

Prime JB*-Triples and Extreme Dual Ball Density

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Abstract. Properties of the extreme points $\partial_e(E_1^*)$ of the closed dual ball E_1^* of a JB*-triple E are studied. It is shown that the canonical mapping from $\partial_e(E_1^*)$ onto the structure space, $\text{Prim}(E)$, of primitive M-ideals of E is an open mapping. This property is utilised to show that $\partial_e(E_1^*)$ is weak* dense in E_1^* if and only if E is an infinite dimensional Hilbert space, an infinite dimensional spin factor or E is prime with zero socle.

1. Introduction

A Banach space E is prime (in the sense of M-ideals) if the set $\partial_e(E_1^*)$ of extreme points of the closed dual ball E_1^* is weak* dense in the dual sphere $S(E_1^*)$. A converse obtains under favourable conditions on the canonical structure map $\psi : \partial_e(E_1^*) \rightarrow \text{Prim}(E)$, where $\text{Prim}(E)$ is the structure space of primitive M-ideals (see Section 2). The converse is far from generally true, however, and the question arises which prime Banach spaces satisfy the extreme dual density condition.

In this paper we seek an answer in the case of the large class of complex Banach spaces whose open unit ball is a bounded symmetric domain. Originating in the study of complex Banach manifolds [23, 33] these complex Banach spaces, known as JB*-triples, which have received considerable recent attention, occur naturally in operator algebras and as images of contractive projections on them [17, 24, 32]. Connections to holomorphy, convexity, quantum physics and Jordan structures is extensively exposed in [11, 30, 31, 33].

Studies of prime JB*-triples are contained in [5, 7, 10, 15, 26]. Related investigations of structure spaces can be found in [8, 9]. We remark prime JB*-triples with non-zero socle are identified as the generalised Cartan factors in [5] and that [26] derives a deep Zel'manovian classification of prime JB*-triples. The present note is inspired by the pervasively influential paper [19].

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We review prime structure in Banach spaces in Section 2. For every JB*-triple E we prove in Section 3 that the structure map $\psi : \partial_e(E_1^*) \rightarrow \text{Prim}(E)$ is an open mapping, via dual space atomic theory [16] and the use of norming subsets. The roles of Hilbert spaces and spin factors are discussed in Section 4. For a JB*-triple E it turns out, modulo the minor exceptions of finite dimensional Hilbert spaces, that $\partial_e(E_1^*)$ is weak* dense in the unit ball if it is weak* dense in the unit sphere. Our main classification result, given in Section 5, is that for a JB*-triple E , $\partial_e(E_1^*)$ is weak* dense in E_1^* if and only if E is an infinite dimensional Hilbert space, an infinite dimensional spin factor or is prime with zero socle.

2. Prime structure in Banach spaces

For a (real or complex) Banach space E let $\partial_e(E_1^*)$ denote the set of extreme points of the norm closed dual ball E_1^* . If X is contained in E_1^* , $\text{co}(X)$ denotes the convex hull of X and \overline{X} denotes the weak* closure of X . By the Krein–Milman theorem \overline{X} contains $\partial_e(E_1^*)$ if and only if $E_1^* = \overline{\text{co}(X)}$. M-structure [1, 4, 21] arises naturally in the consideration of stronger density conditions, as explained below. We refer to the comprehensive and lucid exposition [21] for any unexplained terms involving M-structure.

Since E^* is the l_1 -sum of $P(E^*)$ and $\ker P$ for any L-projection P of E^* , we have $\partial_e(E_1^*)$ is contained in $P(E^*) \cup \ker P$. Thus, if $\partial_e(E_1^*)$ is weak* dense in $S(E_1^*)$ and P is weak* continuous then $E^* = P(E^*) \cup \ker P$ so that $P = 0$ or $P = I$.

Similarly, if $\partial_e(E_1^*)$ is norm dense in $S(E_1^*)$ then all L-projections of E^* are trivial which via [21, 1.9] implies that E^{**} has no non-trivial M-projections.

For an M-ideal J of E we have E^* is the l_1 -sum of

$$J^0 = \{\rho \in E^* : \rho|_J = 0\} \simeq (E/J)^*$$

and

$$J^\# = \{\rho \in E^* : \|\rho|_J\| = \|\rho\|\} \simeq J^*$$

(where \simeq indicates the canonical isometries). We have $\partial_e(E_1^*) = \partial_e(J_1^0) \cup \partial_e(J_1^\#)$ and the restriction map is an isometry from $\partial_e(E_1^*) \setminus J^0$ onto $\partial_e(J_1^*)$. When convenient we shall identify J^0 with $(E/J)^*$.

We say that E is *prime* if whenever I and J are M-ideals with $I \cap J = 0$, then $I = 0$ or $J = 0$.

