{x,y,z\} = (x \circ y^*) \circ z + (y^* \circ z) \circ x - (z \circ x) \circ y^*$, in other words, $Z_2(e)$ is a unital JB*-*algebra*. A JB*-algebra having a predual is called a JBW*-*algebra*. As shown in [36], the self-adjoint parts of JB*-algebras (respectively JBW*-algebras) are exactly the JB-*algebras* (respectively JBW-*algebras*). For definitions and basic results about JB-algebras, we refer the reader to [17]. If $Z = Z_2(e)$, then $e$ is called *unitary*. If $Z_0(e) = \{0\}$, then the tripotent $e$ is called *complete*. Two tripotents $u$ and $v$ are said to be *orthogonal* if $u \,\Box\, v = 0$. The elements of the predual $Z_*$ of a JBW*-triple $Z$ are exactly the *normal* functionals on $Z$, that is, the continuous linear functionals on $Z$ which are additive on orthogonal tripotents.

Given an orthogonal family of tripotents $\{e_i\}_{i \in \Lambda}$ in a JB*-triple $Z$, we can form a *joint Peirce decomposition*

$$Z = \bigoplus_{i,j \in \Lambda} Z_{ij}$$

where Peirce spaces $Z_{ij}$ are defined by

$$\begin{aligned} Z_{ii} &= Z_2(e_i), & Z_{ij} &= Z_1(e_i) \cap Z_1(e_j), \quad i \neq j, \\ Z_{i0} &= Z_1(e_i) \cap \bigcap_{j \neq i} Z_0(e_j), & Z_{00} &= \bigcap_i Z_0(e_i). \end{aligned}$$

We have, for $z_{ij} \in Z_{ij}$ and $e = \sum e_i$,

$$(e_k \,\Box\, e)(z_{ij}) = (e_k \,\Box\, e_k)(z_{ij}) = \begin{cases} 0 & \text{if } k \notin \{i,j\}, \\ \frac{1}{2} z_{ij} & \text{if } k \in \{i,j\}. \end{cases}$$

JBW*-triples have an abundance of tripotents. In fact, given a JBW*-triple $Z$ and $f$ in the predual $Z_*$, there is a unique tripotent $v_f \in Z$, called the *support tripotent* of $f$, such that $f \circ P_2(v_f) = f$ and the restriction $f|Z_2(v_f)$ is a *faithful positive* normal functional.

The Murray-von Neumann classification of the von Neumann algebras can be extended to that of JBW*-triples and a JBW*-triple can be decomposed into a direct sum of type $j$, $j$ =I, II, III, summands (see [18], [20]). A JBW*-triple is called *continuous* if it does not contain a type I summand in which case, it is a direct sum of a JW*-algebra $H(A,\alpha)$ and a weak* closed right ideal of a continuous von Neumann algebra, as shown in [20], where

$$H(A,\alpha) = \{a \in A : \alpha(a) = a\}$$

is the fixed-point set of a period 2 weak* continuous antiautomorphism $\alpha$ of a von Neumann algebra $A$. It follows that continuous JBW*-triples are JW*-triples.

A JBW*-triple $Z$ is called *of type* I if it contains an abelian tripotent $e$ such that $Z = U(e)$ where $U(e)$ denotes the weak* closed triple ideal generated by $e$. We recall that a tripotent $e$ is said to be *abelian* if the Peirce 2-space $P_2(e)(Z)$ is an abelian triple which is equivalent to saying that $P_2(e)(Z)$ is an associative JBW*-algebra in the usual Jordan product $x \circ y = \{x,e,y\}$. Horn (4.14, [19]) has shown that a JBW*-triple is type I if, and only if, every weak*-closed triple ideal contains an abelian tripotent.

An important class of type I JBW*-triples are the following six types of *Cartan factors*:

type 1 $B(H,K)$ *with triple product* $\{x,y,z\} = \frac{1}{2}(xy^*z + zy^*x)$;
type 2 $\{z \in B(H,H) : z^t = -z\}$;
type 3 $\{z \in B(H,H) : z^t = z\}$;
type 4 *spin factor*;
type 5 $M_{1,2}(\mathcal{O})$ *with triple product* $\{x,y,z\} = \frac{1}{2}(x(y^*z) + z(y^*x))$;
type 6 $M_3(\mathcal{O})$;

where $B(H,K)$ is the Banach space of bounded linear operators between complex Hilbert spaces $H$ and $K$, and $z^t$ is the transpose of $z$ induced by a conjugation on $H$. Cartan factors of type 2 and 3 are subtriples of $B(H,H)$, the latter notation is shortened to $B(H)$. The type 3 and 4 are Jordan algebras with the usual Jordan product $x \circ y = \frac{1}{2}(xy + yx)$. A *spin factor* is a Banach space that is equipped with a complete inner product $\langle \cdot , \cdot \rangle$ and a conjugation $j$ on the resulting Hilbert space, with triple product

$$\{x,y,z\} = \frac{1}{2}(\langle x,y\rangle z + \langle z,y\rangle x - \langle x,jz\rangle jy)$$

such that the given norm and the Hilbert space norm are equivalent.

By Horn's result in [18], a JBW*-triple $Z$ is of type I if, and only if, it is linearly isometric to an $\ell^\infty$-sum $\bigoplus_\alpha L^\infty(\Omega_\alpha) \otimes C_\alpha$ where $C_\alpha$ is a Cartan factor. Such a type I JBW*-triple will be called *type* $\mathrm{I}_{\mathrm{fin}}$ if each Cartan factor $C_\alpha$ is finite-dimensional It has been shown in [9] that a JBW*-triple $Z$ is type $\mathrm{I}_{\mathrm{fin}}$ if, and only if, its predual $Z_*$ has the Dunford-Pettis property. We recall that a Banach space $W$ has the *Dunford-Pettis property* if every weakly compact operator on $W$ is completely continuous. Such property is inherited by complemented subspaces.

Horn's type I structural result above also shows that a JBW*-algebra is type I as a JBW*-triple if and only if its self-adjoint part is a type I JBW-algebra in the sense of [17].

LEMMA 1.1. *Let* $Z$ *be a* JBW*-*subtriple of a type* $\mathrm{I}_{\mathrm{fin}}$ JBW*-*triple. Then* $Z$ *is type* $\mathrm{I}_{\mathrm{fin}}$.

