Since $l^1(\mathbb{N})$ is an $l^\infty(\mathbb{N})$-submodule of $\mathbb{C}^{\mathbb{N}}$, it follows that $l^1(\mathbb{N})/c_{00}$ is an $l^\infty(\mathbb{N})$-submodule of E. Let $Q: E \to E/(l^1(\mathbb{N})/c_{00}) = \mathbb{C}^{\mathbb{N}}/l^1(\mathbb{N})$ be the quotient map. Then $Q \circ D: l^\infty(\mathbb{N}) \to \mathbb{C}^{\mathbb{N}}/l^1(\mathbb{N})$ is the desired derivation.

We now return to the second cohomology group $H^2(A, l^1(\mathbb{N}))$ for strongly regular amenable Banach function algebras. Let A be such a Banach function algebra. It is easy to find a set $S = \{x_n \in \Phi_A \mid n \in \mathbb{N}\}$ and a function $f_0 \in A$ such that S is discrete in the relative topology and $f_0(S)$ is infinite. Let $\Theta: A \to l^\infty(\mathbb{N})$ be given by

$$\Theta(f)(n) = f(x_n) \quad \text{for } f \in A, n \in \mathbb{N}.$$

Clearly Θ is an algebra homomorphism and if $l^1(\mathbb{N})$ is a Banach A-bimodule as in Definition 2.3, then for $f \in A$ and $\beta \in l^1(\mathbb{N})$ we have

$$f \cdot \beta = \Theta(f) \cdot \beta. \tag{12}$$

THEOREM 3.26 (**CH**). *Let A be an infinite-dimensional amenable strongly regular Banach function algebra. Then there exists a Banach A-bimodule E such that the comparison map $i_2: \mathscr{H}^2(A, E) \to H^2(A, E)$ is not surjective.*

Proof. Let f_0 and S be as above. Then $\Theta(f_0)$ is a bounded sequence such that the set $\{\Theta(f_0)(n) \mid n \in \mathbb{N}\}$ is infinite. By Theorem 3.25, there exists a derivation $D: l^\infty(\mathbb{N}) \to \mathbb{C}^{\mathbb{N}}/l^1(\mathbb{N})$ with $D(\Theta(f_0)) \neq 0$. Hence $D \circ \Theta: A \to \mathbb{C}^{\mathbb{N}}/l^1(\mathbb{N})$ is a non-zero derivation. By Proposition 2.5 we have $H^2(A, l^1(\mathbb{N})) \neq 0$. Since $\mathscr{H}^2(A, l^1(\mathbb{N})) = 0$ by amenability, the result follows.

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THE DUNFORD–PETTIS PROPERTY IN JB*-TRIPLES

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JB*-triples occur in the study of bounded symmetric domains in several complex variables and in the study of contractive projections on C*-algebras. These spaces are equipped with a ternary product $\{\cdot, \cdot, \cdot\}$, the *Jordan triple product*, and are essentially geometric objects in that the linear isometries between them are exactly the linear bijections preserving the Jordan triple product (cf. [**23**]).

A JB*-triple is a complex Banach space and its open unit ball admits many biholomorphic automorphisms, which play a fundamental role in the theory of JB*-triples and bounded symmetric domains. In fact, a Banach space is a JB*-triple if, and only if, the biholomorphic automorphisms of its open unit ball act transitively [**23**]. Recently, Isidro and Kaup [**22**] studied the question of when these holomorphic automorphisms are weakly continuous, and a notion of *weakly continuous* JB*-*triples* was introduced in [**24**]. The weak continuity of these automorphisms turns out to be closely related to a well-known Banach property, namely the *Dunford–Pettis property* (which will be recalled below). Indeed, using [**22**] one can show that a JB*-triple with a unitary tripotent has the Dunford–Pettis property if, and only if, every biholomorphic automorphism of its open unit ball is *sequentially weakly continuous* in the sense that it preserves weak convergence of sequences (see Proposition 8 below). It is therefore of interest to know which JB*-triples have the Dunford–Pettis property.

In this paper, we characterise JB*-triples having the Dunford–Pettis property. We show that, among other results, a JB*-triple Z has the Dunford–Pettis property if, and only if, for every weakly null sequence (z_n) in Z, the sequence $(\{z_n, z_n, z\})$ is also weakly null for all $z \in Z^{**}$. It follows that the Dunford–Pettis property is inherited by subtriples and that a JBW*-triple W has the Dunford–Pettis property if, and only if, W is an l_∞-sum $\bigoplus_\alpha L^\infty(\Omega_\alpha, \mu_\alpha, C_\alpha)$, where C_α is a Cartan factor and $\sup_\alpha \dim C_\alpha < \infty$. We also show that the predual W_* has the Dunford–Pettis property if, and only if, $W = l_\infty$-sum $\bigoplus_\alpha L^\infty(\Omega_\alpha, \mu_\alpha, C_\alpha)$ with $\dim C_\alpha < \infty$ for all α. These results subsume those in [**5, 9, 10**].

1. *Dunford–Pettis property and weak continuity of automorphisms*

Let $C(X)$ be the Banach space of continuous functions on a compact Hausdorff space X. It is a celebrated result of Grothendieck [**15**] that every weakly compact linear operator on $C(X)$ is completely continuous. He called this property of $C(X)$ the *Dunford–Pettis property* (DPP for short) referring, of course, to an earlier result of Dunford and Pettis [**13**] that L_1-spaces enjoy the same property. Grothendieck [**15**] has also shown that a Banach space E has DPP if, and only if, whenever (x_n) and

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(f_n) are weakly null sequences in E and its dual E^* respectively, then $\lim_{n\to\infty} f_n(x_n) = 0$. Therefore if E^* has DPP so does E, but the converse is false. Note that DPP is not inherited by subspaces or quotients.