Proposition 2.1. *Let E be a Banach space such that $\partial_e(E_1^*)$ is weak* dense in $S(E_1^*)$. Then E is prime.*

Proof. Let I and J be M-ideals of E with $I \cap J = 0$. Since I and J are M-summands of the M-ideal $K = I + J$ [21, 1.17], K^* has a non-trivial weak* continuous L-projection if I and J are non-zero. In which case by the above remarks there exists ρ in $S(K_1^*)$ outside $\overline{\partial_e(K_1^*)}$ so that no extension of ρ in $S(E_1^*)$ can be in

$\overline{\partial_e(E_1^*)}$. Hence, $I = 0$ or $J = 0$. □

The structure space of a Banach space E is the set

$$\text{Prim}(E) = \{\psi(\rho) : \rho \in \partial_e(E_1^*)\}$$

of primitive M-ideals of E where for each ρ in $\partial_e(E_1^*)$, $\psi(\rho)$ is the largest M-ideal in $\ker \rho$. For $S \subset \text{Prim}(E)$ and $X \subset E$, $k(S)$ is the largest M-ideal of E contained in the intersection of the members of S , and $h(X)$ is the set of primitive M-ideals containing X . $\text{Prim}(E)$ is to be regarded as a topological space with the structure topology — the unique topology for which $h(k(S))$ is the closure of $S \subset \text{Prim}(E)$.

Proposition 2.2. ([1, 3.3], [4, 3.17].) *If E is a Banach space the map*

$$\psi : \partial_e(E_1^*) \longrightarrow \text{Prim}(E) \quad (\rho \mapsto \psi(\rho))$$

is continuous (with respect to the weak topology on $\partial_e(E_1^*)$).*

Examples of prime Banach spaces for which $\partial_e(E_1^*)$ is weak* closed and not equal to $S(E_1^*)$ include spaces of rectangular matrices $M_{m,n}(\mathbb{C})$, where $m, n \geq 2$, and the disc algebra. We further remark that the structure map, ψ , of 2.2 need not be open, as the example of the disc algebra again shows.

We shall need the following partial converse of 2.1 in Section 5.

Theorem 2.3. *Let E be a prime Banach space such that*

- (a) *the structure map $\psi : \partial_e(E_1^*) \rightarrow \text{Prim}(E)$ is open;*
- (b) *$\partial_e(E_1^*)$ is weak* dense in $S(E_1^*) \cap \psi(\rho)^0$ for each ρ in $\partial_e(E_1^*)$.*

Then $\partial_e(E_1^)$ is weak* dense in $S(E_1^*)$.*

Proof. By the Bishop–Phelps theorem it is enough to show that $\partial_e(E_1^*)$ is weak* dense in the norm attaining elements of $S(E_1^*)$. Take ϕ in $S(E_1^*)$ and x in E such that $\phi(x) = \|x\| = 1$. Let $0 < \epsilon < 1$ and consider the basic (relatively) open subset of $S(E_1^*)$

$$U = \{\phi' \in S(E_1^*) : |\phi'(a_i) - \phi(a_i)| < \epsilon, \quad i = 1, \dots, n\}$$

where the a_i are non-zero elements of E . We shall show that U has non-empty intersection with $\partial_e(E_1^*)$. There is no loss in assuming that

$$\|a_i\| \leq 1, \quad \text{for } i = 1, \dots, n,$$

since if $\alpha = \max_1^n \|a_i\| > 1$ we may replace the a_i and ϵ with $\alpha^{-1}a_i$ and $\alpha^{-1}\epsilon$ without affecting U . Further, since we can cut down U by intersecting with

$$\{\phi' \in S(E_1^*) : |\phi'(x) - \phi(x)| < \epsilon\},$$

no harm can come from letting $a_1 = x$ in the definition of U .

Let $0 < 6\delta < \epsilon$. Via the Krein–Milman theorem choose $\sigma = \lambda_1\rho_1 + \dots + \lambda_m\rho_m$ where the $\rho_i \in \partial_e(E_1^*)$, $\lambda_i \geq 0$ with $\lambda_1 + \dots + \lambda_m = 1$ such that

$$(2.1) \quad |\sigma(a_i) - \phi(a_i)| < \delta \quad \text{for } i = 1, \dots, n.$$

In particular, we have

$$|\sigma(x)| > \phi(x) - \delta = 1 - \delta.$$

For each $j = 1, \dots, m$ form the open neighbourhood of ρ_j in $\partial_e(E_1^*)$

$$V_j = \{\tau \in \partial_e(E_1^*) : |\tau(a_i) - \rho_j(a_i)| < \delta, i = 1, \dots, n\}.$$

Since E is prime the condition (a) implies that we can choose $\tau = \tau_1$ in V_1 and τ_j in V_j such that $\psi(\tau) = \psi(\tau_j)$ for $j = 2, \dots, m$. Thus with $f = \lambda_1\tau_1 + \dots + \lambda_m\tau_m$ we have $f \in \psi(\tau)^0$ and

$$(2.2) \quad |f(a_i) - \sigma(a_i)| < \delta \quad \text{for } i = 1, \dots, n.$$

In particular,

$$\|f\| \geq |f(x)| > |\sigma(x)| - \delta > 1 - 2\delta.$$

Therefore, $g = \|f\|^{-1}f \in S(E_1^*) \cap \psi(\tau)^0 \subset \overline{\partial_e(E_1^*)}$ by (b) and for each $i = 1, \dots, n$

$$(2.3) \quad |g(a_i) - f(a_i)| \leq \|f\|^{-1}(1 - \|f\|) < 2\delta(1 + 3\delta) < 3\delta.$$

The inequalities (2.1), (2.2) and (2.3) give

$$|g(a_i) - \phi(a_i)| < 5\delta < \epsilon \quad \text{for } i = 1, \dots, n,$$

so that g belongs to $U \cap \overline{\partial_e(E_1^*)}$, whence the result. \square

Remark 2.4. The unit ball E_1 of an infinite dimensional Banach space is the weak closure of its unit sphere $S(E_1)$. Thus $\partial_e(E_1^*)$ is weak* dense in E_1^* if $\partial_e(E_1^*)$ is weak* dense in $S(E_1^*)$ and E is infinite dimensional.