*Proof.* By Corollary 6 of [9], $Z_*$ has the Dunford-Pettis property. ∎

We will begin our investigation of contractive projections in the next section. A contractive projection $P : Z \to Z$ on a JB*-triple $Z$ is a bounded linear map such that $P^2 = P$ and $\|P\| \leqslant 1$. We will exclude the trivial case of $P = 0$ which then implies $\|P\| = 1$. Given such a contractive projection $P$ on $Z$ with the triple product $\{\cdot,\cdot,\cdot\}$, one can show, using the holomorphic characterization of JB*-triples ([22], [30]) that the range $P(Z)$ is also a JB*-triple in the triple product

$$[x,y,z] = P\{x,y,z\}, \quad x,y,z \in P(Z).$$

Moreover, one has the following conditional expectation formula:

$$P\{Px,Py,Pz\} = P\{Px,y,Pz\}, \quad x,y,z \in Z.$$

The above result has also been proved in [14] for subtriples of $C^*$-algebras, via an operator algebra approach which also yields the formula

$$P\{Px,Py,Pz\} = P\{x,Py,Pz\}.$$

A weak* continuous projection on a JBW*-triple is called *normal.*

## 2. CONTRACTIVE PROJECTIONS ON $JW^*$-ALGEBRAS

In this section, we consider a JW*-algebra $A \subset B(H)$ with positive part $A^+$, inheriting various topologies of $B(H)$. A positive linear functional $\varphi$ of $A$ is called a *trace* if $\varphi(sxs) = \varphi(x)$ for all symmetries $s \in A$ and all $x \in A^+$, where a *symmetry* in $A$ is a self-adjoint element $s$ such that $s^2$ is the identity in $A$. A JW*-algebra $A$ is called *reversible* if $a_1, \dots, a_n \in A$ implies $a_1 \cdots a_n + a_n \cdots a_1 \in A$. By Corollary 1.2.10 of [3], every normal trace on a reversible JW*-algebra $A$ can be extended to a normal trace on its enveloping von Neumann algebra. Further, if $\varphi$ is faithful, so is its extension. In the sequel, our JW*-subalgebras need not have the same identity element as the JW*-algebras which contain them.

The following lemma is a special case of Lemma 1.1, but the proof below is *intrinsic* without using the Dunford-Pettis property.

LEMMA 2.1. *Every* JW*-*subalgebra of a type* $\mathrm{I}_{\mathrm{fin}}$ JW*-*algebra is of type* $\mathrm{I}_{\mathrm{fin}}$.

*Proof.* Let $A$ be a JW*-subalgebra of a type $\mathrm{I}_{\mathrm{fin}}$ JW*-algebra $B$. Then $A$ is finite since it is a subalgebra of a finite algebra. Let $p \in A$ be a projection. Then $pAp$ is a subalgebra of the type $\mathrm{I}_{\mathrm{fin}}$ algebra $pBp$. Suppose, for contradiction, that $pAp$ contains no abelian projection. By cutting down to a homogeneous summand, we may assume that $pBp$ is homogeneous. Then by Theorem 17 of [35], $p$ can be decomposed into any number of mutually orthogonal and strongly equivalent projections in $pAp$. Since equivalent projections in $pAp$ are also equivalent in $pBp$, and since in a homogeneous type $\mathrm{I}_{\mathrm{fin}}$ algebra there are at most a fixed number of mutually orthogonal and strongly equivalent projections, we have a contradiction. So $pAp$ contains an abelian projection and $A$ is of type $\mathrm{I}_{\mathrm{fin}}$. ∎

LEMMA 2.2. *Let* $(A,\circ)$ *be a* JW*-*algebra with identity* $1$ *and let* $P : A \to A$ *be a contractive projection. If* $P(A)$ *contains a unitary tripotent* $u$ *of* $P(A)$, *then* $P1 = P(uu^*u^*u) = P(u\circ u^*)$. *In particular, if* $u$ *is a projection in* $A$, *then* $P1 = u$.

*Proof.* Recall that the triple product in $P(A)$ is given by

$$[x,y,z] = P\{x,y,z\}.$$

Since $u$ is a unitary tripotent in $P(A)$, we have by the main identity

$$\begin{aligned} P1 &= [P1,u,[u,u,u]] \\ &= [[P1,u,u],u,u] - [u,[u,P1,u],u] + [u,u,[P1,u,u]] \\ &= P1 - [u,[u,P1,u],u] + P1 \end{aligned}$$

and by the conditional expectation formula,

$$\begin{aligned} P1 &= [u,[u,P1,u],u] \\ &= P\{u,P\{u,P1,u\},u\} \\ &= P\{u,P(u^2),u\} = P\{u,u^2,u\} = P(uu^*u^*u). \end{aligned}$$

Also, $P1 = [u,u,P1] = P\{u,u,P1\} = P\{u,u,1\} = P(u\circ u^*)$. ∎

REMARK 2.3. The above result shows that there is at most one unitary tripotent in $P(A)$ which is a projection in $A$. If $P(A)$ is a JW*-subalgebra of $A$, then the identity in $P(A)$ is a projection in $A$ and $P(1_A) = 1_{P(A)}$.

A JW*-algebra $A$ is said to be *finite* (or *modular* as in [35]) if its projections form a modular lattice in which case $A$ admits a centre-valued trace, and therefore a separating family of normal traces. It has been shown in Theorem 1.1.14 of [3] that a reversible JW*-algebra is finite if, and only if, its enveloping von Neumann algebra is finite. In fact, a reversible JW*-algebra and its enveloping von Neumann algebra have the same type, as shown in Theorem 1.3.2 of [3]. A projection $p$ in a JW*-algebra $A$ is called *finite* if the JW*-algebra $pAp$ is finite.

By 5.3.10 of [17], a JW*-algebra $A$ has a decomposition

$$A = A_1 \oplus A_2$$

where $A_1$ is a reversible JW*-algebra and $A_2$ is a type $\mathrm{I}_2$ JW*-algebra. Further, by 6.3.14 of [17], every type $\mathrm{I}_2$ JW*-algebra is a direct sum of JW*-algebras of the form $C(X, V)$ where $C(X)$ is an abelian W*-algebra and $V$ is a spin factor. Since every infinite-dimensional spin factor $V$ can be embedded as a JW*-subalgebra of a type $\mathrm{II}_1$ W*-factor $R$, with the canonical trace on $V$ extended to the trace of $R$ (cf. Remark 1.2.11, [3]), it follows from the above remarks that a finite (respectively semifinite) JW*-algebra $A$ can be embedded as a JW*-subalgebra of a von Neumann algebra $\mathcal{A}$ which is a direct sum of a finite (respectively semifinite) von Neumann algebra and a type $\mathrm{II}_1$ von Neumann algebra, with the faithful normal finite (respectively semifinite) trace on $A$ extending to a faithful normal finite (respectively semifinite) trace on $\mathcal{A}$.

REMARK 2.4. Let $A$ be a JW*-algebra with the decomposition $A = A_1 \oplus A_2$, where $A_1$ is a reversible JW*-algebra and $A_2$ is a type $\mathrm{I}_2$ JW*-algebra. In the rest of this section, we always assume that $A$ is embedded in the von Neumann algebra $\mathcal{A} = \mathcal{A}_1 \oplus \mathcal{N}$, where $\mathcal{A}_1$ is the enveloping von Neumann algebra of $A_1$ and $\mathcal{N}$ is a type $\mathrm{II}_1$ von Neumann algebra containing $A_2$. The above remarks imply that $A$ is finite if, and only if, $\mathcal{A}_1$ is finite. We also note that if $e$ is a finite projection in $A$, then it is also a finite projection in $\mathcal{A}$.

We recall that, for a net $(x_\alpha)$ in a von Neumann algebra, we say that

$$x_\alpha \to 0 \textit{ strongly} \Leftrightarrow x_\alpha^* x_\alpha \to 0 \textit{ weakly}$$

and

$$x_\alpha \to 0 \textit{ strongly*} \Leftrightarrow x_\alpha^* x_\alpha + x_\alpha x_\alpha^* \to 0 \textit{ weakly},$$

where "weakly" refers to the weak operator topology. Plainly, strong* convergence implies strong convergence.

