ON WEAK SEQUENTIAL CONVERGENCE IN JB*-TRIPLE DUALS

LESLIE J. BUNCE AND ANTONIO M. PERALTA

Abstract. We study various Banach space properties of the dual space E^* of a homogeneous Banach space (alias, a JB*-triple) E. For example, if all primitive M-ideals of E are maximal, we show that E^* has the Alternative Dunford-Pettis property (respectively, the Kadec-Klee property) if and only if all biholomorphic automorphisms of the open unit ball of E are sequentially weakly continuous (respectively, weakly continuous). Those E for which E^* has the weak* Kadec-Klee property are characterised by a compactness condition on E. Whenever it exists, the predual of E is shown to have the Kadec-Klee property if and only if E is atomic with no infinite spin part.

1. Introduction

Let E be a complex Banach space. It is said that E has the Kadec-Klee property (the KKP hereafter) if weak sequential convergence in the unit sphere of norm one elements of E implies norm convergence. In other words, the KKP is the Schur property confined to the unit sphere. When applied to the Dunford-Pettis property this procedure results in its "alternative" introduced and studied in [21]. Thus E is defined to possess the Alternative Dunford-Pettis property (the DP1 in the sequel) if, whenever (x_n) and (ρ_n) are sequences in E and E^*, respectively, where (ρ_n) is weakly null and $x_n \to x$ weakly in E with $\|x_n\| = \|x\| = 1$ for all n, we have $\rho_n(x_n) \to 0$. Plainly, the KKP implies the DP1 and both properties are geometric. The geometry of E is entirely determined by the structure of the group, $\mathcal{G}$, of biholomorphic automorphisms on the open unit ball, D, of E [27]. When $\mathcal{G}$ acts homogeneously on D, E is termed a JB*-triple. The latter comprise an extensive class of complex Banach spaces that includes all Hilbert spaces, spin factors and C*-algebras. More generally, given a complex Hilbert space, every norm closed subspace of $B(H)$ that is also closed under $x \mapsto xx^*x$ is a JB*-triple.

2000 *Mathematics Subject Classification.* Primary 17C65, 46L05, and 46L70.

It was shown in [21] that the DP1 coincides with the usual Dunford-Pettis property on von Neumann algebras. Modulo infinite dimensional Hilbert spaces and spin factors, which possess the DP1 but not the Dunford-Pettis property, this was extended to JBW*-triples in [1]. Recently [10] the present authors were able to establish that a von Neumann algebra is type I if and only if its predual has the DP1 and have proceeded to obtain an analogous characterisation for JBW*-triple preduals [11]. The latter (see §2) represents the starting point of this paper where we study the DP1 and the KKP on dual spaces of JB*-triples elucidating structure and connections with other convergence properties.

We recall that, as defined in [27], a JB*-triple is a complex Banach space E with a continuous triple product $(a,b,c) \mapsto \{a,b,c\}$ that is conjugate linear in b and symmetric bilinear in a and c for which each operator on E of the form $D = D(a,a)$, given by $x \mapsto \{a,a,x\}$, is hermitian with non-negative spectrum satisfying $\|D\| = \|a\|^2$ and

$$D(\{x,y,z\}) = \{D(x),y,z\} - \{x,D(y),z\} + \{x,y,D(z)\}.$$

A tripotent of E is an element u satisfying $\{u,u,u\} = u$, associated with which are the mutually orthogonal Peirce projections

$$P_2(u) = Q_u^2, \;\; P_1(u) = 2(D(u,u) - P_2(u)) \text{ and } P_0(u) = I - P_2(u) - P_1(u),$$

where Q_u is the conjugate linear operator given by $x \mapsto \{u,x,u\}$. A non-zero tripotent u of E is said to be minimal if $P_2(u)(E) = \mathbb{C}u$. If E has a predual, E_*, E is said to be a JBW*-triple. In which case, the predual is unique and the triple product is separately weak*-continuous [4]. If H is a complex Hilbert space, a weak* closed subspace of $B(H)$ that is closed under the triple product $\{a,b,c\} = \frac{1}{2}(ab^*c + cb^*a)$ is a JBW*-triple known as a JW*-triple. The Cartan factors, of which there are six kinds, are key examples of JBW*-triples. The rectangular, hermitian and symplectic Cartan factors, respectively, arise as the weak* closed left ideals of $B(H)$, the symmetric and the antisymmetric operators on H (with respect to a conjugation, where H is a complex Hilbert space). The spin factors comprise a fourth kind. The exceptional factors of dimensions 16 and 27 are the remaining two.