The Dunford–Pettis property plays a useful role in Banach spaces and, as we have mentioned, it also appears in JB*-triples in connection with a weak version of continuity for biholomorphic automorphisms. We refer to **[11]** for an excellent survey of the Dunford–Pettis property and the works of many authors on this subject. We first introduce some background for JB*-triples.

A JB*-*triple* is a complex Banach space Z with a continuous triple product $\{\cdot,\cdot,\cdot\}: Z\times Z\times Z\to Z$ which is linear and symmetric in the outer variables, and conjugate linear in the middle variable, and satisfies

(i) the operator $a\mapsto\{z,z,a\}$ on Z is Hermitian with non-negative spectrum for all $z\in Z$;

(ii) $\|\{z,z,z,\}\| = \|z\|^3$;

(iii) the main identity

$$\{a,b,\{x,y,z\}\} = \{\{a,b,x\},y,z\} - \{x,\{b,a,y\},z\} + \{x,y,\{a,b,z\}\}.$$

A JB*-triple which is a dual Banach space is called a JBW*-*triple*, in which case the predual is unique and the triple product is separately weak* continuous. The second dual of a JB*-triple is a JBW*-triple with a natural triple product **[12]**.

A norm closed subspace of a C*-algebra which is also algebraically closed under the triple product $\{x,y,z\} = \frac{1}{2}(xy^*z+zy^*x)$ is a JB*-triple, called a J*-*algebra* **[18]**. Other examples of JB*-triples include the *Cartan factors of types* 1 to 6, where a *type* 4 Cartan factor is a complex spin factor, *type* 5 is the JB*-triple consisting of 1×2 matrices over the complex Cayley algebra $\mathbb{O}$, and *type* 6 the Hermitian 3×3 matrices over $\mathbb{O}$. The types 1, 2 and 3 are defined as follows for arbitrary complex Hilbert spaces H and K:

$L(H,K)$ is *type* 1, where $L(H,K)$ consists of all bounded linear operators from H to K;

$\{z\in L(H): z = jz^*j\}$ is *type* 2;

$\{z\in L(H): z = -jz^*j\}$ is *type* 3, where $j: H\to H$ is a conjugation.

An element e in a JB*-triple Z is called a *tripotent* if $\{e,e,e\} = e$; it is called *unitary* if $\{e,e,z\} = z$ for all $z\in Z$. If Z has a unitary tripotent u then Z becomes a Jordan algebra with product $z\circ w = \{z,u,w\}$ and involution $z^* = \{u,z,u\}$ such that

$$\{x,y,z\} = x\circ(y^*\circ z) - y^*\circ(z\circ x) + z\circ(x\circ y^*)$$

and we also have $\|x\circ y\| \leqslant \|x\|\,\|y\|$ and $\|x^*\| = \|x\|$. In other words, Z becomes a JB*-algebra with identity u.

The self-adjoint part $\{z\in A: z^* = z\}$ of a JB*-algebra A is a so-called JB-*algebra* which is a real Jordan Banach algebra. A JB-algebra with a (unique) predual is called a JBW-*algebra*. A JB-algebra which can be represented as a (real) Jordan algebra of bounded self-adjoint operators on a (complex) Hilbert space is called a JC-*algebra*. A JBW-algebra with similar representation is called a JW-*algebra*. We refer to **[17]** for a detailed theory of JB-algebras. See also **[14, 32, 33]**. A recent survey of JB*-triples can be found in **[28]**.

Let Z be a JB*-triple with open unit ball D. Then the group $\mathrm{Aut}(D)$ of (biholomorphic) automorphisms of D acts transitively on D. Indeed, given $a\in D$ the map $g_a: D\to D$ defined by

$$g_a(z) = a + B(a,a)^{1/2}(I_Z + z\,\square\,a)^{-1}z$$

is a biholomorphic automorphism satisfying $g_a(0) = a$ and $g_a^{-1} = g_{-a}$, where the *box operator* $z\,\square\,a: Z\to Z$ is defined by $z\,\square\,a(w) = \{z,a,w\}$ and $B(x,y): Z\to Z$ is the *Bergman operator* given by

$$B(x,y)(z) = z - 2\{x,y,z\} + \{x,\{y,z,y\},x\}.$$

In **[22]** Isidro and Kaup described the automorphisms of D which are $\sigma(Z,Z^*)$-continuous and consequently, the weak continuity of all automorphisms of D in a unital C*-algebra A implies that A has the Dunford–Pettis property **[22**, Proposition 2.7**]**. Although this implication need not hold for JB*-triples, we shall clarify the relationship between DPP and weak continuity of automorphisms in Corollary 7 and Proposition 8 below.

For convenience, we shall call a mapping between Banach spaces *sequentially weakly continuous* (s.w.c.) if it preserves weak convergence of sequences.

Let $\mathrm{Aut}_s(D) = \{g\in\mathrm{Aut}(D): g \text{ is s.w.c.}\}$. Following **[22]**, we let

$$\mathrm{CONT}_w(Z) = \{a\in Z: \text{the map } z\longmapsto\{z,a,z\} \text{ is s.w.c. on } Z\}.$$

We begin with a description of $\mathrm{Aut}_s(D)$ analogous to that in Theorem 3.6 of **[22]**. Write $\mathrm{Aut}(Z) = \{g\in\mathrm{Aut}(D): g(0) = 0\}$.

LEMMA 1. $\mathrm{Aut}_s(D) = \{g_a\lambda: a\in D\cap\mathrm{CONT}_w(X), \lambda\in\mathrm{Aut}(Z)\}$.