Given a JW*-algebra $A \subset \mathcal{A}$, we define $A^2 = \{ab : a, b \in A\}$ which is

LEMMA 2.5. *Let $A \subset \mathcal{A}$ be a* JW*-*algebra as in Remark* 2.4. *Then the following conditions are equivalent:*

(i) *$A$ is finite;*
(ii) *the map $x \in A^2 \mapsto x^* \in A^2$ is strongly continuous on bounded spheres;*
(iii) *the net $(y_\alpha x_\alpha) \to 0$ strongly, whenever $(x_\alpha)$ and $(y_\alpha)$ are bounded nets of self-adjoint elements in $A$ with $(x_\alpha y_\alpha) \to 0$ strongly in $A^2$.*

*Proof.* (i) ⇒ (ii) This follows from the above remarks that $\mathcal{A}$ is finite and the fact that the map $x \mapsto x^*$ is continuous on bounded spheres in a finite von Neumann algebra.

(ii) ⇒ (iii) Obvious.

(iii) ⇒ (i) If $A$ is not finite, then by Lemma 23 of [35], there is an infinite orthogonal sequence $\{p_n\}$ of projections in $A$ such that, for every $n$,

$$p_1 = s_n p_n s_n$$

where $s_n$ is a symmetry. Given any normal state $\psi$ of the von Neumann algebra $\mathcal{A}$, we have

$$\sum \psi(p_n) = \psi\left(\sum p_n\right) \leqslant \psi(1) < \infty.$$

So $\psi((s_n p_n)^*(s_n p_n)) = \psi(p_n) \to 0$. But $\psi((p_n s_n)^*(p_n s_n)) = \psi(p_1) \not\to 0$ for some $\psi$, a contradiction. ∎

LEMMA 2.6. *Let $P : A \to A$ be a contractive projection on a* JW*-*algebra $A$ such that $P(A)$ is a* JW*-*subalgebra of $A$. Then:*

(i) *$P(x \circ x^*) \geqslant 0$ for all $x \in A$;*
(ii) *$P(a \circ x) = P(a) \circ x$ for $a \in A, x \in P(A)$;*
(iii) *if $P$ is weak* continuous, then $P$ is strong* continuous on bounded spheres.*

*Proof.* (i) By Lemma 2.2, $P1$ is the identity in $P(A)$. Let $\varphi$ be a state of $P(A)$. Then $\varphi \circ P(1) = \varphi(P1) = 1$ implies that $\varphi \circ P$ is a state of $A$. Hence $\varphi(P(x \circ x^*)) \geqslant 0$. As $\varphi$ was arbitrary, we have $P(x \circ x^*) \geqslant 0$. This implies that $P$ is self-adjoint.

(ii) This is proved in [10]. We give a short alternate proof here. We have

$$\begin{aligned} P(a \circ x) = P(a \circ (x \circ P(1))) &= \frac{1}{2} P\{a, x, P(1)\} + \frac{1}{2} P\{a, P(1), x\} \\ &= \frac{1}{2}(P\{Pa, x, P(1)\} + P\{Pa, P(1), x\}) \\ &= \frac{1}{2}(\{Pa, x, P(1)\} + \{Pa, P(1), x\}) = Pa \circ x. \end{aligned}$$

(iii) Let $x_\alpha \to 0$ strongly*. Then $x_\alpha \circ x_\alpha^* \to 0$ weakly and hence $P(x_\alpha \circ x_\alpha^*) \to 0$ weakly. Using (i), (ii), and the self-adjointness of $P$, we have

$$0 \leqslant P((Px_\alpha - x_\alpha) \circ (Px_\alpha - x_\alpha)^*) = P(x_\alpha \circ x_\alpha^*) - P(x_\alpha) \circ P(x_\alpha)^*$$

which implies that $P(x_\alpha) \circ P(x_\alpha)^* \to 0$ weakly. ∎

A JW*-algebra $A$ is *semifinite* if every nonzero projection in $A$ contains a nonzero finite projection. This is equivalent to saying that $A$ does not contain any

PROPOSITION 2.7. *Let $A$ be a semifinite (respectively finite)* JW**-algebra and $P$ a normal contractive projection on $A$ such that $P(A)$ is a* JW**-subalgebra of $A$. Then $P(A)$ is semifinite (respectively finite).*

*Proof.* Let $A \subset \mathcal{A}$ as in Remark 2.4. Suppose $P(A)$ is of type III. We show that $P(A) = 0$. Let $e \in A$ be a finite projection. Suppose $P(e) \neq 0$. We have $P(e) \geqslant 0$ and by spectral theory there exists a nonzero projection $p \in P(A)$ such that $\lambda p \leqslant P(e)$ for some $\lambda > 0$. Let $(x_\alpha)$ and $(y_\alpha)$ be bounded nets of self-adjoint elements in $pP(A)p$ such that $(x_\alpha y_\alpha)$ converges to 0 strongly. We shall show that $(y_\alpha x_\alpha)$ converges to 0 strongly. Since, as remarked above, $e$ is also a finite projection in $\mathcal{A}$, by pp. 97–98 of [28], the nets $(y_\alpha x_\alpha e)$ and $(e y_\alpha x_\alpha)$ converge to 0 strongly in $\mathcal{A}$. Since the nets $(e x_\alpha y_\alpha)$ and $(x_\alpha y_\alpha e)$ both converge to 0 strongly, we have $(e x_\alpha y_\alpha)$ and $(e y_\alpha x_\alpha)$ both converging to 0 strongly*. Let $z_\alpha = x_\alpha y_\alpha + y_\alpha x_\alpha$ which belongs to $pP(A)p$. Then $(e z_\alpha)$ and $(z_\alpha e)$ both converge strongly* to 0. Therefore $\{e, z_\alpha, P(e)\} \to 0$ strongly* in $A$. Since $P$ is strongly* continuous on bounded spheres and $P(A)$ is in particular a subtriple of $A$, we have

$$P(e) z_\alpha P(e) = P\{Pe, z_\alpha, P(e)\} = P\{e, z_\alpha, P(e)\} \to 0$$

strongly*, and hence strongly. It follows that

$$[pP(e)p + (1-p)] z_\alpha [pP(e)p + (1-p)] = pP(e) z_\alpha P(e) p \to 0$$

strongly, which gives

$$z_\alpha = [pP(e)p + (1-p)]^{-1} pP(e) z_\alpha P(e) p [pP(e)p + (1-p)]^{-1} \to 0$$

strongly and therefore $y_\alpha x_\alpha \to 0$ strongly. By Lemma 2.5, $pP(A)p$ is finite, contradicting that $P(A)$ is type III. Hence $P$ vanishes on every finite projection in $A$ and $P(A) = 0$.

If we apply the above argument to the identity element of a finite $A$ we obtain that $P(A)$ is finite. ∎

It will follow from Theorems 4.2 and 4.4 in Section 4 that Proposition 2.7, and Proposition 2.8 which follows, remain true without the assumption that $P(A)$ is a subalgebra. The proof of Proposition 2.8 is an adaptation to the Jordan algebra setting of the proof for von Neumann algebras in [34].