Let E be a JB*-triple. We habitually regard $E \subset E^{**}$, the latter being a JBW*-triple by [17]. We denote the extreme points of the dual ball E_1^* of E^* by $\partial_e(E_1^*)$. For each ρ in $\partial_e(E_1^*)$ there is a unique minimal tripotent $u(\rho)$ in E^{**} such that $\rho(u(\rho)) = 1$ and all minimal tripotents arise in this way [22]. The M-ideals of E are precisely its norm closed algebraic ideals [4]. By a primitive ideal of E is meant a primitive

M-ideal, the set of all of which is denoted by Prim (E). Thus,

$$\mathrm{Prim}(E) = \{\psi(\rho) : \rho \in \partial_e(E_1^*)\},$$

where for each $\rho \in \partial_e(E_1^*)$, $\psi(\rho)$ denotes the largest norm closed ideal (M-ideal) in $\ker(\rho)$. The corresponding structure map is $\psi : \partial_e(E_1^*) \to \mathrm{Prim}(E)$ $(\rho \mapsto \psi(\rho))$. When $\partial_e(E_1^*)$ has the weak* topology and $\mathrm{Prim}(E)$ has the usual hull-kernel topology, ψ is open and continuous [12]. We refer to [2, 5, 23] for M-ideal theory in Banach spaces.

2. The DP1 and KKP in JB*-triple Duals

By [24] the type I JBW*-triples are the ℓ_∞-sums of $A\overline{\otimes}C$ where A is an abelian von Neumann algebra and C is a Cartan factor. This notation is to be interpreted as follows. If C is an exceptional factor, $A\overline{\otimes}C$ means just $A\otimes C$. Otherwise $A\overline{\otimes}C$ means the weak* closure of $A\otimes C$ in the von Neumann tensor product $A\overline{\otimes}B(H)$ where C is a JW*-subtriple of $B(H)$.

We shall say that a JBW*-triple has *no infinite spin part* if it contains no non-zero ℓ_∞-summand of the form $A\overline{\otimes}C$, where A is an abelian von Neumann algebra and C is an infinite dimensional spin factor. An atomic JBW*-triple is an ℓ_∞-sum of Cartan factors.

Our starting point is the following recently discovered characterisation.

Lemma 2.1. *Let E be a JBW*-triple. Then E_* has the DP1 if and only if E is type I with no infinite spin part.*

Proof. See [11, Theorem 4.5] □

Theorem 2.2. *Let E be a JBW*-triple. Then E_* has the KKP if and only if E is atomic with no infinite spin part.*

Proof. Let E_* have the KKP. Then E_* has the DP1. Thus, by Lemma 2.1 and [24] we may suppose that E is of the form $A\overline{\otimes}C$, where A is an abelian von Neumann algebra and C is a Cartan factor not equal to an infinite dimensional spin factor. Given τ in C_* with $\|\tau\| = 1$, A_* is linearly isometric to the norm closed subspace $A_* \otimes \tau$ of E_* via $\rho \mapsto \rho \otimes \tau$. Since the KKP is inherited by norm closed subspaces, it follows that A_* has the KKP implying that A satisfies Dell'Antonio's property U and so is atomic, by [16, Theorem 2]. Hence, $A\overline{\otimes}C$ is atomic as required.

Conversely, let $E = (\sum C_\alpha)_\infty$, where each C_α is a Cartan factor not equal to an infinite dimensional spin factor. Since, by [21, Theorem 1.9], the KKP is stable under ℓ_1-sums it is enough to show that the predual of each C_α has the KKP. Thus, fixing $C_\alpha = C$, say, it may be supposed that C is an infinite dimensional rectangular, hermitian or symplectic factor. But then C_* is isometric to a subspace of $B(H)_*$, for some complex Hilbert space H, and so C_* has the KKP because $B(H)_*$ does, by [21, 2.3]. We remark that the latter fact, for separable H, may also be deduced from [3, Appendix]. This completes the proof. $\square$

By Theorem 2.2 together with [13] we have the following.