A subset X of E_1^* , where E is a Banach space, is said to be *homogeneous* if λX is contained in X for all scalars λ of modulus one. The subset X is said to *norm* E if, for each x in E , $\|x\| = \sup\{|\rho(x)| : \rho \in X\}$. By the Krein–Milman theorem $\partial_e(E_1^*)$ norms E . More generally, since $\text{co}(X)$ is balanced if X is homogeneous we have the following.

Lemma 2.5. *If X is a homogeneous subset of E_1^* , where E is a Banach space, then X norms E if and only if $\partial_e(E_1^*) \subset \overline{X}$.*

3. JB*-Triples

A JB*-triple E is a complex Banach space with a continuous ternary product $(a, b, c) \mapsto \{abc\}$ conjugate linear in b and symmetric bilinear in a and c satisfying

$$\{ab\{xyz\}\} = \{\{abx\}yz\} + \{xy\{abz\}\} - \{x\{bay\}z\}$$

such that $\|a\|^3 = \|\{aaa\}\|$ and $x \mapsto \{aax\}$ is an hermitian linear operator on E with non negative spectrum.

For each a in E we write $D(a, a)(x) = \{aax\}$ and $Q_a(x) = \{axa\}$ for each x in E .

The geometry and algebra of JB*-triples are intimately related by the property [23] that the surjective linear isometries are the algebraic isomorphisms. A JB*-triple with a (necessarily unique) Banach predual is said to be a JBW*-triple, and in that case the triple product is separately weak* continuous [2]. The second dual E^{**} of a JB*-triple E is a JBW*-triple containing E as a subtriple in the natural embedding [13].

A norm closed subspace E of $B(H, K)$, where H and K are complex Hilbert spaces, for which xx^*x belongs to E whenever x does is a JB*-triple with triple product $\{xyz\} = \frac{1}{2}(xy^*z + zy^*x)$. If closed in the weak operator topology E is a JBW*-triple. Examples of this kind are four of the six types of Cartan factors. Namely, spin factors (see Section 4) and the rectangular, hermitian and symplectic Cartan factors which are, respectively, of the form $B(H, K)$, $\{x \in B(H) : x = jx^*j\}$ and $\{x \in B(H) : x = -jx^*j\}$ where $j : H \rightarrow H$ is a conjugation. The remaining two exceptional factors, both finite dimensional, are representable as certain spaces of 3×3 matrices with complex Cayley number entries.

A tripotent u (i. e. $u = \{uuu\}$) in a JB*-triple E gives rise to the mutually orthogonal Peirce projections

$$P_2(u) = Q_u^2, \quad P_1(u) = 2(D(u, u) - Q_u^2), \quad P_0(u) = I - 2D(u, u) + Q_u^2.$$

A non-zero tripotent u in E is said to be minimal if $P_2(u)(E) = \mathbb{C}u$. Each ρ in $\partial_e(E_1^*)$ is supported by a unique minimal tripotent, $u(\rho)$, in E^{**} and the assignment $\rho \mapsto u(\rho)$ is a bijection from $\partial_e(E_1^*)$ onto the set of minimal tripotents of E^{**} [16, Proposition 4].

Combining Propositions 2 and 4 of [16] we have the following.

Lemma 3.1. *If u is a tripotent in E^{**} where E is a JB*-triple and $\rho \in \partial_e(E_1^*)$ such that $\rho(u) = 1$ then $u(\rho) \in P_2(u)(E^{**})$.*

A subspace I of a JB*-triple E is said to be an ideal if $\{EEI\} + \{EIE\}$ is contained in I and to be an inner ideal if $\{IEI\}$ is contained in I . The norm closed ideals of E are its M-ideals [2]. The norm closed inner ideals of E are characterised in [14] as those JB*-subtriples B of E for which each element in B^* has unique norm preserving extension in E^* .