PROPOSITION 2.8. *Let $P$ be a normal contractive projection on a type* I JW**-algebra $A$ and suppose $P(A)$ is a* JW**-subalgebra of $A$. Then $P(A)$ is of type* I.

*Proof.* By Proposition 2.7, $P(A)$ is a semifinite JW*-algebra. Suppose that $P(A)$ contains a type II summand. By following $P$ by the projection onto the type II part, we can assume that $P(A)$ is of type II. We show $P(A) = 0$. It suffices to show that for any finite projection $q$ in $P(A)$, we have $qP(A)q = 0$. By following $P$ with the projection $q \cdot q$, we may further assume that $P(A)$ is of type II$_1$. Suppose $B = P(A) \neq 0$, we deduce a contradiction.

Let $A \subset \mathcal{A}$ as in Remark 2.4. There are faithful normal semifinite traces $\tau, \tau_0, \tilde{\tau}_0$ on $B, A, \mathcal{A}$ respectively such that $\tau$ is finite and $\tilde{\tau}_0$ is an extension of $\tau_0$. Since $\tau \circ P$ is a normal positive functional on $A$, by the Radon–Nikodym theorem (Theorem 2.4, [2]) there is an element $h \in L^1(A, \tau_0)^+$ such that

$$\tau \circ P(x) = \tau_0(\{h^{1/2} x h^{1/2}\}) \text{ for } x \in A. \tag{2.1}$$

Note that for self-adjoint $x \in A$ and $y \in B$, $\tau \circ P(y^2 \circ x) = \tau(P(y^2 \circ x)) = \tau(y^2 \circ Px)$ by Lemma 2.6(ii). On the other hand, $\tau \circ P(\{yxy\}) = \tau(P\{yxy\}) = \tau(\{y, Px, y\}) = \tau(y^2 \circ Px)$, the latter by [25]. Hence $\tau \circ P(\{yxy\}) = \tau \circ P(y^2 \circ Px)$, for self-adjoint $x \in A, y \in B$. Applying this to a projection $p \in B$ and using the extension property and (2.1), we have

$$\tilde{\tau}_0\left(h\left[\frac{xp+px}{2} - pxp\right]\right) = 0 \quad \text{for every } x \in A.$$

Expanding $\tilde{\tau}_0(h \circ (p \circ x)) = \tilde{\tau}_0(h \circ (pxp))$ and using the associative trace properties of $\tilde{\tau}_0$ yields

$$\tilde{\tau}_0\left(\frac{xhp + xph + xhp + xph}{4}\right) = \tilde{\tau}_0\left(\frac{xphp + xphp}{2}\right).$$

Hence,

$$\tilde{\tau}_0(x(ph + hp - 2php)) = 0,$$

which is the same as $\tau_0(x \circ (ph + hp - 2php)) = 0$. Since this is true for all $x \in A$, we have $ph = php = hp$, so that $h$ is affiliated with $B'$, the commutant of $B$ (see [2]). Note that, since $p$ is a finite projection, all of the strong products above are in $L^1(\tilde{A}, \tilde{\tau}_0)$.

Since $h \in L^1(A, \tau_0)$, we may pick a nonzero finite projection $e \in A \cap B'$ (a spectral projection of $h$). It is easy to see that $eB$ is a JW*-subalgebra of $A$ and that $eB'' = (eB)''$. Since $B$ is reversible, $B$ of type II$_1$ $\Rightarrow$ $B''$ of type II$_1$ $\Rightarrow$ $eB''$ of type II$_1$ $\Rightarrow$ $(eB)''$ of type II$_1$ $\Rightarrow$ $eB$ of type II$_1$. But $eB = eBe \subset eAe$ is of type I$_{\text{fin}}$ by Lemma 2.1, giving a contradiction. Hence $B = 0$. ∎

## 3. CONTRACTIVE PROJECTIONS ON VON NEUMANN ALGEBRAS

PROPOSITION 3.1. *Let $M$ be a von Neumann algebra of type* I *and let $e$ be a partial isometry of $M$. Then the Peirce 2-space $P_2(e)M$ is a* JW**-algebra of type* I.

*Proof.* We note that $P_2(e)M$ is a von Neumann algebra with identity $e$, under the product $x \cdot y = xe^*y$ and involution $x^\sharp = ex^*e$ as well as a JW*-algebra under $x \circ y = \{xey\} = (xe^*y + ye^*x)/2$ and $x^\sharp$. Also, $(P_2(e)M, \cdot)$ is a von Neumann algebra of type I if and only if $(P_2(e)M, \circ)$ is a JW*-algebra of type I.

Now suppose that $v$ is a nonzero central projection in $(P_2(e)M, \cdot)$. Below we shall verify the following:

(i) $v$ is a tripotent in $M$;
(ii) $v \cdot P_2(e)M = P_2(v)M$ as sets;
(iii) the identity map : $(v \cdot P_2(e)M, \cdot) \to (P_2(v)M, \times)$, where $x \times y = xv^*y$ and $x \mapsto vx^*v$ is the involution in $(P_2(v)M, \times)$, is a *-isomorphism of von Neumann algebras;
(iv) $(P_2(v)M, \times)$ has a non-zero abelian projection.

Assuming that (i)–(iv) have been proved, if there is a nonzero central projection $v$ such that $v \cdot P_2(e)M$ is a continuous von Neumann algebra, we obtain a contradiction that it contains a nonzero abelian projection. So $(P_2(e)M,\cdot\,)$ is a von Neumann algebra of type I.

It remains to verify (i)–(iv) above.

Since $v = v\cdot v = v^\sharp$, we have $v = v\cdot v\cdot v = ve^*ve^*v = v(ev^*e)^*v = vv^*v$. This proves (i).

Since $v = ee^*ve^*e$, we have $v = ee^*v = ve^*e$ so that $vv^*e = v(ev^*e)^*e = ve^*ve^*e = ve^*v = v$ and similarly $ev^*v = v$. Hence, for $y \in M$, $v\cdot P_2(e)y = v\cdot P_2(e)y\cdot v = ve^*(P_2(e)y)e^*v = vv^*ee^*(P_2(e)y)e^*ev^*v = P_2(v)y$ (since $v^*v \leqslant e^*e$ and $vv^* \leqslant ee^*$). This proves (ii).

For $x, y \in P_2(v)M$, we have, by (ii), $x = xe^*v$ and $y = ve^*y$. Therefore $x\cdot y = xe^*y = xe^*ve^*ve^*y = xv^*ve^*y = xv^*y = x\times y$. As for the involution, $ex^*e = e(ve^*xe^*v)^*e = ev^*ex^*ev^*e = vx^*v$. This proves (iii).

Let $p = vv^*$. We shall show that:

(a) there is a non-zero abelian projection $h \in M$ with $h \leqslant p$ and $c(h) = c(p)$, where $c(\cdot)$ denotes central support;

(b) with $z = hv$, $z$ is a non-zero abelian projection in the von Neumann algebra $(P_2(v)M, \times)$ and this will prove (iv).