Corollary 2.3. *If E is a JBW*-triple then E_* has the KKP if and only if E_* has the Radon-Nikodym property and E has no infinite spin part.*

Various structure in JB*-triples is brought into focus when above discussed properties are imposed upon dual spaces. A composition series $\{J_\lambda : 0 \leq \lambda \leq \alpha\}$ in a JB*-triple E is a strictly increasing family of norm closed ideals of E indexed by a segment of ordinals satisfying (i) $J_0 = \{0\}$ and $J_\alpha = E$; (ii) if λ is a limit ordinal then J_λ is the norm closure of the union of $\{J_\mu : \mu < \lambda\}$. If $\mathcal{G}$ denotes the group of biholomorphic automorphisms of the open unit ball of E, then E is said to be sequentially weakly continuous if every element of $\mathcal{G}$ is sequentially weakly continuous (i.e. preserves weak sequential limits). If all elements of $\mathcal{G}$ are weakly continuous then E is defined to be weakly continuous.

Weak (sequential) continuity of this kind has been extensively studied in [25, 26, 28] and also in [9].

By a quotient of a JB*-triple E we shall mean E/J for some norm closed ideal J of E. An elementary JB*-triple is the norm closed ideal, $J(C)$, of a Cartan factor C generated by its minimal tripotents. We note that $J(C)^{**} = C$.

Lemma 2.4. *A JB*-triple E has no infinite dimensional spin factor quotients if and only if E^{**} has no infinite spin part.*

Proof. See [9, Theorem 4.4]. $\square$

Proposition 2.5. *The following are equivalent for a JB*-triple E*

- (a) *E^* has the KKP;*
- (b) *E^* has the Radon-Nikodym property and E has no infinite dimensional spin factor quotients;*
- (c) *E has a composition series $\{J_\lambda : 0 \leq \lambda \leq \alpha\}$ such that for each $\lambda < \alpha$, $J_{\lambda+1}/J_\lambda$ is an elementary JB*-triple not equal to an infinite dimensional spin factor.*

Proof. The equivalence of (a) and (b) follows from Corollary 2.3 and Lemma 2.4. The latter together with [7, Theorem 3.4] implies that (b) and (c) are equivalent. □

A DP1 analogue of Proposition 2.5 is availed by the following.

Lemma 2.6. *The following are equivalent for a JB*-triple E.*

(1) *E is sequentially weakly continuous;*
(2) *Every primitive ideal of E is maximal and E^{**} is type I with no infinite spin part;*
(3) *Every primitive quotient of E is an elementary JB*-triple not equal to an infinite dimensional spin factor.*

Proof. See [9, Theorem 5.5]. □

Proposition 2.7. *Let E be a JB*-triple. Then E^* has the DP1 if and only if E has a composition series $\{J_\lambda : 0 \leq \lambda \leq \alpha\}$ such that , for each $\lambda < \alpha$, $J_{\lambda+1}/J_\lambda$ is sequentially weakly continuous.*

Proof. Suppose E^* has the DP1. By Lemma 2.1 and Lemma 2.4 E^{**} is type I and E has no infinite dimensional spin factor quotients. Since every norm closed ideal and quotient of E inherits the latter condition, it follows from [9, Proposition 3.5] that E has a composition series for which each successive quotient satisfies the condition of Lemma 2.6(c), as required.