Proof. Every $\lambda\in\mathrm{Aut}(Z)$ is the restriction of a norm continuous linear map and is therefore $\sigma(Z,Z^*)$-continuous. Similarly, $B(a,a)^{1/2}$ is $\sigma(Z,Z^*)$-continuous for all $a\in D$. Let $a\in D\cap\mathrm{CONT}_w(Z)$. The mapping $z\mapsto(I_Z+z\,\square\,a)^{-1}z$ is then s.w.c. as it is a uniform limit

$$\sum_{n=0}^{\infty}(-z\,\square\,a)^n z$$

of s.w.c. mappings. Therefore g_a is s.w.c. and $g_a\lambda$ is s.w.c. for all $\lambda\in\mathrm{Aut}(Z)$. Conversely, let $g\in\mathrm{Aut}_s(D)$. Let $a = g(0)$ and $\lambda = g_a^{-1}g\in\mathrm{Aut}(Z)$. Since $g_a = g\lambda^{-1}$, it follows that g_a is s.w.c. Moreover, $g_a(-a) = 0$ and by **[22**, Lemma 3.3**]**, we have

$$\{z,a,w\} = -g_a'(0)^{-1}g_a''(0)(z,w) \quad \text{for } z,w\in Z.$$

Clearly, the mapping $z\mapsto\{z,a,z\}$ is s.w.c. if, and only if, the mapping $z\mapsto g_a''(0)(z,z)$ is s.w.c. Since g_a is s.w.c. and $g_a''(0)$ is the local uniform limit of s.w.c. mappings, it follows that $z\mapsto\{z,a,z\}$ is s.w.c. and $a\in\mathrm{CONT}_w(Z)$.

REMARKS. (1) Lemma 1 proves that g is s.w.c. if, and only if, $g(0)\in\mathrm{CONT}_w(Z)$. Since $\mathrm{CONT}_w(Z)$ is a characteristic ideal in Z, it follows that g is s.w.c. if, and only if, g^{-1} is s.w.c. and hence that $\mathrm{Aut}_s(D)$ is a subgroup of $\mathrm{Aut}(D)$.

(2) By transitivity of $\mathrm{Aut}(D)$, we have $\mathrm{Aut}(D) = \mathrm{Aut}_s(D)$ if, and only if, $\mathrm{CONT}_w(Z) = Z$. Therefore, **[22**, Proposition 2.7**]** can be restated as follows.

PROPOSITION 2. *Let A be a unital* C*-*algebra with open unit ball D. The following conditions are equivalent:*

(i) *A has the Dunford–Pettis property;*

(ii) *every $g\in\mathrm{Aut}(D)$ is sequentially weakly continuous.*

We shall show in Proposition 8 that the above result is true for JB*-algebras with identity. However, the result is false for JB*-triples without unitary tripotent. Indeed, the C*-algebra $K(H)$ of compact operators on an infinite-dimensional Hilbert space H does not have DPP, but every automorphism of its open unit ball is weakly continuous [**22**, Corollary 3.9].

2. JB-*triples with Dunford–Pettis property*

The Dunford–Pettis property for C*-algebras has been characterized in [**8**, **10**]. See also [**5**, **16**]. We extend these characterizations to JB*-triples using more elaborate arguments involving Grothendieck's inequality and Jacobson's coordinatization theorem for Jordan algebras.

Let W be a JBW*-triple with predual W_*. For $f \in W_*$ with $\|f\| = 1$, there is a tripotent e_f in W such that $f(e_f) = 1$ and furthermore, given $w \in W$ with $f(w) = 1$, we have $f\{z, z, e_f\} = f\{z, z, w\}$ for all $z \in W$ [**2**, Proposition 1.2]. Therefore the pre-Hilbert space norm $\|\cdot\|_f = \sqrt{f\{\cdot, \cdot, e_f\}}$ is well-defined on W. The topology $s(W, W_*)$ generated by these semi-norms $\|\cdot\|_f$, where $f \in W_*$ and $\|f\| = 1$, is called the *strong*-topology* or *s*-topology* on W (cf. [**3**, **27**]).

LEMMA 3. *Let Z be a* JB*-*triple and let $T: Z \to Z^*$ be a bounded conjugate linear operator. Then T is weakly compact.*

Proof. Let Z_c be the JB*-triple obtained from Z by changing only its scalar multiplication to $(\lambda, z) \mapsto \bar{\lambda} z$. Then $T: Z_c \to Z^*$ is a linear map and by [**7**, Lemma 5], T is weakly compact.

Given a JB*-triple C, we shall denote by $L^\infty(\Omega, \mu, C)$ the JB*-triple of all essentially bounded weakly measurable C-valued functions on a finite measure space (Ω, μ) (cf. [**20**]).

LEMMA 4. *Let W be a* JBW*-*triple without summands $L^\infty(\Omega, \mu, C^5)$ and $L^\infty(\Omega', \mu', C^6)$, where C^5 and C^6 are the type 5 and type 6 Cartan factors respectively.*

Let (f_n) be a $\sigma(W_*, W)$-null sequence in W_* and let (w_n) be an $s(W, W_*)$-null sequence in W. Then we have

$$\lim_{n\to\infty} \sup\{|f_k(w_n)| : k = 1, 2, \ldots\} = 0.$$

Proof. By [**7**, Corollary 3], W embeds as a subtriple in a von Neumann algebra M with W_* complemented in M_*. We first show that $w_n^* w_n + w_n w_n^* \to 0$ in the $\sigma(M, M_*)$-topology. Let $f \in M_*$ and let

$$N = \{w \in W : f(w^*w + ww^*) = 0\}.$$

Define an inner product on the quotient W/N by $\langle w + N, z + N \rangle = f(z^*w + wz^*)$. The natural quotient map T from W to the completion of W/N is w^*-w^*-continuous as $f \in M_*$ and by the 'little Grothendieck Theorem' [**2**; **9**, Proposition 4]) there exists $\phi \in W_*$ such that

$$\sqrt{f(w^*w + ww^*)} = \|T(w)\| \leqslant 2^{1/2} \|T\| \sqrt{\phi\{w, w, e_\phi\}},$$

where $e_\phi \in W$ is a tripotent such that $\|\phi\| = \phi(e_\phi) = 1$.