Proposition 3.2. *Let I be a norm closed inner ideal of a JB*-triple E . Let ρ be in $\partial_e(E_1^*)$ and let σ denote its restriction to I . If $\sigma \neq 0$, then $\|\sigma\|^{-1}\sigma$ lies in $\partial_e(I_1^*)$.*

Proof. By Propositions 2 and 7 of [16] there is a tripotent u in I^{**} with $\sigma(u) = \|\sigma\|$ and $\rho P_2(u) = \lambda\tau$, where $\tau \in \partial_e(E_1^*)$ and $0 < \lambda \leq 1$. We have

$$\lambda\tau(u) = \sigma(u) = \|\rho I^{**}\| \geq \|\rho P_2(u)\| = \lambda,$$

so that $\tau(u) = 1$. Hence, $u(\tau) \in I^{**}$, as follows from 3.1, giving

$$\|\sigma\|^{-1}\sigma = \tau|I \in \partial_e(I_1^*). \quad \square$$

Let E be a JB*-triple and let $\rho \in \partial_e(E_1^*)$. The weak* closed ideal E_ρ^{**} generated by $u(\rho)$ in E^{**} is a Cartan factor [12]. Let

$$P_\rho : E^{**} \longrightarrow E_\rho^{**}$$

denote the induced M -projection (and homomorphism) onto E_ρ^{**} . In the notation of 2.3, the primitive ideal $\psi(\rho)$ equals $E \cap \ker P_\rho$ [2, 3.6]. We shall write

$$[\rho] = \{ \tau \in \partial_e(E_1^*) : P_\tau = P_\rho \}.$$

Via $\tau \mapsto \tau P_\rho$, $[\rho]$ identifies with the extreme points of the predual ball of A_ρ^{**} .

We shall show that the continuous surjection

$$\psi : \partial_e(E_1^*) \longrightarrow \text{Prim}(E)$$

is an open mapping. For each ρ in $\partial_e(E_1^*)$ we note that

$$[\rho] \subset \psi^{-1}(\psi(\rho)) \subset \partial_e(E_1^*) \cap \psi(\rho)^0.$$

Lemma 3.3. *Let (J_i) be a family of norm closed ideals of a JB^* -triple E . For each i let S_i be a subset of J_i^0 norming E/J_i . Then $\cup S_i$ norms E/J where $J = \cap J_i$. Further, $J^0 \cap \partial_e(E_1^*)$ is contained in the weak* closure of $(\cup J_i^0) \cap \partial_e(E_1^*)$.*

Proof. The first statement follows from the usual isometric embedding of E/J into the l_∞ -sum of the E/J_i . Together with 2.5 this implies the second statement. \square

Lemma 3.4. *Let E be a JB^* -triple, let ρ be in $\partial_e(E_1^*)$ and let V be an open subset of $\partial_e(E_1^*)$ that has non-empty intersection with $\psi(\rho)^0$. Then*

- (a) $[\rho]$ is weak* dense in $\psi(\rho)^0 \cap \partial_e(E_1^*)$;
- (b) V has non-empty intersection with $\psi^{-1}(\psi(\rho))$.

Proof. (a) Identify $[\rho]$ with $\partial_e(M_{*,1})$ where $M = E_\rho^{**}$. By [16, 2.11] for τ in M_* with $\|\tau\| = 1$ we may write

$$\tau = \sum_{n=1}^{\infty} \lambda_n \rho_n,$$

where the ρ_n are in $[\rho]$ and the $\lambda_n \geq 0$ with $\sum_{n=1}^{\infty} \lambda_n = 1$, giving

$$|\tau(x)| \leq \sup |\rho_n(x)|, \quad \text{for each } x \in M.$$

Therefore, $[\rho]$ norms M and hence norms $P_\rho(E)$. Now, by the canonical isometries

$$P_\rho(E) \simeq E/\psi(\rho) \quad \text{and} \quad \psi(\rho)^0 \simeq (E/\psi(\rho))^*$$

together with 2.5, we have that $\psi(\rho)^0 \cap \partial_e(E_1^*)$ is the weak* closure of $[\rho]$ in $\partial_e(E_1^*)$.

- (b) Since $[\rho]$ is contained in $\psi^{-1}(\psi(\rho))$ this follows from (a). \square

Theorem 3.5. *Let E be a JB^* -triple. Then $\psi : \partial_e(E_1^*) \rightarrow \text{Prim}(E)$ is open.*

Proof. Let V be open in $\partial_e(E_1^*)$ and let $T = \{Q \in \text{Prim}(E) : Q^0 \cap V = \emptyset\}$. Let S be the union of the sets $\partial_e(E_1^*) \cap Q^0$ as Q ranges over T . Then

$$\partial_e(E_1^*) \cap k(T)^0 \subset \bar{S},$$

by 3.3. Since V is disjoint from S , it follows that V and $k(T)^0$ are disjoint. Hence, $\psi(V)$ is contained in $\text{Prim}(E) \setminus h(k(T))$.

Conversely, let $\rho \in \partial_e(E_1^*)$ such that $\psi(\rho)$ does not contain $k(T)$. Then $\psi(\rho)$ does not belong to T so that, by construction, $V \cap \psi(\rho)^0$ is non-empty. Therefore, $V \cap \psi^{-1}(\psi(\rho))$ is non-empty, by 3.4(b), implying that $\psi(\rho) \in \psi(V)$. Together with the first part of the proof this gives

$$\psi(V) = \text{Prim}(E) \setminus h(k(T)),$$

proving that ψ is open. \square

Remark 3.6. The socle, $K(E)$, of a JB*-triple E is the norm closed ideal of E generated by its minimal tripotents. We note that if E is prime such that $P_\rho(E)$ has non-zero socle for all ρ in $\partial_e(E_1^*)$, then E has non-zero socle. This follows from the fact (cf. [9, §3]) that there is in E a (necessarily prime) non-zero norm closed inner ideal I isometric to $C_0(X)$ for some locally compact Hausdorff space X , so that I must be one dimensional and hence generated by a minimal tripotent of E . In which case, by [5, Theorem 16], we have $K(M) \subset E \subset M$ (giving $K(M) = K(E)$) for some Cartan factor M .