Conversely, if E has a composition series of the kind described in the statement, then E^{**} is linearly isometric to the ℓ_∞-sum of the $(J_{\lambda+1}/J_\lambda)^{**}$ so that E^{**} is type I with no infinite spin part, as follows from Lemma 2.6(a) $\Rightarrow$ (c), whence E^* has the DP1 by Lemma 2.1. □

By a standard argument (c.f. [18, 4.3.5]) if E is a JB*-triple with a composition series $\{J_\lambda : 0 \leq \lambda \leq \alpha\}$ and F is a JB*-subtriple of E, then $\{F \cap J_\lambda : 0 \leq \lambda \leq \alpha\}$ is a composition series of F and each $(F \cap J_{\lambda+1})/(F \cap J_\lambda)$ is realisable as a JB*-subtriple of $J_{\lambda+1}/J_\lambda$.

Corollary 2.8. *Let F be a JB*-subtriple of a JB*-triple E.*

(a) *If E^* has the DP1, then F^* has the DP1.*
(b) *If E^* has the KKP, then F^* has the KKP.*

Proof. (a) Since sequential weak continuity is inherited by JB*-subtriples and by quotients this follows from the preceding remark and Proposition 2.7.

(b) If E^* has the KKP then it has the DP1 so that via (a) and its argument together with Proposition 2.5 (a) $\Leftrightarrow$ (c), F has a composition series in which successive quotients are JB*-subtriples of non-infinite dimensional spin factor elementary JB*-triples. But JB*-subtriples of the latter kind are themselves C_0-sums of JB*-triples of the same kind [8]. It follows that F has a composition series of the kind described in Proposition 2.5(c), whence the result. $\square$

Remark. The analogues of 2.8$(a), (b)$ for preduals of JBW*-triples are false. Any non-type I JW*-triple E can be realised as a JW*-subtriple of $B(H)$ for some complex Hilbert space H. But $B(H)_*$ has the KKP whereas E_* (by Lemma 2.1) does not ever have the DP1.

In the next result part (a) is a consequence of Lemma 2.1 together with Lemmma 2.6 $(b) \Rightarrow (a)$ and (b) follows from Theorem 2.2 combined with [28, Theorem 5.7] to give a direct comparison of above phenomena in a significant case.

Proposition 2.9. *Let E be a JB*-triple for which every primitive ideal is maximal. Then*

- (a) *E^* has the DP1 if and only if E is sequentially weakly continuous.*
- (b) *E^* has the KKP if and only if E is weakly continuous.* $\square$

A JBW*-triple E is said to be σ-finite if every family of mutually orthogonal tripotents in E is at most countable. Such JBW*-triples have been studied in [20]. On the bidual of a JB*-triple σ-finiteness is a strong condition revealing structures similar to those discussed above, as we shall now see.

First we recall, [27], that the JB*-subtriple generated by an element x in a JB*-triple E is linearly isometric to $C_0(S_x)$, where S_x, the triple spectrum of x, is a locally compact Hausdorff space of $[0, +\infty)$ with $S_x \cup \{0\}$ compact. This notation is retained in the next result and thereafter.

Theorem 2.10. *The following are equivalent for a JB*-triple E.*

- (a) *E^{**} is σ-finite,*
- (b) *E^{**} is atomic and σ-finite,*
- (c) *E has a countable composition series $(J_\lambda)_{0\leq\lambda\leq\alpha}$, where each $J_{\lambda+1}/J_\lambda$ is an elementary JB*-triple of countable rank.*

Hence, if E is separable, then E^* has the KKP if and only if E^{**} is σ-finite with no infinite spin part.

Proof. If (c) holds then E^{**} is linearly isometric to the countable ℓ_∞-sum of the necessarily σ-finite Cartan factors $(J_{\lambda+1}/J_\lambda)^{**}$, implying the condition (b). The implication $(b) \Rightarrow (a)$ being obvious, it remains to show $(a) \Rightarrow (c)$.

Let E^{**} be σ-finite and let $x \in E$. Since $C_0(S_x)$ is linearly isometric to a JB*-subtriple of E, $C_0(S_x)^{**}$ is linearly isometric to a JBW*-subtriple of E^{**} and so is σ-finite. Since the support projections, in $C_0(S_x)^{**}$, of the evaluation maps on $C_0(S_x)$ are mutually orthogonal, S_x must be countable. Thus, (c) now follows from [8, Theorem 3.4] and [9, Theorem 4.5]. $\square$

Remark 2.11. Non-separable spin factors and rectangular Cartan factors of the form $B(H, K)$ where H is non-separable and K is separable, are σ-finite with non-separable elementary ideal. All other σ-finite Cartan factors have separable elementary ideal. Thus, if E^{**} is σ-finite and contains no Cartan factor ℓ_∞-summands of the first kind mentioned above, then E^* is norm separable. Hence, in this case, if E^{**} is σ-finite then E is separable if and only if E^* is separable.