Since $w_n \to 0$ in $s(W, W_*)$, the above inequality implies that $f(w_n^* w_n + w_n w_n^*) \to 0$. Thus we have shown that the sequence $(w_n^* w_n + w_n w_n^*)$ is $\sigma(M, M_*)$-null in M and now we can apply [**29**, Lemma III 5.5] to conclude that

$$\lim_{n\to\infty} \sup\{|f_k(w_n)| : k = 1, 2, \ldots\} = 0.$$

We are now ready to derive the criteria for DPP in JB*-triples. We usually embed Z into Z^{**} and the topology $s(Z^{**}, Z^*)$ restricted to Z will be denoted by $s(Z, Z^*)$.

THEOREM 5. *Let Z be a* JB*-*triple. The following conditions are equivalent:*
(i) *Z has the Dunford–Pettis property;*
(ii) *for any weakly null sequence (z_n) in Z, the sequence $(\{z_n, z_n, z\})$ tends to 0 weakly for all $z \in Z^{**}$;*
(iii) *every weakly null sequence in Z is $s(Z, Z^*)$-null.*

Proof. (i) $\Rightarrow$ (ii). Let $z_n \to 0$ weakly and let $z \in Z^{**}$. Let $f \in Z^*$. Define $T: Z \to Z^*$ by $T(w)(y) = f\{y, w, z\}$ for $w, y \in Z$. Then T is conjugate linear and hence is weakly compact by Lemma 3. By DPP of Z, we have $\|T(z_n)\| \to 0$ which gives $f\{z_n, z_n, z\} \to 0$, as (z_n) is norm bounded.

(ii) $\Rightarrow$ (iii). Let $z_n \to 0$ weakly. For any $f \in Z^*$ with $\|f\| = 1 = f(e_f)$ and $e_f \in Z^{**}$, we have $\|z_n\|_f = \sqrt{f\{z_n, z_n, e_f\}} \to 0$ by (ii). So $z_n \to 0$ in the $s(Z, Z^*)$-topology.

(iii) $\Rightarrow$ (i). Let $z_n \to 0$ in $\sigma(Z, Z^*)$ and let $f_n \to 0$ in $\sigma(Z^*, Z^{**})$. We show that $f_n(z_n) \to 0$ which will yield the DPP for Z. We have

$$Z^{**} = l_\infty\text{-sum}(W \oplus L_5 \oplus L_6), \qquad (\dagger)$$

where $L_j = L^\infty(\Omega_j, \mu_j, C^j)$ or 0 (for $j = 5, 6$), and W is a JBW*-triple without summand $L^\infty(\Omega_j, \mu_j, C^j)$ for $j = 5$ or 6. Also Z^* is the l_1-sum of the corresponding preduals. By decomposing (z_n) and (f_n) according to (†) and by noting the preduals $L^1(\Omega_j, \mu_j, C^j_*)$ (for $j = 5, 6$) have DPP, we can reduce our arguments to the case $f_n \in W_*$ and $z_n \in W$. By (iii), $z_n \to 0$ in $s(W, W^*)$ and by Lemma 4, we have $f_n(z_n) \to 0$ as $n \to \infty$, completing the proof.

REMARK. Let Z be a JB*-triple and let $C(X, Z)$ be the JB*-triple of Z-valued continuous functions on a compact Hausdorff space X. A simple application of Theorem 5 shows that $C(X, Z)$ has DPP if, and only if, Z has DPP. The structure of compact type symmetric manifolds associated to $C(X, Z)$ has been studied in [**25**, **26**]. It may be of interest to investigate how DPP is related to the geometric structure of these manifolds.

COROLLARY 6. *Let Z be a* JB*-*triple with* DPP *and let Z_1 be a subtriple of Z. Then Z_1 has* DPP.

COROLLARY 7. *Let Z be a* JB*-*triple with* DPP *and open unit ball D. Then every $g \in \mathrm{Aut}(D)$ is sequentially weakly continuous.*

Proof. By the remark following Lemma 1, it is equivalent to show that $\mathrm{CONT}_w(Z) = Z$. Fix $z \in Z$ and let $z_n \to 0$ weakly. As in the proof of Theorem 5, given $f \in Z^*$, the weak compactness of the linear map $T: Z \to Z^*$ defined by $T(w)(y) = f\{w, z, y\}$ yields $f\{z_n, z, z_n\} \to 0$ as $n \to \infty$. So $z \in \mathrm{CONT}_w(Z)$.

We have the following converse.

Proposition 8. *Let A be a* JB*-*algebra with identity e and open unit ball D. The following conditions are equivalent:*

(i) *A has the Dunford–Pettis property;*

(ii) *if (z_n) is a weakly null sequence in A then the sequence $(\{z_n, z_n, e\}) = (z_n^* \circ z_n)$ is weakly null;*

(iii) *every $g \in \operatorname{Aut}(D)$ is sequentially weakly continuous.*

Proof. (ii) $\Rightarrow$ (iii). We show that $\mathrm{CONT}_w(A) = A$. It suffices to show that $e \in \mathrm{CONT}_w(A)$, since $\mathrm{CONT}_w(A)$ is an ideal in A by [**22**, Proposition 2.6]. Let $z_n \to 0$ weakly and write $z_n = u_n + iv_n$ with u_n and v_n self-adjoint. Then (u_n) and (v_n) are weakly null and (ii) implies that $(\{u_n, u_n, e\})$, $(\{v_n, v_n, e\})$ and $(\{u_n+v_n, u_n+v_n, e\})$ are all weakly null which gives $\{z_n, e, z_n\} \to 0$ weakly; that is, $e \in \mathrm{CONT}_w(A)$.