Since $pMp$ is of type I, there is a non-zero abelian projection $h \in pMp$ such that $c_{pMp}(h) = p$. Since $h \leqslant p$, we have $c(h) \leqslant c(p)$. To show equality here, take a central projection $r \in M$ with $h \leqslant r$. Then $pr$ is a central projection in $pMp$, so that $pr = prp \geqslant c_{pMp}(h) = p$ and thus $p \leqslant r$ gives $c(p) \leqslant r$. Taking $r = c(h)$, we get $c(p) \leqslant c(h)$. This proves (a).

We next show that $z$ is a non-zero projection in $(P_2(v)M, \times)$. We have $z = hv = phvv^*v = vv^*hvv^*v \in P_2(v)M$, $vz^*v = v(vv^*hvv^*v)^*v = vv^*hv = hv = z$, $z\times z = zv^*z = hvv^*hv = hphv = hv = z$, and $zz^* = hvv^*h = h \neq 0$.

It remains to show that for $x, y \in M$, we have

$$\begin{aligned}&[z\times(vv^*xv^*v)\times z]v^*[z\times(vv^*yv^*v)\times z]\\ &\qquad = [z\times(vv^*yv^*v)\times z]v^*[z\times(vv^*xv^*v)\times z].\end{aligned} \tag{3.1}$$

The left and right sides of (3.1) collapse to $hxv^*hyv^*hv$ and $hyv^*hxv^*hv$ respectively, which are equal since $hMh$ is an abelian subalgebra of $M$. For example, the left side is equal to

$$zv^*vv^*xv^*vv^*zv^*zv^*vv^*yv^*vv^*z = hvv^*xv^*hvv^*hvv^*yv^*hv = hxv^*hyv^*hv.$$

This proves (b), hence (iv) and the Proposition 3.1. ∎

Let $P$ be a normal contractive projection on a JBW*-triple $Z$ and let $f \in P_*(Z_*)$ have the support tripotent (partial isometry in this case) $v_f$. Let $P_k = P_k(v_f)$ denote the Peirce projections induced by $v_f$. The following commutativity formulas were proved in [16]. These will be used freely in the remainder of the paper.

(i) $P_2P = P_2PP_2, PP_2 = PP_2P\,;$
(ii) $PP_0 = P_0PP_0 = PP_0P\,;$
(iii) $PP_1 = PP_1P, P_1PP_0 = 0\,.$

In the next lemma, we shall use these formulas to extend the first two of them to the case where the tripotent is not assumed to be the support of a normal functional. We shall use the fact that, by Zorn's lemma, every tripotent in a JBW*-triple $Z$ is the sum of an orthogonal family of tripotents which are support tripotents of normal functionals on $Z$.

The following lemma is needed in the next section. In this section, it will be used only in the case that $Z$ is a von Neumann algebra, considered as a JW*-triple under $\frac{1}{2}(xy^*z + zy^*x)$.

LEMMA 3.2. *Let $P$ be a normal contractive projection on a* JBW*-*triple $Z$ and suppose that $v$ is a tripotent of the* JBW*-*triple $P(Z)$. Choose a set $S = \{f_i : i \in I\}$ of pairwise orthogonal normal functionals on $P(Z)$ such that $v = \sum_{i\in I} v_{f_i}$, where $v_{f_i}$ is the support tripotent of $f_i$ in $P(Z)$. Let $w_i$ be the support tripotent of $f_i$ in $Z$, necessarily pairwise orthogonal, and let $w$ be the partial isometry $\sum_{i\in I} w_i$. Then*

$$P_2(w)P = P_2(w)PP_2(w), \quad PP_2(w) = PP_2(w)P.$$

*Proof.* Since $w = \sum w_i$, we have $P_2(w) = \sum_i P_2(w_i) + \sum_{j\neq k} P_1(w_j)P_1(w_k)$ and therefore

$$\begin{aligned}&P_2(w)PP_2(w)\\ &\quad = \sum_{i,i'} P_2(w_i)PP_2(w_{i'}) + \sum_{j\neq k, j'\neq k'} P_1(w_j)P_1(w_k)PP_1(w_{j'})P_1(w_{k'})\\ &\qquad + \sum_{j'\neq k',\ \text{all } i} P_2(w_i)PP_1(w_{j'})P_1(w_{k'}) + \sum_{j\neq k,\ \text{all } i'} P_1(w_j)P_1(w_k)PP_2(w_{i'}).\end{aligned}$$

Because $P_2(w_i)P = P_2(w_i)PP_2(w_i)$, by properties of the joint Peirce decomposition, the first sum reduces to $\sum_i P_2(w_i)P$ and each term in the third sum is zero.

Each term in the fourth sum is zero as well. Indeed, since in the following we may assume $k \neq i'$,

$$\begin{aligned}P_1(w_j)P_1(w_k)PP_2(w_{i'}) &= P_1(w_j)[I - P_2(w_k) - P_0(w_k)]PP_2(w_{i'})\\ &= P_1(w_j)[PP_2(w_{i'}) - P_2(w_k)PP_2(w_{i'}) - P_0(w_k)PP_2(w_{i'})]\\ &= P_1(w_j)[PP_2(w_{i'}) - 0 - P_0(w_k)PP_0(w_k)P_2(w_{i'})]\\ &= P_1(w_j)[PP_2(w_{i'}) - PP_0(w_k)P_2(w_{i'})]\\ &= P_1(w_j)[PP_2(w_{i'}) - PP_2(w_{i'})] = 0.\end{aligned}$$

The second sum reduces to $\sum_{j\neq k} P_1(w_j)P_1(w_k)P$. Indeed, if $k \notin \{j', k'\}$, then $[P_1(w_k)PP_1(w_{j'})]P_1(w_{k'}) = 0$ since we have $P_1(w_{j'})P_1(w_{k'})Z \subset P_0(w_k)Z$ and $P_1(w_k)PP_0(w_k) = 0$. Thus, it follows that the second sum is reduced to

$\sum_{j\neq k} P_1(w_j)P_1(w_k)PP_1(w_j)P_1(w_k)$. However,

$$\begin{aligned} &P_1(w_j)P_1(w_k)PP_1(w_j)P_1(w_k)\\ &\quad= P_1(w_j)P_1(w_k)P[I - P_2(w_j) - P_0(w_j)]P_1(w_k)\\ &\quad= P_1(w_j)P_1(w_k)PP_1(w_k) - P_1(w_j)P_1(w_k)PP_2(w_j)P_1(w_k)\\ &\qquad - P_1(w_j)P_1(w_k)PP_0(w_j)P_1(w_k)\\ &\quad= P_1(w_j)P_1(w_k)P[I - P_2(w_k) - P_0(w_k)] + 0 + 0\\ &\quad= P_1(w_j)P_1(w_k)P - P_1(w_j)P_1(w_k)PP_2(w_k) - P_1(w_j)P_1(w_k)PP_0(w_k)\\ &\quad= P_1(w_j)P_1(w_k)P - P_1(w_k)P_1(w_j)PP_2(w_k)\\ &\quad= P_1(w_j)P_1(w_k)P. \end{aligned}$$

This proves the first formula.