We remark that for a Cartan factor M of rectangular, hermitian or symplectic type $K(M)$ is the ideal of compact operators in M ; all other Cartan factors are reflexive with $K(M) = M$.

Lemma 3.7. *Let M be a Cartan factor. Then $K(M)^0 \cap S(M_1^*)$ is contained in the $\sigma(M^*, M)$ closure of $\partial_e(M_{*,1})$.*

Proof. We may suppose $K(M) \neq M$. Let M be rectangular and let $\tau \in M^*$ such that τ vanishes on $K(M)$ and $\|\tau\| = 1$. By the Bishop-Phelps theorem we may suppose that $\tau(x) = \|x\| = 1$ for some x in M . Since the weak* closed subtriple of M generated by x can be realised as a commutative W*-algebra in which x is positive [20, 23] we have $\tau(u) = 1$ for some tripotent u of M . With $N = P_2(u)M$, τ vanishes on $K(N)$ and there is a surjective isometry $\phi : N \rightarrow B(H)$ with $\phi(u) = 1$, where H is a complex Hilbert space. It follows from [19, Theorem 2] that $\tau\phi^{-1}$ is the weak* limit of extreme points of the predual ball of $B(H)$. Therefore $\tau|N$ is the $\sigma(N^*, N)$ limit of a net (ρ_α) in $\partial_e(N_{*,1})$ with $\rho_\alpha(u) = 1$ for all α . Hence, $(\rho_\alpha P_2(u))$ lies in $\partial_e(M_{*,1})$ with τ as its $\sigma(M^*, M)$ limit. The other cases are handled similarly. \square

Given ρ in $\partial_e(E_1^*)$ where E is a JB*-triple we write

$$E(\rho) = (K(E_{\rho^{**}}) \oplus \ker P_\rho) \cap E$$

and

$$\gamma(E_1^*) = \cup\{E(\rho)^0 : \rho \in \partial_e(E_1^*)\} \cap S(E_1^*).$$

We continue to identify $[\rho]$ with $\partial_e(M_{*,1})$ where $M = A_\rho^{**}$.

Corollary 3.8. *Let E be a JB*-triple with ρ in $\partial_e(E_1^*)$. Then $E(\rho)^0 \cap S(E_1^*)$ is contained in the weak* closure of $[\rho]$. If E is prime with zero socle, then $\gamma(E_1^*)$ and $\partial_e(E_1^*)$ have the same weak* closure.*

Proof. We may suppose that $E(\rho) \neq E$. With $K = K(M)$ where $M = A_\rho^{**}$ we have

$$E/E(\rho) \simeq (P_\rho(E) + K)/K \text{ via the isometry } a + E(\rho) \mapsto P_\rho(a) + K.$$

Thus a norm one functional τ on E vanishing on $E(\rho)$ induces a norm one functional $\bar{\tau}$ on $P_\rho(E) + K$ vanishing on K such that $\tau(a) = \bar{\tau}P_\rho(a)$, for all a in E . In turn, $\bar{\tau}$ extends to τ' in $S(M_1^*)$ vanishing on K . By 3.7 there is a net (ρ_α) in $[\rho]$ with τ' as $\sigma(M^*, M)$ limit. Thus, for a in E ,

$$\rho_\alpha(a) = \rho_\alpha P_\rho(a) \longrightarrow \bar{\tau}P_\rho(a) = \tau(a)$$

proving the first statement.

Let E be prime with zero socle. We have $\overline{\gamma(E_1^*)} \subset \overline{\partial_e(E_1^*)}$ (weak* closure) by above. To show the opposite inclusion it is enough by 3.3 to show that

$$J = \cap \{E(\rho) : \rho \in \partial_e(E_1^*)\} = 0.$$

By construction if J is non-zero $P_\rho(J)$ has non-zero socle for each ρ in $\partial_e(J_1^*)$ so that (see 3.6) J has non-zero socle as therefore does E , a contradiction. \square

4. Spin factors and Hilbert spaces

Lemma 4.1. *Let I be a norm closed inner ideal in a JB^* -triple E . If $\partial_e(E_1^*)$ is weak* dense in $S(E_1^*)$ then $\partial_e(I_1^*)$ is weak* dense in $S(I_1^*)$.*

Proof. Let $\partial_e(E_1^*)$ be weak* dense in $S(E_1^*)$. Let $\rho \in S(I_1^*)$ with extension τ in $S(E_1^*)$ and let (τ_α) be a net in $\partial_e(E_1^*)$ with weak* limit τ . The corresponding net (ρ_α) of restrictions to I has weak* limit ρ . We may suppose all ρ_α are non-zero. Since $\|\rho_\alpha\| \leq 1$ for all α and $\|\rho\| = 1$ we have $\|\rho_\alpha\| \rightarrow 1$ so that ρ is the weak* limit of the net $(\|\rho_\alpha\|^{-1}\rho_\alpha)$ which, by 3.2 is contained in $\partial_e(I_1^*)$. The proof of the remaining statement is similar. \square