(iii) $\Rightarrow$ (i). Let $I = \{z \in A$: the map $a \mapsto \{a, a, z\}$ is s.w.c.$\}$. Using the identity

$$\{\{z_n, z_n, a\}, z, z\} - \{a, \{z_n, z_n, z\}, z\} = \{z_n, \{z, a, z_n\}, z\} - \{\{a, z, z_n\}, z_n, z\}$$

and the main identity, together with polarization, one can show that $\{a, z, z\} \in I$ whenever $z \in I$ and $a \in A$. Hence I is an ideal in A by [**6**, Proposition 1.3]. By Theorem 5, DPP will follow from $I = A$. We complete the proof by showing that $e \in I$. Indeed, let $z_n \to 0$ weakly with $z_n = u_n + iv_n$ where u_n and v_n are self-adjoint. Then (u_n) and (v_n) are weakly null. Hence $(\{u_n, e, u_n\})$ and $(\{v_n, e, v_n\})$ are weakly null, since $e \in \mathrm{CONT}_w(A)$. It follows that $\{z_n, z_n, e\} = u_n^2 + v_n^2 \to 0$ weakly. This shows that $e \in I$.

3. JBW*-*triples with Dunford–Pettis property*

We now characterize JBW*-triples having the Dunford–Pettis property. Given a tripotent e in a JB*-triple Z, there corresponds to a *Peirce decomposition*:

$$Z = Z_1(e) \oplus Z_{\frac{1}{2}}(e) \oplus Z_0(e),$$

where $Z_k(e) = \{z \in Z : \{e, e, z\} = kz\}$ is the k-eigenspace of $e \square e$. The tripotent e is *minimal* if the Peirce 1-space $Z_1(e)$ is one-dimensional, in which case the Peirce $\frac{1}{2}$-space $Z_{\frac{1}{2}}(e)$ has rank at most 2. Every Cartan factor has a minimal tripotent. The following crisp proof suggested by the referee is shorter than our original arguments.

Lemma 9. *Let C be a Cartan factor whose predual has the Dunford–Pettis property. Then $\dim C < \infty$.*

Proof. If C is infinite-dimensional and $e \in C$ is a minimal tripotent, then the complemented Peirce $\frac{1}{2}$-space $C_{\frac{1}{2}}(e)$ does not have the DPP as it is reflexive and infinite-dimensional.

By [**20, 21**], every JBW*-triple W can be decomposed into the following l_∞-sum:

$$W = \bigoplus_\alpha L^\infty(\Omega_\alpha, \mu_\alpha, C_\alpha) \oplus R \oplus H(M, \beta),$$

where C_α is a Cartan factor, R is a w^*-closed right ideal of a continuous von Neumann algebra N, and $\beta: M \to M$ is a linear period 2 $*$-antiautomorphism of a continuous

von Neumann algebra M with $H(M, \beta) = \{a \in M : \beta(a) = a\}$. Furthermore, the self-adjoint part $A = H(M, \beta)_{sa} = \{a \in H(M, \beta) : a^* = a\}$ is a continuous (real) JW-algebra under the Jordan product $a \circ b = \frac{1}{2}(ab + ba)$. Recall that a *continuous* JW-algebra is one without type I summand. We need some structure theory of continuous JW-algebras A. The first is the following 'Halving Lemma'. A *symmetry* is an element $s \in A$ such that $s^2 = 1$ where 1 is the identity of A.

Lemma 10. *Let A be a continuous* JW-*algebra and let $p \in A$ be a projection (that is, $p^2 = p$). Then there are orthogonal projections $q, r \in A$ such that $p = q + r$, and q and r are exchanged by a symmetry s which means that $r = \{s, q, s\} = sqs$.*

Proof. See [**30**, Theorem 17] or [**17**, 5.2.14].

Lemma 11. *Let A be a continuous* JW-*algebra and let $p \in A$ be a projection. Then for each n, one can write p as the sum of 2^n orthogonal projections, any two of which are exchanged by a symmetry.*

Proof. We use induction. Suppose that we have $p = p_1 + p_2 + \cdots + p_{2^n}$, where $p_j = s_j p_1 s_j$ with s_j a symmetry. Write $p_1 = q_1 + q_2$ where q_1 and q_2 are orthogonal projections exchanged by a symmetry. Let $q_{2j-1} = s_j q_1 s_j$ and $q_{2j} = s_j q_2 s_j$. Then q_{2j-1} and q_{2j} are orthogonal with $q_{2j-1} + q_{2j} = s_j(q_1 + q_2) s_j = p_j$. So $p = q_1 + q_2 + \cdots + q_{2^{n+1}}$ and each q_k is exchanged by a symmetry with either q_1 or q_2. By [**30**, Theorem 7 and Proposition 10], every pair of the q_k are exchanged by a symmetry.

Given a $*$-algebra B (as defined in [**17**, 2.1.1]), we let $H_n(B)$ denote the algebra of $n \times n$ Hermitian matrices with entries from B, where a matrix (b_{ij}) is *Hermitian* if $(b_{ij}) = (b_{ji}^*)$, and the algebra product is the usual Jordan product of matrices. If p, q are orthogonal projections in a JW-algebra A exchanged by a symmetry, then they are *strongly connected*, that is, there exists $a \in \{p, A, q\}$ such that $a^2 = p + q$ [**1**, 6.6]. Let $p \in A$ be a projection which is the sum of 2^n orthogonal projections exchanged by symmetries. Then by Jacobson's coordinatization theorem [**17**, 2.8.9], $\{p, A, p\} = pAp$ is Jordan isomorphic to the JB-algebra $H_{2^n}(B)$ of $2^n \times 2^n$ Hermitian matrices over a unital $*$-algebra B for $n \geqslant 2$.