For the second formula, we have

$$\begin{aligned} PP_2(w)P &= P(\sum_i P_2(w_i) + \sum_{j\neq k} P_1(w_j)P_1(w_k))P\\ &= \sum_i PP_2(w_i)P + \sum_{j\neq k} PP_1(w_j)P_1(w_k)P\\ &= \sum_i PP_2(w_i) + \sum_{j\neq k} PP_1(w_j)P_1(w_k) = PP_2(w) \end{aligned}$$

since

$$\begin{aligned} PP_1(w_j)P_1(w_k)P &= (PP_1(w_j)P)P_1(w_k)P\\ &= PP_1(w_j)(PP_1(w_k)) = PP_1(w_j)P_1(w_k). \quad ∎ \end{aligned}$$

The following lemma is probably known. We include a proof for completeness.

LEMMA 3.3. *Let $p$ be a projection in a JB$^*$-algebra $A$ and let $A_1(p)$ be the Peirce 1-space. Then $A_1(p) \cap A^+ = 0$.*

*Proof.* If $x \in A_1(p) \cap A^+$, then let $y = x^{1/2} \in A_{sa}$ and let $y = y_2 + y_1 + y_0$ be its Peirce decomposition with respect to $p$. Then $x = y_2^2 + y_1^2 + y_0^2 + 2(y_2 + y_0)y_1$. Since $x \in A_1(p)$, we have $y_2^2 + y_1^2 + y_0^2 = 0$ and because the JB-algebra $A_{sa}$ is formally real, $y = 0$. ∎

LEMMA 3.4. *Let $P, Z, v, S, w$ be as in Lemma 3.2. Then:*

*(i) the map $Q = P_2(w)P : Z_2(w) \to Z_2(w)$ is a normal faithful unital contractive projection with range $P_2(w)P(Z)$;*

*(ii) the map $P_2(w)$ is a linear surjective isometry of $P(Z)_2(v)$ onto $P_2(w)P(Z)$.*

*Proof.* (i) By Lemma 3.2, $Q^2 = P_2(w)PP_2(w)P = P_2P = Q$ and $Q(Z_2(w)) = P_2(w)PP_2(w)(Z) = P_2(w)P(Z)$. To show that $Q$ is unital, note first that by Lemma 2.7 of [11], $v_i = w_i + P_0(w_i)v_i$ so that $w_i \perp (v_i - w_i)$. By taking sums and limits, one obtains $(v - w) \perp w$ and $\|v - w\| \leqslant 1$. Indeed, it is easy to see

that for any finite set $F$ of indices, $\sum_F w_i$ is the support tripotent of the normal functional $\sum_F f_i$. Hence, $\sum_F w_i \perp \sum_F (v_i - w_i)$ so that $\sum_F (v_i - w_i) \in Z_0(\sum_F w_i)$ and $\left\|\sum_F w_i \pm \sum_F (v_i - w_i)\right\| = 1$. By passing to the limit and noting that each $f_i$ has the value 1 on $w \pm (v - w)$, we have $\|w \pm (v - w)\| = 1$, and since $P_2(w)$ is contractive, $\|w \pm P_2(w)(v - w)\| \leqslant 1$, and since $w$ is an extreme point of the unit ball of $Z_2(w)$, $P_2(w)(v - w) = 0$, that is, $P_2(w)v = w$. Now $v = w + P_1(w)v + P_0(w)v$, so by Lemma 1.6 of [15], $P_1(w)v = 0$ and thus $v = w + P_0(w)v$ and $v = Pv = Pw + PP_0(w)x$ so that $Qw = P_2(w)Pw = P_2(w)(v - PP_0(w)v) = P_2(w)v = w$ and $Q$ is unital.

Finally, we show that $Q$ is faithful. Suppose that $b \in Z$, $P_2(w)b \geqslant 0$, and $P_2(w)Pb = 0$. We shall show that $P_2(w)b = 0$. In the first place, since $P_2(w_i)$ is a positive operator on the JB$^*$-algebra $P_2(w)Z$ (3.3.6, [17]), $P_2(w_i)b = P_2(w_i)P_2(w)b \geqslant 0$ for every $i \in I$. Since $P_1(w_k)P_1(w_l)b \perp w_i$, we have

$$\begin{aligned} 0 &= \langle P_2(w)Pb, f_i\rangle = \langle PP_2(w)Pb, f_i\rangle = \langle PP_2(w)b, f_i\rangle = \langle P_2(w)b, f_i\rangle\\ &= \sum_j \langle P_2(w_j)b, f_i\rangle + \sum_{k\neq l} \langle P_1(w_k)P_1(w_l)b, f_i\rangle = \langle P_2(w_i)b, f_i\rangle. \end{aligned}$$

Hence $P_2(w_i)b = 0$ for all $i$. Therefore $P_2(w)b = \sum_i P_2(w_i)b + \sum_{j\neq k} P_1(w_j)P_1(w_k)b = \sum_{j\neq k} P_1(w_j)P_1(w_k)b = 0$ by Lemma 3.3, since each $P_1(w_k)P_1(w_l)b$ must be positive. This proves that $Q$ is faithful, and hence (i) holds.

(ii) Let $B$ denote the JBW$^*$-algebra $P(Z)_2(v)$. Then, by definition, $B = \{\{v, \{vxv\}_{P(Z)}, v\}_{P(Z)} : x \in P(Z)\}$. But

$$\{v, \{vxv\}_{P(Z)}, v\}_{P(Z)} = P\{v, P\{vxv\}, v\} = P\{v, \{vxv\}, v\} = PQ(v)^2x$$

so that $B = PQ(v)^2P(Z)$ and

$$P_2(w)B = P_2(w)PQ(v)^2P(Z) = P_2(w)PP_2(w)Q(v)^2P(Z) = P_2(w)P(Z).$$

Now let $F_v$ be the normal state space of $B$, that is

$$F_v = \{\ell \in B_* : \|\ell\| = 1 = \ell(v)\}.$$

Recall from the first part of the proof that $v = Pw + PP_0(w)v$. Also, $P(w) = P\{v, Pw, v\} = P(w)^\sharp$ implies that for $\ell \in F_v$, $\ell(P(w))$ is real and $1 = \ell(v) = \ell(P(w)) + \ell(PP_0(w)v)$. Therefore $\ell(P(w)) \geqslant 0$, so that in fact $0 \leqslant P(w) \leqslant v$, that is, $v - P(w) \in B^+$. Now for each $i$, we have $f_i(v - Pw) = f_i(PP_0(w)v) = f_i(P_0(w)v) = f_i(P_2(w_i)P_0(w)v) = 0$. It follows, using Lemma 3.3 as above, that $v - Pw = 0$.

Now, for arbitrary $\ell \in F_v$, as $Pw = v$, we have $\ell(w) = \ell(P(w)) = \ell(v) = 1$ and by Lemma 3.1 of [11],

$$\ell = P_2(w)_*\ell. \tag{3.2}$$

By linearity and the Jordan decomposition for self-adjoint functionals, (3.2) extends to all $\ell \in B_*$. Hence for $b \in B$, we have $\|b\| = \sup\{|\ell(b)| : \|\ell\| = 1, \ell \in B_*\} = \sup\{|\ell(P_2(w)b)| : \|\ell\| = 1, \ell \in B_*\} \leqslant \|P_2(w)b\|$. This proves (ii). ∎

PROPOSITION 3.5. *Let $P$ be a normal contractive projection on a von Neumann algebra $M$ of type* I. *Then $P(M)$ is a* JW$^*$*-triple of type* I.