We conclude this section with two observations.

Proposition 2.12. *Let E be a JB*-triple. Then*

(a) *E has a largest norm closed ideal J for which J^* has the DP1;*
(b) *E has a largest norm closed ideal J for which J^* has the KKP.*

Proof. (a) Let K be the largest weak* closed ideal of E^{**} that is a type I JBW*-triple with no infinite spin part and let $J = E \cap K$. Then J^* has the DP1, by Lemma 2.1, since J^{**} is a weak* closed ideal of K. Conversely, let I be a norm closed ideal of E such that I^* has the DP1. A further application of Lemma 2.1 gives $I^{**} \subset K$ so that $I = I^{**} \cap E \subset K \cap E = J$.

(b) Via Theorem 2.2, the proof is similar. $\square$

Proposition 2.13. *Let E be a JB*-triple with the KKP. Then E is finite dimensional or a spin factor or a Hilbert space.*

Proof. Given $x \in E$, the commutative C*-algebra $C_0(S_x)$ has the KKP and so is finite dimensional by [21, Theorem 3.4] implying that S_x is finite. By, [8, Proposition 4.5 (iii)] and [14, Theorem 6], this implies that E is reflexive. In particular, E is a JBW*-triple and the result is now immediate from [1, Corollary 3]. $\square$

3. The weak* Kadec-Klee Property

If X is a Banach space, let $S(X_1^*)$ denote the unit sphere of norm one elements in X^*.

Definition 3.1. Let X be a Banach space. The dual space, X^*, is said to have the *weak* Kadec-Klee property* (W*KKP, in the sequel) if weak* sequential convergence in $S(X_1^*)$ implies norm convergence.

By [19, Theorem 2.6], if I is a norm closed inner ideal of a JB*-triple E, then each $\rho \in S(I_1^*)$ has a unique extension $\overline{\rho} \in S(E_1^*)$. We retain this notation in the following, the statement and proof of which is reminiscent of [15, Lemma 1].

Lemma 3.2. *Let I be a norm closed inner ideal in a JB*-triple E. Then the unique extension map, from $S(I_1^*)$ to $S(E_1^*)$, is weak*-continuous.*

Proof. Let $\rho_\alpha \to \rho$ in the $\sigma(I^*, I)$-topology in $S(I_1^*)$. To show continuity it is enough to show that there is a subnet $\overline{\rho}_\beta \to \overline{\rho}$ in the $\sigma(E^*, E)$-topology. But there is a subnet $\overline{\rho}_\beta \to \tau$ in the $\sigma(E^*, E)$-topology with $\tau \in E_1^*$. Since $\tau|_I = \rho$, we have $\|\tau\| = 1$ so that $\tau = \overline{\rho}$ by the above mentioned uniqueness, as required. □

Corollary 3.3. *Let I be a norm closed inner ideal in a JB*-triple E such that E^* has the W*KKP. Then I^* has the W*KKP.*

Proof. This follows from Lemma 3.2. □

We recall that a JB*-triple E is defined to be a *compact* JB*-triple if the conjugate linear operator, $x \mapsto \{a, x, a\}$, is compact for each $a \in E$. Such JB*-triples have been studied in [8, 6]. By [8, Theorems 3.4, 3.6] or [6, Theorem 18] a JB*-triple E is compact if and only if E is a C_0-sum of elementary JB*-triples E_i, where no E_i is an infinite dimensional spin factor. In which case, E^{**} is the ℓ_∞-sum of the Cartan factors E_i^{**}. Since, then, E is an ideal of E^{**} it follows from Lemma 3.2, for example, that weak*-convergence in $S(E_1^*)$ implies weak convergence. Therefore by Theorem 2.2, we have the following.