Proposition 12. *Let A be a continuous* JW-*algebra. Then the l_∞-sum $\bigoplus_{n \geqslant 2} H_{2^n}(\mathbb{R})$ embeds as a Jordan subalgebra of A.*

Proof. Write the orthogonal sum $1 = p_1 + p_2 + \cdots + p_n + \cdots$, where each p_n is the sum of 2^n orthogonal projections exchanged by symmetries. By the above remark, we have the following natural embeddings of l_∞-sums:

$$\bigoplus_{n \geqslant 2} H_{2^n}(\mathbb{R}) \subset \bigoplus_{n \geqslant 2} H_{2^n}(B_n) \simeq \bigoplus_{n \geqslant 2} \{p_n, A, p_n\} \subset A.$$

Corollary 13. *Let A be a continuous* JW-*algebra. Then A does not have the Dunford–Pettis property.*

Proof. Suppose otherwise. Then the JB*-algebra $A + iA$ has DPP and, by Corollary 6 and Proposition 12, $\bigoplus_n H_{2^n}(\mathbb{R})$ would have DPP. This is impossible since $\bigoplus_n H_{2^n}(\mathbb{R})$ contains as a complemented subspace the l_∞-sum $\bigoplus_n l_2^{2^n-1}$ of the Hilbert spaces $l_2^{2^n-1}$ and $\bigoplus_n l_2^{2^n-1}$ does not have DPP [**11**, p. 22].

THEOREM 14. *Let W be a JBW*-triple. The following conditions are equivalent:*
(i) *W has the Dunford–Pettis property;*
(ii) *$W = l_\infty$-sum $\bigoplus_\alpha L^\infty(\Omega_\alpha, \mu_\alpha, C_\alpha)$ with $\sup_\alpha \dim C_\alpha < \infty$, where each C_α is a Cartan factor.*

Proof. (ii) $\Rightarrow$ (i). Since $n_\alpha := \dim C_\alpha < \infty$, each $L^\infty(\Omega_\alpha, \mu_\alpha, C_\alpha)$ is linearly isomorphic to the l_∞-sum of n_α copies of $L^\infty(\Omega_\alpha, \mu_\alpha, \mathbb{C})$ and hence has DPP. Since $\sup_\alpha \dim C_\alpha < \infty$, it follows that

$$\bigoplus_\alpha L^\infty(\Omega_\alpha, \mu_\alpha, C_\alpha) = \sum_j \big(\sum_\alpha L^\infty(\Omega_\alpha, \mu_\alpha, C_\alpha^j)\big)$$

has DPP, where C_α^j is a Cartan factor of type j.

(i) $\Rightarrow$ (ii). As remarked after Lemma 9, we can write

$$W = \bigoplus_\alpha L^\infty(\Omega_\alpha, \mu_\alpha, C_\alpha) \oplus R \oplus H(M, \beta).$$

By Corollary 13, $H(M,\beta) = 0$. If $R \neq 0$, then $R = pN$ for some continuous von Neumann algebra N and non-zero projection $p \in N$. By Corollary 6, the von Neumann algebra pNp has DPP since R has DPP. By [**8**, Theorem 3], pNp is type I. But pNp is continuous (see for example [**31**, Corollary 11]) which is a contradiction. So $R = 0$.

It remains to show that $\sup_\alpha \dim C_\alpha < \infty$. By Lemma 9, we have $\dim C_\alpha < \infty$ for all α. We need to consider only the supremum over the first 4 types of Cartan factors.

We first show that every such C_α contains a complemented subspace linearly isomorphic to a Hilbert space $l_2^{n_\alpha}$ where $(n_\alpha + 1)^2 \geqslant \dim C_\alpha$ and the isomorphism ϕ has bound $\|\phi\|\,\|\phi^{-1}\| \leqslant 2$.

If C_α is type 1, say the factor M_{mn} of all complex $m \times n$ matrices, then the Hilbert spaces l_2^m and l_2^n embed as complemented subspaces of M_{mn} by a norm-1 projection. If C_α is of type 2 consisting of symmetric complex $n \times n$ matrices, then $\dim C_\alpha = \frac{1}{2}n(n+1)$ and l_2^{n-1} embeds in C_α as a complemented subspace by the projection $p: C_\alpha \to C_\alpha$, given by $p(a_{ij}) = (b_{ij})$, where

$$b_{ij} = \begin{cases} a_{ij} & \text{if } \min(i,j) = 1 \text{ and } i \neq j, \\ 0 & \text{otherwise,} \end{cases}$$

and $\|p\| \leqslant 2$. Likewise l_2^{n-1} complements in the type 3 Cartan factor of $n \times n$ skew-symmetric matrices which has dimension $\frac{1}{2}n(n-1)$.

If C_α is type 4 of dimension n, then it admits an inner product $\langle\,,\rangle$ such that $\frac{1}{2}\|a\|^2 \leqslant \langle a, a\rangle \leqslant \|a\|^2$ for all $a \in C_\alpha$. It follows that there is a linear isomorphism $\phi: C_\alpha \to l_2^n$ with $\|\phi\|\,\|\phi^{-1}\| \leqslant \sqrt{2}$.

Now the l_∞-sum $\bigoplus_\alpha C_\alpha$ is a subtriple of $\bigoplus_\alpha L^\infty(\Omega_\alpha, \mu_\alpha, C_\alpha)$ and therefore has DPP.

If $\sup_\alpha \dim C_\alpha = \infty$, then by the above remarks the l_∞-sum $\bigoplus_\alpha l_2^{n_\alpha}$ is linearly isomorphic to a complemented subspace in $\bigoplus_\alpha C_\alpha$. Hence $\bigoplus_\alpha l_2^{n_\alpha}$ has DPP, which is impossible since $n_\alpha \uparrow \infty$ [**11**, p. 22]. So $\sup_\alpha \dim C_\alpha < \infty$.