*Proof.* Let $v$ be any nonzero tripotent of $P(M)$ and choose $w \in M$ as in Lemma 3.2. By Proposition 3.1, $M_2(w)$ is a JW$^*$-algebra of type I. By Lemma 3.4 (i) and Corollary 1.5 of [10], $P_2(w)P(M) = Q(M_2(w))$ is a JW$^*$-subalgebra of $M_2(w)$, where $Q = P_2(w)P$. By Proposition 2.8, $Q(M_2(w))$ is a JBW$^*$-algebra of type I, and by Lemma 3.4(ii), $P(M)_2(v)$ (the Peirce 2-space of the tripotent $v$ of the JW$^*$-triple $P(M)$) is also of type I since a unital surjective linear isometry is a Jordan $^*$-isomorphism. One can now choose $v$ to be a complete tripotent of $P(Z)$ to obtain from 4.14 of [19] that $P(M)$ is a JW$^*$-triple of type I. ∎

## 4. CONTRACTIVE PROJECTIONS ON JW*-TRIPLES

PROPOSITION 4.1. *Let $Z$ be a* JBW$^*$*-triple of type* I *and let $v$ be a tripotent in $Z$. Then $P_2(v)Z$ is a* JBW$^*$*-algebra of type* I.

*Proof.* By Horn's structure theorem, we may assume that $Z = L^\infty(\Omega, C)$ where $C$ is a Cartan factor. If $C$ is of types 1, 2, or 3, then there is a normal contractive projection $Q$ on $L^\infty(\Omega, \widetilde{C})$, where $\widetilde{C}$ is the von Neumann envelope of $C$ with range $Z$. Since $P_2(v)Q$ is a normal contractive projection from the type I von Neumann algebra $L^\infty(\Omega, \widetilde{C})$ onto $P_2(v)Z$, the latter is of type I by Proposition 3.5. If $C$ is of type 4, then $P_2(v)Z = L^\infty(\Omega_2, C) \oplus L^\infty(\Omega_1)$, where $\Omega_k = \{\omega \in \Omega : \text{rank of } v(\omega) \text{ is } k\}$, $k = 1, 2$. Indeed, if $f \in Z$ and $g = P_2(v)f$, then $g = 0$ on $\Omega_0$, $g(\omega) = \langle f(\omega), \widehat{v(\omega)}\rangle v(\omega)$ for $\omega \in \Omega_1$ and $g = f$ on $\Omega_2$. Here we use the notation $\hat{v}$ for the normal functional with support tripotent $v$. It follows that the map $g = P_2(v)f \in P_2(v)Z \mapsto (g_2, g_1) \in L^\infty(\Omega_2, C) \oplus L^\infty(\Omega_1)$, where $g_1(\omega) = \langle f(\omega), \widehat{v(\omega)}\rangle$ for $\omega \in \Omega_1$ and $g_2 = g|\Omega_2$, is a surjective linear isometry.

If $C$ is of types 5 or 6, then it is finite-dimensional and $L^\infty(\Omega, C)$ is of type $\mathrm{I}_{\mathrm{fin}}$. By Lemma 1.1, the subtriple $P_2(v)(Z)$ is of the same type. ∎

THEOREM 4.2. *Let $P$ be a normal contractive projection on a* JW$^*$*-triple $Z$ of type* I. *Then $P(Z)$ is of type* I.

*Proof.* By 4.14 of [19], we need only to show that $P(Z)_2(v)$ is of type I for a complete tripotent $v \in P(Z)$. Choose $w \in Z$ as in Lemma 3.2. By Proposition 4.1, $P_2(w)Z$ is a JW$^*$-algebra of type I. One can now argue exactly as in the proof of Proposition 3.5, using Lemma 3.4, to show that $P_2(w)P$ is a faithful, normal, unital contractive projection of $P_2(w)Z$ onto $P_2(w)P(Z)$ (which is again a subalgebra by Corollary 1.5 of [10]) and that $P_2(w)$ is a unital isometry of $P(Z)_2(v)$ onto $P_2(w)P(Z)$. As in the proof of Proposition 3.5 and using Proposition 2.8, $P_2(w)P(Z)$ is of type I and so is $P(Z)_2(v)$. ∎

PROPOSITION 4.3. *Let $Z$ be a semifinite* JW$^*$*-triple and let $v$ be a partial isometry in $Z$. Then $Z_2(v)$ is a semifinite* JW$^*$*-algebra.*

*Proof.* We prove this first in the case that $Z$ is a von Neumann algebra $M$. If $M_2(v)$ had a type III part, we could follow $P_2(v)$ by the projection of $M_2(v)$ onto that type III part and obtain a Peirce 2-space of $M$ of type III. So we may assume that $M_2(v)$ is of type III. Let $p$ be a finite nonzero projection in $M$ dominated by $v^*v$. Then $vp$ is a nonzero projection in $M_2(v)$ dominated by $v$ (cf. Proposition 3.1). We shall show that $M_2(vp)$ is finite by showing that its involution is strongly continuous on bounded spheres. Recall that $M_2(vp) = \{x = vp(vp)^*a(vp)^*vp = vpv^*ap : a \in M\}$ is a von Neumann algebra under $x \cdot y = x(vp)^*y$ and $x^\sharp = vpx^*vp$.

Let $x_\alpha$ be a bounded net in $M_2(vp)$. Then

$$x_\alpha \xrightarrow{s} 0 \text{ in } M_2(vp) \Rightarrow vpx_\alpha^*vp(vp)^*x_\alpha \xrightarrow{w} 0 \Rightarrow vpx_\alpha^*x_\alpha \xrightarrow{w} 0$$

$$\Rightarrow px_\alpha^*x_\alpha \xrightarrow{w} 0 \Rightarrow px_\alpha^*x_\alpha p \xrightarrow{w} 0 \Rightarrow x_\alpha p \xrightarrow{s} 0 \Rightarrow \quad \text{(by pp. 97–98 of [28])}$$

$$x_\alpha^* = px_\alpha^* \xrightarrow{s} 0 \Rightarrow x_\alpha x_\alpha^* \xrightarrow{w} 0 \Rightarrow x_\alpha x_\alpha^* vp \xrightarrow{w} 0 \Rightarrow x_\alpha(vp)^*vpx_\alpha^*vp \xrightarrow{w} 0$$

$$\Rightarrow x_\alpha \cdot x_\alpha^\sharp \xrightarrow{w} 0 \Rightarrow x_\alpha^\sharp \xrightarrow{s} 0 \text{ in } M_2(vp).$$

Thus $vp$ is a finite projection which is a contradiction.