Lemma 3.4. *If E is a compact JB*-triple, then E^* has the W*KKP.*

Elements ρ, τ in $\partial_e(E_1^*)$, where E is a JB*-triple, are *orthogonal* if the tripotents $u(\rho), u(\tau)$ are orthogonal, in which case $\|\rho - \tau\| = 2$ (since $(\rho - \tau)(u(\rho) - u(\tau)) = 2$).

Lemma 3.5. *Let E be a separable JB*-triple such that E^* has the W*KKP. Then E is a compact JB*-triple.*

Proof. Since E^* has the W*KKP it has the KKP and so contains a non-zero elementary compact JB*-triple ideal by Proposition 2.5 ($(a) \Rightarrow (c)$). Let J be the C_0-sum of all such ideals of E. Then J is compact and we must show that $J = E$. In order to obtain a contradiction, suppose that $J \neq E$.

We have $\partial_e(E_1^*) = X \cup Y$, where

$$X = \{\rho \in \partial_e(E_1^*) | \rho(J) \neq 0\} \text{ and } Y = \{\rho \in \partial_e(E_1^*) | \rho(J) = 0\},$$

the latter being weak* closed in $\partial_e(E_1^*)$. If X is weak* closed in $\partial_e(E_1^*)$ then Y is open so that $\psi(Y)$ is open in Prim(E), where

$$\psi : \partial_e(E_1^*) \to Prim(E)$$

is the structure map (see Introduction). This would imply that $E = I \oplus J$ for some non-zero norm closed ideal I of E. Since I^* has the W*KKP, this would further imply that I contains a non-zero compact elementary ideal orthogonal to J, a contradiction. Therefore, X is not weak*-closed and so, since the unit ball of E_1^* is metrisable, there is a sequence (ρ_n) in X with weak* limit ρ in Y. But ρ is orthogonal to each ρ_n giving $\|\rho_n - \rho\| = 2$, for all n, and we have arrived at the desired contradiction. □

Lemma 3.6. *Let E be a compact JB*-triple and let (ρ_n) be an infinite mutually orthogonal sequence in $\partial_e(E_1^*)$. Then (ρ_n) is weak* null.*

Proof. For each n, let e_n denote $u(\rho_n)$, the support tripotent of ρ_n, and let e denote $\sum_n e_n$ (in E^{**}). Let f be a minimal tripotent of E^{**} and let $\rho \in \partial_e(E_1^*)$ with $\rho(f) = 1$. We have $\rho(e) = \sum_n \rho(e_n)$, so that $\rho(e_n) \to 0$. Hence, via [22, Lemma 2.2], since $\rho_n, \rho \in \partial_e(E_1^*)$ and e_n and f are respectively, their support tripotents, we have $|\rho_n(f)| = |\rho(e_n)| \to 0$.

Given $x \in E$, let $\varepsilon > 0$. Since E is compact there exists $y \in E$ such that y is a linear combination of a finite number of minimal tripotents in E with $\|x - y\| \leq \varepsilon$. By above, $\rho_n(y) \to 0$, so that for all n large enough

$$|\rho_n(x)| \leq |\rho_n(x - y)| + |\rho_n(y)| < 2\varepsilon.$$

Hence, (ρ_n) is weak* null. □

Given a JB*-algebra E and $x \in E$ with $0 \leq x \leq 1$, then $r(x)$ will denote the range projection of x in E^{**} (i.e. the least projection in E^{**} majorising x).

Lemma 3.7. *Let J be a norm closed compact ideal of a non-compact JB*-algebra E and let $E = J + \mathbb{C}x$, where x is a positive norm one element of E with $x(x-1) \in J$ and $r(x) = 1$ (in E^{**}). Then E^* does not have the W*KKP.*