REMARK. It is interesting to compare Theorem 14 with a recent result in [**19**] that, given a type I JBW*-triple W, then every (bounded) derivation on W is inner if, and only if, $W = \bigoplus_\alpha L^\infty(\Omega_\alpha, \mu_\alpha, C_\alpha)$ with $\sup_{\alpha \in K} \dim C_\alpha < \infty$, where $K = \{\alpha : C_\alpha$ is type 1 non-square or type 4$\}$.

We now consider the predual W_* of a JBW*-triple W. The *w*-topology* on W refers to the topology $\sigma(W, W_*)$.

LEMMA 15. *Let W be a JBW*-triple. The following conditions are equivalent:*
(i) *W_* has the Dunford–Pettis property;*
(ii) *Given a $\sigma(W, W^*)$-null sequence (w_n) in W, the sequence $(\{w_n, w_n, w\})$ is w*-null for all $w \in W$.*

Proof. (i) $\Rightarrow$ (ii). Let (w_n) be a weakly null sequence in W. Fix $w \in W$ and $f \in W_*$. Define the linear functional g_n on W by $g_n(\cdot) = f\{\cdot, w_n, w\}$. Then (g_n) is $\sigma(W_*, W)$-null in W_*, since the triple product $\{\cdot,\cdot,\cdot\}$ is separately w^*-continuous [**2**]. By DPP, we have $f(\{w_n, w_n, w\}) = g_n(w_n) \to 0$ as $n \to \infty$.

(ii) $\Rightarrow$ (i). Let (f_n) be a $\sigma(W_*, W)$-null sequence in W_* and let (w_n) be a $\sigma(W, W^*)$-null sequence in W. By (ii), (w_n) is $s(W, W_*)$-null. Separating out the possible summands $L^\infty(\Omega, \mu, C^5)$ and $L^\infty(\Omega', \mu', C^6)$ of W as in the proof of Theorem 5, we get $f_n(w_n) \to 0$ as $n \to \infty$ by Lemma 4. So W_* has DPP.

COROLLARY 16. *Let W be a JBW*-triple whose predual W_* has the Dunford–Pettis property. Let U be any JBW*-subtriple of W. Then U_* has the Dunford–Pettis property.*

Bunce [**5**] has shown that the predual of a type II_1 von Neumann algebra M does not have DPP, by noting that M contains an infinite spin system. For JB*-triples, we shall use the fact that every continuous JW-algebra contains an infinite spin system which is shown below.

Given $a \in H_n(\mathbb{R})$ and $b \in H_m(\mathbb{R})$, we define, as in [**17**, p. 139], elements $a \otimes 1_m$, $1_n \otimes b$ and $a \otimes b$ in $H_{nm}(\mathbb{R})$ by the following:

$$a \otimes 1_m = \begin{pmatrix} a & & 0 \\ & \ddots & \\ 0 & & a \end{pmatrix},$$

$$1_n \otimes b = (b_{ij} 1_n) = \begin{pmatrix} b_{11} & & 0 & \cdots & b_{1m} & & 0 \\ & \ddots & & & & \ddots & \\ 0 & & b_{11} & \cdots & 0 & & b_{1m} \\ \vdots & & \vdots & & \vdots & & \vdots \\ b_{m1} & & 0 & \cdots & b_{mm} & & 0 \\ & \ddots & & & & \ddots & \\ 0 & & b_{m1} & \cdots & 0 & & b_{mm} \end{pmatrix},$$

$$a \otimes b = (a \otimes 1_m)(1_n \otimes b).$$

The inclusion $H_n(\mathbb{R}) \subset H_{4n}(\mathbb{R})$ means that $H_n(\mathbb{R})$ is identified as a JB-subalgebra of $H_{4n}(\mathbb{R})$ via the embedding $a \in H_n(\mathbb{R}) \mapsto a \otimes 1_4 \in H_{4n}(\mathbb{R})$. We also identify $H_n(H_4(\mathbb{R}))$ with $H_{4n}(\mathbb{R})$ by deleting additional parentheses.

LEMMA 17. *Let A be a continuous JW-algebra containing $H_n(\mathbb{R})$ as a Jordan subalgebra, where $n > 2$. Then we have $A \supset H_{4n}(\mathbb{R}) \supset H_n(\mathbb{R})$.*

Proof. We may assume that the identity e in $H_n(\mathbb{R})$ coincides with the identity 1 of A, for otherwise, we can consider $H_n(\mathbb{R}) \subset eAe$. There are mutually orthogonal projections $p_1, \ldots, p_n \in H_n(\mathbb{R})$, any two of which are exchanged by a symmetry, such that $1 = p_1 + \cdots + p_n$.

By coordinatization [17, 2.8.9], A is Jordan isomorphic to $H_n(R)$, for some unital $*$-algebra R, with matrix unit (e_{ij}), such that the Peirce component $\{xe_{11} : x \in R_{sa}\}$ of the self-adjoint part $R_{sa} = \{x \in R : x^* = x\}$ is Jordan isomorphic to $p_1 A p_1$ [17, 2.8.17 and 2.8.19]. Since $p_1 A p_1$ does not contain a type I summand, using Lemma 11 and coordinatization again, we have $p_1 A p_1 = H_4(R') \supset H_4(\mathbb{R})$ for some unital $*$-algebra R'. Therefore we have

$$A = H_n(R) \supset H_n(H_4(\mathbb{R})) = H_{4n}(\mathbb{R}) \supset H_n(\mathbb{R}).$$

Let $(A, \circ)$ be a unital real Jordan algebra. A *spin system* in A is a set P of at least two symmetries not equal to ± 1 such that $s \circ t = 0$ for $s \neq t$ in P.