To prove the general case, write $Z = Z_{\mathrm{I}} \oplus Z_{\mathrm{II}}$ where $Z_{\mathrm{I}}$ is of type I and $Z_{\mathrm{II}}$ is of type II. Since $P_2(v)Z = P_2(v_1)Z_{\mathrm{I}} \oplus P_2(v_2)Z_{\mathrm{II}}$ for suitable partial isometries $v_1 \in Z_{\mathrm{I}}$ and $v_2 \in Z_{\mathrm{II}}$, and we already know that $P_2(v_1)Z_{\mathrm{I}}$ is of type I, we may assume by [20] that $Z$ is triple isomorphic to $pM \oplus H(N, \alpha)$, where $M$ and $N$ are von Neumann algebras of type II. Accordingly, $v_2 = v_2' + v_2''$ so that $P_2(v_2)Z_{\mathrm{II}} = P_2(v_2')(pM) \oplus P_2(v_2'')(H(N, \alpha)) = M_2(v_2') \oplus H(N_2(v_2''), \alpha)$ and $Q(N_2(v_2'')) = H(N_2(v_2''), \alpha)$ where $Q$ is the projection $Q(x) = (x + \alpha(x))/2$ for $x \in N$. By the first part of the proof, both $M_2(v_2')$ and $N_2(v_2'')$ are semifinite. Then by Proposition 2.7, $P_2(v_2'')H(N, \alpha)$ is semifinite and the result follows. ∎

THEOREM 4.4. *Let $P$ be a normal contractive projection on a semifinite* JW$^*$*-triple $Z$. Then $P(Z)$ is a semifinite* JW$^*$*-triple.*

*Proof.* By passing to the type III part of $P(Z)$, assuming it is nonzero for contradiction, and using [20], we may assume that $P(Z) = pM \oplus H(N, \alpha)$ where $M$ and $N$ are von Neumann algebras of type III. As in the proof of Proposition 3.5, using Lemma 3.4, Proposition 4.3, and Proposition 2.7, one shows that $P(Z)_2(v)$ is semifinite for any tripotent $v$ of $P(Z)$. Choosing $v = 0 \oplus 1_N$ leads to $P(Z)_2(v) = H(N, \alpha)$, a contradiction unless $H(N, \alpha) = 0$. Choosing $v = p \oplus 0$ leads to $P(Z)_2(v) = pMp$, again a contradiction unless $pMp = 0$, which implies that $p = 0$, another contradiction. ∎

## 5. CONTRACTIVE PROJECTIONS ON JBW*-TRIPLES

In this section we extend Theorems 4.2 and 4.4 to arbitrary JBW*-triples and make some remarks on the atomic case. A close examination of the proof of Theorem 4.2 reveals that it carries over to the case of JBW*-triples if we can show that the range of a faithful normal positive unital projection on a type I JBW*-algebra is olso type I. This is proved in the following proposition, of which (i) was proved in [10] for JC-algebras.

PROPOSITION 5.1. *Let $P$ be a faithful, positive, unital, normal projection on a* JBW*-*algebra $A$. Then:*

(i) *$P(A)$ is a *-subalgebra of $A$;*
(ii) *if $A$ is of type* I, *then $P(A)$ is of type* I.

*Proof.* Using the Kadison-Schwarz inequality, extended to JB-algebras in Theorem 1.2 of [26], if $x \in P(A)$, then $P(x^2) - x^2 \geqslant P(x)^2 - x^2 = 0$ (cf. Lemma 1.2 (3), [10]). Since $P$ is faithful, and $P(P(x^2) - x^2) = 0$, we have $P(x^2) - x^2 = 0$ so that $x^2 \in P(A)$. Since $P$ is self-adjoint, this proves (i).

To prove (ii), let $A$ be of type I, and suppose for contradiction, that $P(A)$ has type II or III summands. As before, by following $P$ by the projection onto these summands, we may assume that $P(A)$ is of type II or III. In each of these cases, $P(A)$ remains a subalgebra of $A$. By the halving lemma (5.2.15, [17]) there exist four orthogonal projections with sum the identity of $P(A)$, with each pair exchanged by a symmetry (see p. 122, [17]). By the argument in 5.2.8 of [17], the elements of each such pair are strongly connected. Since $P$ is unital and $P(A)$ is a subalgebra of $A$, the above also holds in $A$. By the coordinatization theorem (2.8.9, [17]) $A$ is a Jordan matrix algebra, that is, $A$ is isomorphic to $H_4(R)$ for some *-algebra $R$. By 2.7.6 of [17], $R$ is associative and therefore $A$ is a JW*-algebra. But this leads to a contradiction because we have already shown in Proposition 2.8 that $P(A)$ is of type I in this case. ∎

Now, proceeding exactly as in the proof of Theorem 4.2 we have the result for JBW*-triples.

THEOREM 5.2. *Let $P$ be a normal contractive projection on a* JBW*-*triple $Z$ of type* I. *Then $P(Z)$ is of type* I.

As noted before, a type II JBW*-triple is a JW*-triple. It follows from Proposition 4.1 and Theorem 4.4 that if $Z$ is a semifinite JBW*-triple, then $P_2(e)Z$ is also a semifinite JBW*-algebra. Using this fact, now there is no difficulty of extending the proof of Theorem 4.4 to the case of JBW*-triples.

THEOREM 5.3. *Let $P$ be a normal contractive projection on a semifinite* JBW*-*triple $Z$. Then $P(Z)$ is a semifinite* JBW*-*triple.*

Tomiyama ([34]) has proved that a von Neumann algebra which is the range of a normal contractive projection on an atomic von Neumann algebra is itself atomic. It is also known (see Exercise 8, p. 334, [33]) that a von Neumann algebra $M \subset B(H)$ is atomic if and only if there is a faithful family of normal conditional expectations of $B(H)$ onto $M$. We end with a very simple proof of a result which extends Tomiyama's theorem to JBW*-triples. The proof follows from a result in [7] which states that a JBW*-triple is atomic if, and only if, its predual has the Radon–Nikodym property. It is clearly false without the normality assumption on $P$. We first state a lemma of independent interest.

LEMMA 5.4. *Let $Z$ be a type* $\mathrm{I_{fin}}$ JBW*-*triple and let $P : Z \to Z$ be a normal contractive projection. Then $P(Z)$ is a type* $\mathrm{I_{fin}}$ JBW*-*triple.*

*Proof.* We note that $P(Z)$ is norm-closed. By weak* continuity of $P$ and the Krein–Smulyan Theorem, $P(Z)$ is also weak* closed. Also, $P$ induces a contractive projection $P_* : f \in Z_* \mapsto f \circ P \in Z_*$ on the predual $Z_*$. By [9], $Z_*$ has the Dunford–Pettis property. The predual of $P(Z)$ identifies with $Z_*/P_*^{-1}(0)$ which is linearly isometric to the complemented subspace $P_*(Z_*)$ of $Z_*$, and therefore has the Dunford–Pettis property. Hence by [9] again, $P(Z)$ is of type $\mathrm{I_{fin}}$. ∎

PROPOSITION 5.5. *Let $Z$ be an atomic* JBW*-*triple and let $P : Z \to Z$ be a normal contractive projection. Then the range $P(Z)$ is (linearly isometric to) an atomic* JBW*-*triple.*

*Proof.* As in the proof of Lemma 5.4, the predual of $P(Z)$ is linearly isometric to a complemented subspace of the predual $Z_*$ which has the Radon-Nikodym property. So $P(Z)$ is atomic by [7]. ∎

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