A complex Hilbert space H of dimension at least two with conjugation $x \mapsto \bar{x}$ induces the *spin factor*, $V = \mathbb{C}1 \oplus H$, which is a JB^* -algebra via

$$\begin{aligned} (\alpha 1 + h) \circ (\beta 1 + k) &= (\alpha\beta - \langle h, \bar{k} \rangle)1 + \beta h + \alpha k, \\ (\alpha 1 + h)^* &= \bar{\alpha}1 - \bar{h} \end{aligned}$$

and

$$\|x\|^2 = \langle x, x \rangle + (\langle x, x \rangle^2 - |\langle x, \bar{x} \rangle|^2)^{1/2},$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product on V when considered as the orthogonal sum of the Hilbert spaces \mathbb{C} and H , and $\bar{x} = \bar{\alpha}1 + \bar{h}$ for α in \mathbb{C} and h in H .

The triple product is given by

$$\{xyz\} = \langle x, y \rangle z + \langle z, y \rangle x - \langle x, \bar{z} \rangle \bar{y}$$

and we have $\|\cdot\|_2 \leq \|\cdot\| \leq \sqrt{2}\|\cdot\|_2$.

If u is a minimal tripotent of V then $\langle u, \bar{u} \rangle = 0$ giving $2\langle u, u \rangle = 1$ and $2|\langle u, x \rangle|^2 \leq \langle x, x \rangle$ for all x in V . If u is a non-zero and non-minimal tripotent in V then u is unitary, that is, $P_2(u)V = V$.

The real spin factor, $V_{sa} = \mathbb{R}1 \oplus K$, where $K = H_{sa} = \{h \in H : h = h^*\}$, has norm given by

$$\|\alpha 1 + k\| = |\alpha| + \|k\|_2, \quad \text{for all } \alpha \in \mathbb{R} \text{ and } k \in K.$$

Via the Riesz theorem the weak* compact set of states of V

$$B = \{\rho \in S(V_1^*) : \rho(1) = 1\} = \{\rho^k : k \in K, \|k\| \leq 1\}$$

where for k in K with $\|k\| \leq 1$ and all α in \mathbb{C} and h in H we have

$$\rho^k(\alpha 1 + h) = \alpha + \langle h, k \rangle \quad (= \langle \alpha 1 + h, 1 + k \rangle).$$

The minimal projections of V are the elements $\frac{1}{2}(1 + k)$ where k is in K with $\|k\| = 1$ and

$$\partial_e(B) = \{\rho^k : k \in K, \|k\| = 1\} \subset \partial_e(V_1^*).$$

Proposition 4.2. *Let V be an infinite dimensional spin factor. Then $\partial_e(V_1^*)$ is weak* dense in V_1^* .*

Proof. Let V be infinite dimensional and let $\tau \in S(V_1^*)$. By [16, Proposition 4] and [22, 4.2] there is a unitary tripotent v in V with $\tau(v) = 1$. Passing to the homotope $P_2(v)V$ we may suppose that $\tau(v) = 1$ so that, in the above notation, $\tau = \rho^k$ for some k in $K = H_{sa}$ with $\|k\| \leq 1$. Choose an infinite orthonormal sequence (k_n) in the real Hilbert space K with $\langle k_n, k \rangle = 0$, for all n . The norm one elements of K

$$y_n = (1 - \|k\|^2)^{1/2} k_n + k$$

converge weakly to k in K and hence in $H = K + iK$. Now each ρ^{y_n} is in $\partial_e(V_1^*)$ and for α in \mathbb{C} and h in H we have

$$\rho^{y_n}(\alpha 1 + h) = \alpha + \langle h, y_n \rangle \longrightarrow \alpha + \langle h, k \rangle = \tau(\alpha 1 + h).$$

Therefore, $\partial_e(V_1^*)$ is weak* dense in $S(V_1^*)$ and hence is weak* dense in V_1^* by 2.4. \square

We note that $\partial_e(E_1^*)$ is weak* closed if E is a finite dimensional JB*-triple. For suppose, in this case, that ρ in $S(E_1^*)$ is the weak* limit of a sequence (ρ_n) in $\partial_e(E_1^*)$. By compactness, passing to a subsequence we may suppose that $u_n \rightarrow u$ uniformly, where $u_n = u(\rho_n)$ and u is a tripotent in E . It follows that $P_2(u_n) \rightarrow P_2(u)$ uniformly. Therefore, for each x in E ,

$$\rho(x)u = \lim \rho_n(x)u_n = \lim P_2(u_n)(x) = P_2(u)x.$$

Hence, u is a minimal tripotent and, since $\rho(u) = 1$, we have $\rho \in \partial_e(E_1^*)$.