Proof. We note that J is an essential ideal of E. Indeed, suppose $I \cap J = 0$ for some ideal I of E. Then $E = I \oplus J$ and hence I is one dimensional which is a contradiction since E is non-compact. We also note that $x + J$ is a minimal projection of E/J. Further, $x(1 - x) = \sum_n \lambda_n e_n$, for some mutually orthogonal sequence (e_n) of minimal tripotents in J and non-negative null sequence (λ_n). We have

$$r(1 - x) = r(x)\, r(1 - x) = r(x(1 - x)) \leq \sum e_n,$$

so that $r(1-x)$ is a σ-finite projection in E^{**}. If $1-r(1-x)$ is of finite rank then E^{**} must be σ-finite so that E is separable since E is, by construction, a JB*-algebra with no infinite spin factor quotients (see Remark 2.11). In this case the result follows from Lemma 3.5. Thus we may suppose that there exists an infinite mutually orthogonal sequence, (f_n), of minimal projections in E^{**} such that $f_n \leq 1-r(1-x)$, for all n. For each n, let $\rho_n \in \partial_e(E_1^*)$ with support f_n.

Then, for each n, ρ_n is a *pure state* of E and $f_n \leq 1-r(1-x) \leq x$, so that $\rho_n(x) = 1$. Since, by Lemma 3.6, (ρ_n) is weakly null on J, we have that (ρ_n) has weak* limit $\tau \in S(E_1^*)$. But $\|\rho_n - \rho_m\| = 2$ for $n \neq m$ so that (ρ_n) is not norm convergent. □

Given a JB*-triple E and $x \in E$ the norm closed inner ideal, $E(x)$, generated by x in E can be realised as a JB*-algebra containing x as a positive element.

We are now ready to prove the converse of Lemma 3.4.

Theorem 3.8. *Let E be a JB*-triple. Then E^* has the W*KKP if and only if E is compact.*

Proof. Let E^* have the W*KKP. As in the proof of Lemma 3.5, E contains an essential norm closed compact ideal, J, equal to the norm closed ideal generated by the minimal tripotents of E. Suppose that $J \neq E$. Since $(E/J)^*$ has the W*KKP we can choose a norm one element x of $E \backslash J$ such that $x + J$ is a minimal tripotent of E/J. Let I denote $J \cap E(x)$, where $E(x)$ is the norm closed inner ideal of E generated by x. We have

$$J + E(x) = J + \mathbb{C}x$$

so that

$$E(x) = I + \mathbb{C}x,$$

via the natural linear isometry between $E(x)/I$ and $(J + E(x))/J$.

Passing to the JB*-algebra, $E(x)$, we have $x \geq 0$ in $E(x)$ and $r(x)$ is the identity element of $(E(x))^{**}$. Further, $x + J$ is a projection in $E(x)/I$ so that $x(1-x) \in I$, and I is a compact ideal of $E(x)$.

Moreover, $E(x)$ is not compact else x lies in the norm closed linear span of the minimal projections of $E(x)$ and, since the latter is an inner ideal of E, this would imply the contradiction that x is in I. Thus, by Lemma 3.7, $E(x)^*$ does not have the W*KKP and so, by Corollary 3.3, neither does E^*. Therefore, $I = E$, as required. □

Acknowledgements. The first author wishes to express his gratitude for generous financial support from Junta de Andalucía grant FQM 0199, and hospitality from the Departamento de Análisis

Matemático of the University of Granada, where this work was done. The second author was partially supported by D.G.I.C.Y.T. project no. PB 98-1371, and Junta de Andalucía grant FQM 0199.

References

[1] Acosta, M. D. and Peralta, A. M., An alternative Dunford-Pettis property for JB*-triples, Quart. J. Math. Oxford Ser. **52**, 391-401 (2001).

[2] Alfsen, E. M. and Effros, E. G., Structure in real Banach spaces. I, II. Ann. of Math. (2) **96**, 98–128 and 129–173 (1972).

[3] Arazy, J., More on convergence in unitary matrix spaces, Proc. Amer. Math. Soc. **83**, 44–48 (1981).

[4] Barton, T. and Timoney, R., Weak* continuity of Jordan triple product and applications, Math. Scand. **59**, 177-191 (1986).

[5] Behrends, E., *M-structure and the Banach-Stone theorem*, Lecture Notes in Mathematics, **736**. Springer, Berlin, 1979.