PROPOSITION 18. *Every continuous* JW-*algebra A contains an infinite spin system.*

Proof. By Lemma 17, A contains the following JW-factors

$$H_4(\mathbb{R}) \subset H_{16}(\mathbb{R}) \subset \cdots \subset H_{4^n}(\mathbb{R}) \subset \cdots \subset A.$$

We construct an infinite spin system in A as follows.

Let $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\sigma_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in H_2(\mathbb{R})$. Define

$$s_1 = \sigma_2 \otimes \sigma_1,\ s_2 = \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_1, \ldots,\ s_n = \underbrace{\sigma_2 \otimes \cdots \otimes \sigma_2}_{(2n-1)\text{-times}} \otimes \sigma_1.$$

Then $P = \{s_n : n = 1, 2, \ldots\}$ is an infinite spin system in A.

COROLLARY 19. *Let A be a continuous* JW-*algebra. Then its predual A_* does not have the Dunford–Pettis property.*

Proof. Let P be an infinite spin system in A. Then P generates an infinite-dimensional (real) spin factor V in A. Since V_* does not have DPP, Corollary 16 implies that A_* does not have DPP.

We conclude with the following result.

THEOREM 20. *Let W be a* JBW*-*triple. The following conditions are equivalent:*

(i) *W_* has the Dunford–Pettis property;*

(ii) *$W = l_\infty$-sum $\bigoplus_\alpha L^\infty(\Omega_\alpha, \mu_\alpha, C_\alpha)$ where each C_α is a Cartan factor and* $\dim C_\alpha < \infty$.

Proof. (i) $\Rightarrow$ (ii). Write $W = \bigoplus_\alpha L^\infty(\Omega_\alpha, \mu_\alpha, C_\alpha) \oplus R \oplus H(M, \beta)$ as before, where $R = pN$ for some continuous von Neumann algebra N with projection p, and $H(M, \beta)$ is a continuous JW*-algebra. If $R \neq 0$, then $pNp \neq 0$, and R_* has DPP which implies that $(pNp)_*$ also has DPP by Corollary 16. By [5] and [8, Proposition 6], pNp is a type I finite von Neumann algebra, contradicting the fact that N is continuous. So $R = 0$. The self-adjoint part $H(M, \beta)_{sa}$ is a continuous JW-algebra whose predual does

not have DPP by Corollary 19. Hence $H(M, \beta) = 0$. Finally, C_α is identified as a JBW*-subtriple of $L^\infty(\Omega_\alpha, \mu_\alpha, C_\alpha)$ and the DPP of the predual $L^\infty(\Omega_\alpha, \mu_\alpha, C_\alpha)_*$ is passed onto the predual of C_α, by Corollary 16. So $\dim C_\alpha < \infty$ by Lemma 9.

(ii) $\Rightarrow$ (i). Since $\dim C_\alpha < \infty$, the predual $L^\infty(\Omega_\alpha, \mu_\alpha, C_\alpha)_* = L^1(\Omega_\alpha, \mu_\alpha, (C_\alpha)_*)$ has DPP. Hence $W_* = l_1 - \text{sum} \sum_\alpha L^1(\Omega_\alpha, \mu_\alpha, (C_\alpha)_*)$ has DPP, by arguments similar to those in [9, p. 62].

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Note added in proof. In condition (ii) of Theorem 5, Z^{**} can be replaced by Z.

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COORDINATES FOR THE REGULAR COMPLEX POLYGONS

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ABSTRACT

Certain projections of the real polytopes {3, 3, 4}, {3, 4, 3}, {3, 3, 5} suggest highly symmetric coordinates for the self-reciprocal complex polygons 3{3}3, 4{3}4, 3{4}3, 5{3}5 and 3{5}3. Although there are a number of interesting complications, this suggestion is essentially correct and leads to elegant coordinates for all the sporadic complex polygons. Among the by-products of producing these coordinates we count most significant our new insights about 2{6}3 and our simple proof that the 600 vertices of the real polytope {5, 3, 3} are quite unrelated to the 600 vertices of either 5{6}2 or 5{4}3.

1. *Introduction*

The unitary transformation

$$(u, v) \longrightarrow (u, v)\begin{pmatrix} e^{2\pi i/p} & 0 \\ 0 & 1 \end{pmatrix}$$

of period $p \geqslant 2$ which fixes the points of the line $u = 0$, affords a natural generalization of the real reflection

$$(u, v) \longrightarrow (u, v)\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

which has this line for its mirror. By allowing complex reflections of period $p > 2$ we can extend the notion of a regular real polygon to that of a regular complex polygon. The specimen denoted $p_1\{q\}p_2$ has p_1 collinear vertices on each edge and p_2 edges through each vertex. Its symmetry group is generated by complex reflections R_1 and R_2 where R_1 fixes the centroid of an edge and cycles the vertices on it, while R_2 fixes a vertex of this edge and cycles the edges through it.

The middle number q in the symbol $p_1\{q\}p_2$ appears in the defining relations

$$R_1^{p_1} = R_2^{p_2} = 1, \quad R_1 R_2 R_1 \cdots = R_2 R_1 R_2 \cdots \quad (q \text{ factors on each side})$$

for the symmetry group $p_1[q]p_2 = p_2[q]p_1$ of our polygon. (If q is odd, it follows that R_1 and R_2 are conjugate, $p_2 = p_1 = p$, and the relation $R_2^p = 1$ is superfluous.) The number q also has a direct geometrical interpretation explained in [**9**]. It is the length of a minimal cycle of vertices in which successive members belong to an edge and successive edges so determined, are distinct. If the polygon is real, $p_1 = p_2 = 2$ and q is equal to the total number of vertices.

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