Corollary 4.3. *Let E be a JB*-triple with non-zero socle. Then $\partial_e(E_1^*)$ is weak* dense in E_1^* if and only if E is an infinite dimensional Hilbert space or an infinite dimensional spin factor.*

Proof. Let $\partial_e(E_1^*)$ be weak* dense in E_1^* . Then E is infinite dimensional and by 2.1 is prime so that $K(M) \subset E \subset M$ for some Cartan factor M , by [5, Theorem 16]. If I is a finite dimensional inner ideal of M then $\partial_e(I_1^*) = S(I_1^*)$, by 4.1 and the above remark, implying that I is a Hilbert space. It follows that M is a Hilbert space or a spin factor so that $E = M$, as required. \square

5. Characterisations of extreme dual density

A JB*-triple E is said to be *primitive* if the zero ideal of E is a primitive M-ideal, and to be *simple* if E has no non-trivial norm closed ideals. We continue to employ notation introduced in Section 3.

Theorem 5.1. *Let E be a non-zero JB*-triple. Then $[\rho]$ is weak* dense in E_1^* for some ρ in $\partial_e(E_1^*)$ if and only if E satisfies one of the following conditions.*

- (a) E is primitive with zero socle.
- (b) E is an infinite dimensional Hilbert space.
- (c) E is an infinite dimensional spin factor.

Proof. If E satisfies (b) or (c) then for every ρ in $\partial_e(E_1^*)$ we have $[\rho] = \partial_e(E_1^*)$, which is weak* dense in E_1^* by 4.3. Let E satisfy condition (a). Then, since for some ρ in $\partial_e(E_1^*)$ P_ρ is isometric on E and since (in the notation given prior to 3.8) $P_\rho(E(\rho)) \subset K(E_\rho^{**})$, we have $E(\rho) = 0$.

It follows from 3.8 that $S(E_1^*)$ is contained in the weak* closure of $[\rho]$. Thus $[\rho]$ is weak* dense in E_1^* by 2.4.

Conversely let $[\rho]$ be weak* dense in E_1^* for some ρ in $\partial_e(E_1^*)$. Since $\psi(\rho)^0 \cap \partial_e(E_1^*)$ is weak* closed in $\partial_e(E_1^*)$ and contains $[\rho]$ we have $\psi(\rho) = 0$, by 3.4 (a), so that E is primitive. The desired conclusion follows from this and 4.3. \square

Since a JB*-triple E is simple if and only if the zero ideal is the only primitive ideal of E , which is equivalent to the property that P_ρ is isometric on E for all ρ in $\partial_e(E_1^*)$, the above arguments verify the following statement.

Corollary 5.2. *Let E be a non-zero JB*-triple. Then $[\rho]$ is weak* dense in E_1^* for all ρ in $\partial_e(E_1^*)$ if and only if E satisfies one of the following conditions.*

- (a) E is simple with zero socle.
- (b) E is an infinite dimensional Hilbert space.
- (c) E is an infinite dimensional spin factor.

If u is a tripotent in a JB*-triple E , the operator on E

$$S(u) = I - 2P_1(u) = I - 4D(u, u) + 4Q_u^2$$

is an automorphism of E . For any x in E we write

$$S(x) = I - 4D(x, x) + 4Q_x^2.$$

Lemma 5.3. *Let E be a JB*-triple and let $x \in E$ with $\|x\| \leq 1$. Then $\|S(x)\| \leq 1$.*

Proof. Since each primitive quotient of E is isometric to a JB^* -subtriple of some $B(H)$ or of the finite dimensional exceptional JB^* -algebra factor M_3^8 , it is enough to validate the statement for these two special cases. In the first case we have

$$\|1 - 2xx^*\| \leq 1, \quad \|1 - 2x^*x\| \leq 1 \quad \text{and} \quad S(x)(y) = (1 - 2xx^*)y(1 - 2x^*x),$$

for each y in E , giving the required result.

Next, let E be M_3^8 . Upon passing to an appropriate unitary homotope of E we may suppose that $x \geq 0$. Since E is finite dimensional $S(x)$ attains its norm at an extreme point of the unit ball of E and hence at a unitary v of E . By [34], the JB^* -subalgebra, A , of E generated by x and v is isometric to a JB^* -subalgebra of some $B(H)$. Therefore, since $S(x)$ is invariant on A , we have that $\|S(x)\| \leq 1$ by the first case, above. \square

For a JB^* -triple E and ρ in $\partial_e(E_1^*)$ we make the definition

$$S_\rho(E_1^*) = \{\lambda\rho S(x) : |\lambda| = 1, x \in E, \|x\| \leq 1\},$$

and we note that $S_\rho(E_1^*)$ is contained in $\psi(\rho)^0 \cap E_1^*$.