[6] Bouhya, K. and Fernandez Lopez, A., Jordan-$*$-triples with minimal inner ideals and compact JB*-triples, Proc. London Math. Soc. (3) **68**, no. 2, 380–398 (1994).

[7] Bunce, L. J. and Chu, C.-H., Dual spaces of JB*-triples and the Radon-Nikodym property, Math. Z. **208**, 327-334 (1991).

[8] Bunce, L. J. and Chu, C.-H., Compact operations, multipliers and Radon-Nikodym property in JB*-triples, Pacific J. Math. **153**, 249-265 (1992).

[9] Bunce, L. J., Chu, C.-H., and Zalar, B., Classification of sequentially weakly continuous JB*-triples, Math. Z. **234**, no. 1, 191-208 (2000).

[10] Bunce, L. J. and Peralta, A. M., The alternative Dunford-Pettis property in C*-algebras and von Neumann preduals, Proc. Amer. Math. Soc. to appear.

[11] Bunce, L. J. and Peralta, A. M., Images of Contractive Projections on Operator Algebras, J. Math. Anal. Appl. to appear.

[12] Bunce, L. J. and Sheppard, B., Prime JB*-triple and Extreme Dual Ball Density, Math. Nachr. to appear.

[13] Chu, C.-H. and Iochum, B., On the Radon-Nikodym property in Jordan triples, Proc. Amer. Math. Soc. **99**, 462-464 (1987).

[14] Chu, C.-H. and Iochum, B., Complementation of Jordan triples in von Neumann algebras, Proc. Amer. Math. Soc. **108**, 19-24 (1990).

[15] Chu, C.-H. and Kusuda, M., On factor states of C^*-algebras and their extensions, Proc. Amer. Math. Soc. **124** (1996), 207–215 (1996).

[16] Dell'Antonio, G. F., On the limits of sequences of normal states, Comm. Pure Appl. Math. 20, 413–429 (1967).

[17] Dineen, S., The second dual of a JB*-triple system, *Complex Analysis, Functional Analysis and Approximation Theory*, vol. 125, North Holland Maths. Studies, 67-69, 1986.

[18] Dixmier, J., *C^*-algebras*, North-Holland, Amsterdam-New York-Oxford, 1977.

[19] Edwards, C. M. and Rüttimann, G. T, A characterization of inner ideals in JB*-triples, Proc. Amer. Math. Soc. **116**, no. 4, 1049-1057 (1992).

[20] Edwards, C. M. and Rüttimann, G. T, Exposed faces of the unit ball in a JBW*-triple, Math. Scand. **82**, no. 2, 287–304 (1998).

[21] Freedman, W., An alternative Dunford-Pettis property, Studia Math. 125, 143–159 (1997).

[22] Friedman, Y. and Russo, B., Structure of the predual of a JBW*-triple, J. Reine u. Angew. Math. **356**, 67-89 (1985).

[23] Harmand, P., Werner, D. and Werner, W., *M-ideals in Banach spaces and Banach algebras*, Lecture Notes in Mathematics, **1547**, Springer-Verlag, Berlin, 1993.

[24] Horn, G., Classification of JBW*-Triples of type I, Math. Z. **196**, 271-291 (1987).

[25] Isidro, J. M. and Kaup, W., Weak continuity of holomorphic automorphisms in JB*-triples, Math. Z. **210**, no. 2, 277-288 (1992).

[26] Isidro, J. M. and Stacho, L. L., On weakly and weakly* continuous elements in Jordan triples, Acta Sci. Math. (Szeged) **57**, no. 1-4, 555–567 (1993).

[27] Kaup, W., A Riemann mapping theorem for bounded symmetric domains in complex Banach spaces, Math. Z. **183**, 503-529 (1983).

[28] Kaup, W. and Stacho, L. L., Weakly continuous JB*-triples, Math. Nachr. **166**, 305–315 (1994).

University of Reading, Reading RG6 2AX, Great Britain.
E-mail address: L.J.Bunce@reading.ac.uk

Departamento de Análisis Matemático, Facultad de Ciencias, Universidad de Granada, 18071 Granada, Spain.
E-mail address: aperalta@goliat.ugr.es