Lemma 5.4. ([16, Corollary 2.5].) *Let v and w be minimal tripotents in a Cartan factor M . Then $\lambda S(u)v = w$ for some $|\lambda| = 1$ and tripotent u of M .*

Proof. This is [16, Corollary 2.5] for non-orthogonal v and w . If v and w are orthogonal then, being minimal projections in the spin factor $P_2(e)M$, where $e = v + w$, v and w are exchanged by a symmetry y in $P_2(e)M$ and we find that $S(u)v = w$ where u is given by $2u = e - y$. \square

Lemma 5.5. *Let E be a JB^* -triple with ρ in $\partial_e(E_1^*)$. Then $\psi(\rho)^0 \cap \partial_e(E_1^*)$ is contained in the weak* closure $S_\rho(E_1^*)$.*

Proof. By 3.4(a) it is enough to show that the weak* closure of $S_\rho(E_1^*)$ contains $[\rho]$. Given τ in $[\rho]$ we have $u(\tau)$ and $u(\rho)$ are in E_ρ^{**} so that

$$\lambda S(u)u(\tau) = u(\rho)$$

for some tripotent u in E_ρ^{**} and $|\lambda| = 1$. Since

$$(\lambda\rho S(u))u(\tau) = 1$$

we have $\tau = \lambda\rho S(u)$, by [16, Proposition 4]. By [3, Corollary 3.3] there is a net (x_α) in E with $\|x_\alpha\| \leq 1$ for all α such that $x_\alpha \rightarrow u$ in the strong* topology (see [3]) on E^{**} . By [29] (see also [27, 28]) it follows that $S(x_\alpha)$ converges to $S(u)$ in the pointwise strong* topology. In particular,

$$\lambda\rho S(x_\alpha)(x) \longrightarrow \lambda\rho S(u)(x) = \tau(x)$$

for all x in E , as required. \square

We can now classify the JB^* -triples E for which $\partial_e(E_1^*)$ is weak* dense in E_1^* .

Theorem 5.6. *Let E be a non-zero JB^* -triple. Then $\partial_e(E_1^*)$ is weak* dense in E_1^* if and only if E is an infinite dimensional Hilbert space, an infinite dimensional spin factor or E is prime with zero socle.*

Proof. In view of 4.3 and 3.5 it remains only to show that if E is prime with zero socle then E satisfies the condition (b) of 2.3 since it will then follow from 2.4 that $\partial_e(E_1^*)$ is weak* dense in E_1^* .

Let E be prime with zero socle and let $\rho \in \partial_e(E_1^*)$. We show that $\psi(\rho)^0 \cap S(E_1^*)$ is contained in the weak* closure of $\partial_e(E_1^*)$.

Let $\tau(\psi(\rho)) = \{0\}$ where $\tau \in S(E_1^*)$. By 5.5 together with the Krein–Milman theorem there is a net (τ_α) in the convex hull of $S_\rho(E_1^*)$ such that $\tau_\alpha \rightarrow \tau$ (in the weak* topology). We note that $\|\tau_\alpha\| \rightarrow 1$.

Let $0 < \epsilon < 1$. Choose α_0 such that for all $\alpha \geq \alpha_0$

$$2\|\tau_\alpha\| > 2 - \epsilon.$$

Now fix $\alpha \geq \alpha_0$. We have

$$\tau_\alpha = \mu_1 \lambda_1 \rho S(x_1) + \dots + \mu_n \lambda_n \rho S(x_n)$$

for some $|\lambda_i| = 1$, $\|x_i\| \leq 1$ and $\mu_i \geq 0$ with $\mu_1 + \dots + \mu_n = 1$.

By 3.8 there is a net (ϕ_β) in $\gamma(E_1^*)$ with $\phi_\beta \rightarrow \rho$. This gives

$$\psi_\beta = \mu_1 \lambda_1 \phi_\beta S(x_1) + \dots + \mu_n \lambda_n \phi_\beta S(x_n) \longrightarrow \tau_\alpha.$$

By the lower semicontinuity of the norm on E^* there exists β_0 such that

$$2\|\psi_\beta\| \geq 2\|\tau_\alpha\| - \epsilon, \quad \text{for all } \beta \geq \beta_0.$$

Since the $S(x_i)$ are invariant on ideals it follows from 3.8 that, for all $\beta \geq \beta_0$, $\sigma_\beta = \|\psi_\beta\|^{-1} \psi_\beta$ is contained in $\overline{\partial_e(E_1^*)}$ (weak* closure) and

$$1 + \epsilon > 2(2 - \epsilon)^{-1} \geq \|\tau_\alpha\|^{-1} \geq \lambda_\beta \geq \|\psi_\beta\| > 1 - \epsilon, \quad \text{where } \lambda_\beta = \|\psi_\beta\| \|\tau_\alpha\|^{-1}.$$

Since

$$\|\tau_\alpha\|^{-1} \tau_\alpha = \text{weak}^* \lim (\|\tau_\alpha\|^{-1} \psi_\beta) = \text{weak}^* \lim (\lambda_\beta \psi_\beta),$$

upon passing to a convergent subnet of (σ_β) it now follows that $\|\tau_\alpha\|^{-1} \tau_\alpha$ lies in $[1 - \epsilon, 1 + \epsilon] \cdot \overline{\partial_e(E_1^*)}$, for every $\alpha \geq \alpha_0$.

Since $\|\tau_\alpha\|^{-1} \tau_\alpha \rightarrow \tau$ we have in the same way that τ belongs to $[1 - \epsilon, 1 + \epsilon] \cdot \overline{\partial_e(E_1^*)}$ and in turn to $\overline{\partial_e(E_1^*)}$ by arbitrariness of ϵ , as required. \square

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